Metrisability of Painleve equations and Hamiltonian systems of hydrodynamic type

Felipe Contatto

Department of Applied Mathematics and Theoretical Physics
University of Cambridge

felipe.contatto@damtp.cam.ac.uk

Joint work with Maciej Dunajski.

arXiv:1510.01906 - "First integrals of affine connections and Hamiltonian systems of hydrodynamic type"
arXiv:1604.03579 - "Metrisability of Painlevé equations"

28 July 2016

Overview

1. Metrisability of projective structures
2. Deriving first integrals
3. Killing forms
4. Hydrodynamic-type systems

Overview

1. Metrisability of projective structures
2. Deriving first integrals
3. Killing forms
4. Hydrodynamic-type systems

Problems: (i)Consider the projective structures defined by the Painlevé equations, which of them are metrisable (if any)?

Overview

1. Metrisability of projective structures
2. Deriving first integrals
3. Killing forms
4. Hydrodynamic-type systems

Problems: (i)Consider the projective structures defined by the Painlevé equations, which of them are metrisable (if any)?
(ii) How many first integrals linear in the momenta does a geodesic flow admit?

Overview

1. Metrisability of projective structures
2. Deriving first integrals
3. Killing forms
4. Hydrodynamic-type systems

Problems: (i)Consider the projective structures defined by the Painlevé equations, which of them are metrisable (if any)?
(ii) How many first integrals linear in the momenta does a geodesic flow admit?
(iii) Given a HT system, how many hamiltonian formulations (local sense) does it admit?

Metrisability problem (R. Liouville, 1889)

- Given a set of curves on $\mathbb{R}^n$, are they traces of geodesics of some metric?

Metrisability problem (R. Liouville, 1889)

- Given a set of curves on $\mathbb{R}^n$, are they traces of geodesics of some metric?
- On the (x, y)-plane, the necessary and sufficient condition for the curves to be the geodesics of an affine connection: to be the integral curves of an ODE $y''(x) = \mathcal{F}(x, y, y')$, where $\partial^4_{y'}\mathcal{F}(x, y, y') = 0$.

Metrisability problem (R. Liouville, 1889)

- Given a set of curves on $\mathbb{R}^n$, are they traces of geodesics of some metric?
- On the (x, y)-plane, the necessary and sufficient condition for the curves to be the geodesics of an affine connection: to be the integral curves of an ODE $y''(x) = \mathcal{F}(x, y, y')$, where $\partial^4_{y'}\mathcal{F}(x, y, y') = 0$.

Equivalence relation: two torsion-free connections are *projectively equivalent* ($\Gamma \sim \hat{\Gamma}$) if they share the same unparametrised geodesics.

Metrisability problem (R. Liouville, 1889)

- Given a set of curves on $\mathbb{R}^n$, are they traces of geodesics of some metric?
- On the (x, y)-plane, the necessary and sufficient condition for the curves to be the geodesics of an affine connection: to be the integral curves of an ODE $y''(x) = \mathcal{F}(x, y, y')$, where $\partial^4_{y'}\mathcal{F}(x, y, y') = 0$.

Equivalence relation: two torsion-free connections are *projectively equivalent* $(\Gamma \sim \hat{\Gamma})$ if they share the same unparametrised geodesics.

Definition

A **projective structure** $[\Gamma]$ *is the class of torsion-free connections that are projectively equivalent to* Γ.

Metrisability problem (R. Liouville, 1889)

- Given a set of curves on $\mathbb{R}^n$, are they traces of geodesics of some metric?
- On the (x, y)-plane, the necessary and sufficient condition for the curves to be the geodesics of an affine connection: to be the integral curves of an ODE $y''(x) = \mathcal{F}(x, y, y')$, where $\partial^4_{y'}\mathcal{F}(x, y, y') = 0$.

Equivalence relation: two torsion-free connections are *projectively equivalent* ($\Gamma \sim \hat{\Gamma}$) if they share the same unparametrised geodesics.

Definition

A **projective structure** $[\Gamma]$ *is the class of torsion-free connections that are projectively equivalent to* Γ.

- Metrisability problem: given a projective structure, does it contain a metric connection?

- *The problem is (almost) solved in* $n = 2$ *dimensions*:

- *The problem is (almost) solved in $n = 2$ dimensions*: the necessary condition for the existence of a metric involves the vanishing of some invariants of differential order at least 5 in the connection [Bryant-Dunajski-Eastwood].
- For the construction of a metric, the solution to the metrisability equations must satisfy the non-degeneracy condition.

Hamiltonian description of geodesics

- Consider the metric $g = g_{11}dx^2 + 2g_{12}dxdy + g_{22}dy^2$

and the geodesic Hamiltonian $H = \frac{1}{2}g_{ab}p^a p^b$.

Hamiltonian description of geodesics

- Consider the metric $g = g_{11}dx^2 + 2g_{12}dxdy + g_{22}dy^2$

and the geodesic Hamiltonian $H = \frac{1}{2}g_{ab}p^a p^b$.

- Hamilton's equations

$$\dot{x}^a(t) = \frac{\partial H}{\partial p_a} \quad \dot{p}_a(t) = -\frac{\partial H}{\partial x^a}$$

Hamiltonian description of geodesics

- Consider the metric $g = g_{11}dx^2 + 2g_{12}dxdy + g_{22}dy^2$

and the geodesic Hamiltonian $H = \frac{1}{2}g_{ab}p^a p^b$.

- Hamilton's equations

$$\dot{x}^a(t) = \frac{\partial H}{\partial p_a} \quad \dot{p}_a(t) = -\frac{\partial H}{\partial x^a}$$

- Geodesic equations

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0, \quad a = 1, 2.$$

Hamiltonian description of geodesics

- Consider the metric $g = g_{11}dx^2 + 2g_{12}dxdy + g_{22}dy^2$

and the geodesic Hamiltonian $H = \frac{1}{2}g_{ab}p^a p^b$.

- Hamilton's equations

$$\dot{x}^a(t) = \frac{\partial H}{\partial p_a} \quad \dot{p}_a(t) = -\frac{\partial H}{\partial x^a}$$

- Geodesic equations

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0, \quad a = 1, 2.$$

- Unparametrised geodesics (divide $(a = 2)/(a = 1)$)

$$y'' = A_3(x, y)y'^3 + A_2(x, y)y'^2 + A_1(x, y)y' + A_0(x, y) = \mathcal{F}(x, y, y'),$$

Hamiltonian description of geodesics

- Consider the metric $g = g_{11}dx^2 + 2g_{12}dxdy + g_{22}dy^2$

and the geodesic Hamiltonian $H = \frac{1}{2}g_{ab}p^a p^b$.

- Hamilton's equations

$$\dot{x}^a(t) = \frac{\partial H}{\partial p_a} \quad \dot{p}_a(t) = -\frac{\partial H}{\partial x^a}$$

- Geodesic equations

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0, \quad a = 1, 2.$$

- Unparametrised geodesics (divide $(a = 2)/(a = 1)$)

$$y'' = A_3(x, y)y'^3 + A_2(x, y)y'^2 + A_1(x, y)y' + A_0(x, y) = \mathcal{F}(x, y, y'),$$

where $A_0 = -\Gamma^2_{11}, \quad A_1 = \Gamma^1_{11} - 2\Gamma^2_{12}, \quad A_2 = 2\Gamma^1_{12} - \Gamma^2_{22}, \quad A_3 = \Gamma^1_{22}.$

- If K is a Killing vector $\Rightarrow$ conserved quantity: $K_a\dot{x}^a = g_{ab}K^b\dot{x}^a$

- If K is a Killing vector $\Rightarrow$ conserved quantity: $K_a \dot{x}^a = g_{ab} K^b \dot{x}^a$
- Eliminate t using $H = \frac{1}{2}(g_{11}\dot{x}^2 + 2g_{12}\dot{x}\dot{y} + g_{22}\dot{y}^2)$ and $y'(x) = \dot{y}/\dot{x}$:

$$I(x, y, y') := \frac{1}{(K_1 + K_2 y')^2} \left(g_{11} + 2g_{12}y' + g_{22}y'^2\right)$$

is a first integral of the unparametrised geodesic equation quadratic in y'.

- If K is a Killing vector $\Rightarrow$ conserved quantity: $K_a\dot{x}^a = g_{ab}K^b\dot{x}^a$
- Eliminate t using $H = \frac{1}{2}(g_{11}\dot{x}^2 + 2g_{12}\dot{x}\dot{y} + g_{22}\dot{y}^2)$ and $y'(x) = \dot{y}/\dot{x}$:

$$I(x, y, y') := \frac{1}{(K_1 + K_2 y')^2}\left(g_{11} + 2g_{12}y' + g_{22}y'^2\right)$$

is a first integral of the unparametrised geodesic equation quadratic in y'.

- The Painlevé equations define projective structures.

$(PI)\quad y'' = 6y^2 + x\ ,\quad (PII)\quad y'' = 2y^3 + xy + \alpha\ ,\ \dots\ ,\ (PVI).$

- If K is a Killing vector $\Rightarrow$ conserved quantity: $K_a\dot{x}^a = g_{ab}K^b\dot{x}^a$
- Eliminate t using $H = \frac{1}{2}(g_{11}\dot{x}^2 + 2g_{12}\dot{x}\dot{y} + g_{22}\dot{y}^2)$ and $y'(x) = \dot{y}/\dot{x}$:

$$I(x, y, y') := \frac{1}{(K_1 + K_2 y')^2}\left(g_{11} + 2g_{12}y' + g_{22}y'^2\right)$$

is a first integral of the unparametrised geodesic equation quadratic in y'.

- The Painlevé equations define projective structures.

$(PI) \quad y'' = 6y^2 + x\ , \quad (PII) \quad y'' = 2y^3 + xy + \alpha\ ,\ \ldots\ ,\ (PVI).$

- Satisfy the necessary conditions of [Bryant-Dunajski-Eastwood].

- If K is a Killing vector $\Rightarrow$ conserved quantity: $K_a \dot{x}^a = g_{ab} K^b \dot{x}^a$
- Eliminate t using $H = \frac{1}{2}(g_{11}\dot{x}^2 + 2g_{12}\dot{x}\dot{y} + g_{22}\dot{y}^2)$ and $y'(x) = \dot{y}/\dot{x}$:

$$I(x, y, y') := \frac{1}{(K_1 + K_2 y')^2} \left(g_{11} + 2g_{12}y' + g_{22}y'^2\right)$$

is a first integral of the unparametrised geodesic equation quadratic in y'.

- The Painlevé equations define projective structures.

$(PI) \quad y'' = 6y^2 + x \ , \quad (PII) \quad y'' = 2y^3 + xy + \alpha \ , \ \ldots \ , (PVI).$

- Satisfy the necessary conditions of [Bryant-Dunajski-Eastwood].
- Still need to check non-degeneracy.

Metrisability of Painlevé equations

Results: their projective structures are metrisable for (PIII), (PV) and (PVI) when they are **projectively flat** (equiv to $Y''(X) = 0$) or for

- PIII and PV when they admit an algebraic first integral.

Results: their projective structures are metrisable for (PIII), (PV) and (PVI) when they are **projectively flat** (equiv to $Y''(X) = 0$) or for

- PIII and PV when they admit an algebraic first integral.

These first integrals are derivable from Killing vectors.

E.g. (PV)

$$y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right) y'^2 - \frac{1}{x} y' + \frac{(y-1)^2}{x^2}\left(\alpha y + \frac{\beta}{y}\right)$$

First integral:

$$I = \frac{1}{y}\left(\frac{xy'}{y-1}\right)^2 + \frac{2\beta}{y} - 2\alpha y.$$

Killing 1-forms of affine connections

- Given an affine connection Γ on a surface Σ, its geodesics are the solutions to

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0, \quad a = 1, 2$$

Killing 1-forms of affine connections

- Given an affine connection Γ on a surface Σ, its geodesics are the solutions to

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0, \quad a = 1, 2$$

$$I = K_a\dot{x}^a \quad \text{is a first integral} \quad \Leftrightarrow \quad \nabla_{(a}K_{b)} = 0$$

Killing 1-forms of affine connections

- Given an affine connection Γ on a surface Σ, its geodesics are the solutions to

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0, \quad a = 1, 2$$

$$I = K_a\dot{x}^a \quad \text{is a first integral} \quad \Leftrightarrow \quad \nabla_{(a}K_{b)} = 0$$

- First integrals linear in the momenta are equivalent to Killing 1-forms.

Killing 1-forms of affine connections

- Given an affine connection Γ on a surface Σ, its geodesics are the solutions to

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0, \quad a = 1, 2$$

$$I = K_a\dot{x}^a \quad \text{is a first integral} \quad \Leftrightarrow \quad \nabla_{(a}K_{b)} = 0$$

- First integrals linear in the momenta are equivalent to Killing 1-forms.
- Conditions for their existence will be established by prolongation.

Killing 1-forms of affine connections

- Given an affine connection Γ on a surface Σ, its geodesics are the solutions to

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0, \quad a = 1, 2$$

$$I = K_a\dot{x}^a \quad \text{is a first integral} \quad \Leftrightarrow \quad \nabla_{(a}K_{b)} = 0$$

- First integrals linear in the momenta are equivalent to Killing 1-forms.
- Conditions for their existence will be established by prolongation.
- Useful decomposition of the Riemann tensor:

$$R_{ab}{}^c{}_d = \delta_a{}^c \mathrm{P}_{bd} - \delta_b{}^c \mathrm{P}_{ad} + B_{ab}\delta_d{}^c,$$

where $P_{ab} = \frac{2}{3}R_{ab} + \frac{1}{3}R_{ba}$ and $B_{ab} = -2P_{[ab]} = -\frac{2}{3}R_{[ab]}$.

Killing 1-forms of affine connections

- Given an affine connection Γ on a surface Σ, its geodesics are the solutions to

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0, \quad a = 1, 2$$

$$I = K_a\dot{x}^a \quad \text{is a first integral} \quad \Leftrightarrow \quad \nabla_{(a}K_{b)} = 0$$

- First integrals linear in the momenta are equivalent to Killing 1-forms.
- Conditions for their existence will be established by prolongation.
- Useful decomposition of the Riemann tensor:

$$R_{ab}{}^c{}_d = \delta_a{}^c \mathrm{P}_{bd} - \delta_b{}^c \mathrm{P}_{ad} + B_{ab}\delta_d{}^c,$$

where $P_{ab} = \frac{2}{3}R_{ab} + \frac{1}{3}R_{ba}$ and $B_{ab} = -2P_{[ab]} = -\frac{2}{3}R_{[ab]}$.
- Introduce a volume form ϵ_{ab} and its derivative $\nabla_c\epsilon_{ab} = \theta_c\epsilon_{ab}$.

- Define the inverse volume form $\epsilon^{ab}\epsilon_{cb} = \delta^a_c$.

- Define the inverse volume form $\epsilon^{ab}\epsilon_{cb} = \delta^a_c$.

Theorem

There is a one–to–one correspondence between solutions to the Killing equations and parallel sections of the prolongation connection D on a rank–three vector bundle $E = \Lambda^1(\Sigma) \oplus \Lambda^2(\Sigma) \to \Sigma$ defined by

$$D_a \begin{pmatrix} K_b \\ \mu \end{pmatrix} = \begin{pmatrix} \nabla_a K_b - \epsilon_{ab}\mu \\ \nabla_a \mu - \left(\mathrm{P}^b{}_a + \frac{1}{2}\epsilon^{ef} B_{ef} \delta^b{}_a\right) K_b + \mu\theta_a \end{pmatrix}.$$

- Define the inverse volume form $\epsilon^{ab}\epsilon_{cb} = \delta^a_c$.

Theorem

There is a one–to–one correspondence between solutions to the Killing equations and parallel sections of the prolongation connection D on a rank–three vector bundle $E = \Lambda^1(\Sigma) \oplus \Lambda^2(\Sigma) \to \Sigma$ defined by

$$D_a \begin{pmatrix} K_b \\ \mu \end{pmatrix} = \begin{pmatrix} \nabla_a K_b - \epsilon_{ab}\mu \\ \nabla_a \mu - \left(\mathrm{P}^b{}_a + \frac{1}{2}\epsilon^{ef} B_{ef}\delta^b{}_a\right) K_b + \mu\theta_a \end{pmatrix}.$$

- The integrability conditions for the existence of parallel sections of this connection will lead to a set of invariants of the affine connection Γ.

Theorem

The necessary condition for a C^4 torsion–free affine connection Γ on a surface Σ to admit a linear first integral is the vanishing, on Σ, of two scalars denoted by I_N and I_S of differential order 3 and 4 in Γ.
Locally,

- $I_N = I_S = 0$ *are necessary and sufficient for the existence of a Killing 1-form.*
- *there are precisely 2 Killing forms* $\Leftrightarrow$ $T_a{}^b = 0$ *and* $R_{[ab]} \neq 0$, *where* T *is a rank-2 tensor of differential order 3 in* Γ.
- *there are 3 independent Killing forms* $\Leftrightarrow$ Γ *is projectively flat and* $R_{[ab]} = 0$.

Theorem

The necessary condition for a C^4 torsion–free affine connection Γ on a surface Σ to admit a linear first integral is the vanishing, on Σ, of two scalars denoted by I_N and I_S of differential order 3 and 4 in Γ.
Locally,

- $I_N = I_S = 0$ *are necessary and sufficient for the existence of a Killing 1-form.*
- *there are precisely 2 Killing forms* $\Leftrightarrow$ $T_a{}^b = 0$ *and* $R_{[ab]} \neq 0$, *where T is a rank-2 tensor of differential order 3 in Γ.*
- *there are 3 independent Killing forms* $\Leftrightarrow$ Γ *is projectively flat and* $R_{[ab]} = 0$.

- This does not hold globally.
  Counter example: the flat torus $S^1 \times S^1$ admits precisely 2 global Killing forms (or vectors).

Theorem

The necessary condition for a C^4 torsion–free affine connection Γ on a surface Σ to admit a linear first integral is the vanishing, on Σ, of two scalars denoted by I_N and I_S of differential order 3 and 4 in Γ.
Locally,

- $I_N = I_S = 0$ *are necessary and sufficient for the existence of a Killing 1-form.*
- *there are precisely 2 Killing forms* $\Leftrightarrow$ $T_a{}^b = 0$ *and* $R_{[ab]} \neq 0$, *where* T *is a rank-2 tensor of differential order 3 in* Γ.
- *there are 3 independent Killing forms* $\Leftrightarrow$ Γ *is projectively flat and* $R_{[ab]} = 0$.

- This does not hold globally.
  Counter example: the flat torus $S^1 \times S^1$ admits precisely 2 global Killing forms (or vectors).
- For special connections ($R_{[ab]} = 0$), I_N and I_S become, essentially, Liouville's projective invariants ν_5 and w_1, respectively.

- Recall: $R_{[ab]} = \partial_{[a}\Gamma^c_{b]c} = 0 \Leftrightarrow \nabla$ preserves a volume form.

- Recall: $R_{[ab]} = \partial_{[a}\Gamma^c_{b]c} = 0 \Leftrightarrow \nabla$ preserves a volume form.
- $[\Gamma]$ admits a traceless representative Π^a_{bc}, i.e., $\Pi^a_{ab} = 0$. It is given by

$$\Pi^a_{bc} = \Gamma^a_{bc} - \frac{1}{3}\delta^a_b\Gamma^d_{dc} - \frac{1}{3}\delta^a_c\Gamma^d_{db}.$$

- Recall: $R_{[ab]} = \partial_{[a}\Gamma^c_{b]c} = 0 \;\Leftrightarrow \nabla$ preserves a volume form.
- $[\Gamma]$ admits a traceless representative Π^a_{bc}, i.e., $\Pi^a_{ab} = 0$. It is given by

$$\Pi^a_{bc} = \Gamma^a_{bc} - \frac{1}{3}\delta^a_b\Gamma^d_{dc} - \frac{1}{3}\delta^a_c\Gamma^d_{db}.$$

- Π^a_{bc} are called Thomas symbols. It preserves $\epsilon_{12} = 1$.

- Recall: $R_{[ab]} = \partial_{[a}\Gamma^c_{b]c} = 0 \Leftrightarrow \nabla$ preserves a volume form.
- $[\Gamma]$ admits a traceless representative Π^a_{bc}, i.e., $\Pi^a_{ab} = 0$. It is given by

$$\Pi^a_{bc} = \Gamma^a_{bc} - \frac{1}{3}\delta^a_b\Gamma^d_{dc} - \frac{1}{3}\delta^a_c\Gamma^d_{db}.$$

- Π^a_{bc} are called Thomas symbols. It preserves $\epsilon_{12} = 1$.
- Question: what can we say about the unparametrised geodesics if its associated Thomas symbols admit a Killing form?

- Recall: $R_{[ab]} = \partial_{[a}\Gamma^c_{b]c} = 0 \Leftrightarrow \nabla$ preserves a volume form.
- $[\Gamma]$ admits a traceless representative Π^a_{bc}, i.e., $\Pi^a_{ab} = 0$. It is given by

$$\Pi^a_{bc} = \Gamma^a_{bc} - \frac{1}{3}\delta^a_b\Gamma^d_{dc} - \frac{1}{3}\delta^a_c\Gamma^d_{db}.$$

- Π^a_{bc} are called Thomas symbols. It preserves $\epsilon_{12} = 1$.
- Question: what can we say about the unparametrised geodesics if its associated Thomas symbols admit a Killing form?
- The answer is given by the following theorem, which is partially due to [Liouville,1889].

Theorem

The ODE $y'' = A_0(x,y) + A_1(x,y)y' + A_2(x,y)(y')^2 + A_3(x,y)(y')^3$ *defining a projective structure admits coordinates* (X,Y) *such that* $Y_{XX} = f(X,Y)$ *for some function* f *if and only if* $I_N = I_S = 0$ *for any special connection. Moreover, this is also equivalent to the fact that the connection with Thomas symbols admits a Killing* 1*-form given by* dX*.*

Theorem

The ODE $y'' = A_0(x,y) + A_1(x,y)y' + A_2(x,y)(y')^2 + A_3(x,y)(y')^3$ *defining a projective structure admits coordinates* (X, Y) *such that* $Y_{XX} = f(X,Y)$ *for some function* f *if and only if* $I_N = I_S = 0$ *for any special connection. Moreover, this is also equivalent to the fact that the connection with Thomas symbols admits a Killing* 1*-form given by* dX*.*

Proof.

- Understand how Thomas symbols transform under coordinate transformations.

$\square$

Theorem

The ODE $y'' = A_0(x,y) + A_1(x,y)y' + A_2(x,y)(y')^2 + A_3(x,y)(y')^3$ *defining a projective structure admits coordinates* (X, Y) *such that* $Y_{XX} = f(X, Y)$ *for some function* f *if and only if* $I_N = I_S = 0$ *for any special connection. Moreover, this is also equivalent to the fact that the connection with Thomas symbols admits a Killing* 1*-form given by* dX*.*

Proof.

- Understand how Thomas symbols transform under coordinate transformations.
- Understand how Killing tensors of Thomas symbols change under coordinate transformation.

□

Theorem

The ODE $y'' = A_0(x,y) + A_1(x,y)y' + A_2(x,y)(y')^2 + A_3(x,y)(y')^3$ *defining a projective structure admits coordinates* (X,Y) *such that* $Y_{XX} = f(X,Y)$ *for some function* f *if and only if* $I_N = I_S = 0$ *for any special connection. Moreover, this is also equivalent to the fact that the connection with Thomas symbols admits a Killing* 1*-form given by* dX*.*

Proof.

- Understand how Thomas symbols transform under coordinate transformations.
- Understand how Killing tensors of Thomas symbols change under coordinate transformation.
- Use these facts to show that one can choose coordinates (X, Y) s.t. the Killing form is dX.

$\square$

Theorem

The ODE $y'' = A_0(x,y) + A_1(x,y)y' + A_2(x,y)(y')^2 + A_3(x,y)(y')^3$ *defining a projective structure admits coordinates* (X,Y) *such that* $Y_{XX} = f(X,Y)$ *for some function* f *if and only if* $I_N = I_S = 0$ *for any special connection. Moreover, this is also equivalent to the fact that the connection with Thomas symbols admits a Killing* 1*-form given by* dX.

Proof.

- Understand how Thomas symbols transform under coordinate transformations.
- Understand how Killing tensors of Thomas symbols change under coordinate transformation.
- Use these facts to show that one can choose coordinates (X,Y) s.t. the Killing form is dX.
- Check that this is equivalent to having $Y_{XX} = f(X,Y)$.

$\square$

- **Claim:** Degenerate solutions to the metrisability equations correspond to Killing forms of special connections. Equivalently, $I_N = I_S = 0$ or $\nu_5 = w_1 = 0$.

- **Claim:** Degenerate solutions to the metrisability equations correspond to Killing forms of special connections. Equivalently, $I_N = I_S = 0$ or $\nu_5 = w_1 = 0$.

Corollary (Babich,Bordag1999)

The Painlevé equations can be put in the form $y'' = f(x, y)$ under point transformation.

- **Claim:** Degenerate solutions to the metrisability equations correspond to Killing forms of special connections. Equivalently, $I_N = I_S = 0$ or $\nu_5 = w_1 = 0$.

Corollary (Babich,Bordag1999)

The Painlevé equations can be put in the form $y'' = f(x, y)$ under point transformation.

Example: (PIII): $y'' = \alpha e^{x+y} + \beta e^{x-y} + \gamma e^{2(x+y)} + \delta e^{2(x-y)}$

- **Claim:** Degenerate solutions to the metrisability equations correspond to Killing forms of special connections. Equivalently, $I_N = I_S = 0$ or $\nu_5 = w_1 = 0$.

Corollary (Babich,Bordag1999)

The Painlevé equations can be put in the form $y'' = f(x, y)$ under point transformation.

Example: (PIII): $y'' = \alpha e^{x+y} + \beta e^{x-y} + \gamma e^{2(x+y)} + \delta e^{2(x-y)}$

Remark: Scalars I_N and I_S, along with [B-D-E], answer the question about degeneracy in metrisability $\Rightarrow$

Metrisability problem itself is completely solved in 2D.

Hydrodynamic-type (HT) systems

Definition (HT system (our case))

A system of PDEs is of HT if it has the form

$$\partial_t u^a = v^a{}_b(u)\partial_x u^b, \quad a, b = 1, 2$$

where $u^a = u^a(x,t)$ *and* v *is a diagonalisable matrix with distinct real eigenvalues* $\lambda_1(u)$ *and* $\lambda_2(u)$.

Hydrodynamic-type (HT) systems

Definition (HT system (our case))

A system of PDEs is of HT if it has the form

$$\partial_t u^a = v^a{}_b(u)\partial_x u^b, \quad a, b = 1, 2$$

where $u^a = u^a(x, t)$ *and* v *is a diagonalisable matrix with distinct real eigenvalues* $\lambda_1(u)$ *and* $\lambda_2(u)$.

Theorem (Riemann invariants)

A HT system admits coordinates $R^i(u)$ *(called* **Riemann invariants***) such that*

$$\partial_t R^i = \lambda^i(u(R))\partial_x R^i, \quad i = 1, 2 \qquad \textit{(no summation).}$$

Question: does my HT system admit a Hamiltonian formulation under a Poisson bracket of **Dubrovin-Novikov type**?

$$\{F, G\} = \int_{\mathbb{R}} \frac{\delta F}{\delta u^a}\left(g^{ab}(u)\frac{\partial}{\partial x} + b^{ab}_c(u)\frac{\partial u^c}{\partial x}\right)\frac{\delta G}{\delta u^b}dx.$$

Question: does my HT system admit a Hamiltonian formulation under a Poisson bracket of **Dubrovin-Novikov type**?

$$\{F, G\} = \int_{\mathbb{R}} \frac{\delta F}{\delta u^a}\left(g^{ab}(u)\frac{\partial}{\partial x} + b^{ab}_c(u)\frac{\partial u^c}{\partial x}\right)\frac{\delta G}{\delta u^b}dx.$$

Or

$$\frac{\partial u^a}{\partial t} = \Omega^{ab}\frac{\delta H}{\delta u^b} = \underbrace{g^{ab}\nabla_b\nabla_c\mathcal{H}}_{v^a{}_c}\frac{\partial u^c}{\partial x},$$

where ∇ is the Levi-Civita connection of g, $H[u^1, u^2] = \int \mathcal{H}(u^1, u^2)dx$ and

$$\Omega^{ab} = g^{ab}\frac{\partial}{\partial x} + b^{ab}_c\frac{\partial u^c}{\partial x}$$

Answer **[Ferapontov91]:** It does iff there exists a flat diagonal metric $k^{-1}\,d\big(R^1\big)^2 + f^{-1}\,d\big(R^2\big)^2$ satisfying the following system of PDEs

$$\partial_2 k + 2Ak = 0, \quad \partial_1 f + 2Bf = 0,$$

where

$$A = \frac{\partial_2 \lambda^1}{\lambda^2 - \lambda^1}, \quad B = \frac{\partial_1 \lambda^2}{\lambda^1 - \lambda^2}, \quad \text{and} \quad \partial_i = \partial/\partial R^i$$

Answer [Ferapontov91]: It does iff there exists a flat diagonal metric $k^{-1}\, d\big(R^1\big)^2 + f^{-1}\, d\big(R^2\big)^2$ satisfying the following system of PDEs

$$\partial_2 k + 2Ak = 0, \quad \partial_1 f + 2Bf = 0,$$

where

$$A = \frac{\partial_2 \lambda^1}{\lambda^2 - \lambda^1}, \quad B = \frac{\partial_1 \lambda^2}{\lambda^1 - \lambda^2}, \quad \text{and} \quad \partial_i = \partial/\partial R^i$$

And, by flatness,

$$(\partial_2 A + A^2) f + (\partial_1 B + B^2) k + \frac{1}{2} A \partial_2 f + \frac{1}{2} B \partial_1 k = 0.$$

Answer [Ferapontov91]: It does iff there exists a flat diagonal metric $k^{-1}\, d\left(R^1\right)^2 + f^{-1}\, d\left(R^2\right)^2$ satisfying the following system of PDEs

$$\partial_2 k + 2Ak = 0, \quad \partial_1 f + 2Bf = 0,$$

where

$$A = \frac{\partial_2 \lambda^1}{\lambda^2 - \lambda^1}, \quad B = \frac{\partial_1 \lambda^2}{\lambda^1 - \lambda^2}, \quad \text{and} \quad \partial_i = \partial/\partial R^i$$

And, by flatness,

$$(\partial_2 A + A^2) f + (\partial_1 B + B^2) k + \frac{1}{2} A \partial_2 f + \frac{1}{2} B \partial_1 k = 0.$$

- These are the compatibility conditions of the overdetermined system for $\mathcal{H}$

$$g^{ab} \nabla_b \nabla_c \mathcal{H} = v^a{}_c$$

Claim: The above overdetermined system of PDEs is equivalent to the Killing equations

$$\tilde{\nabla}_{(a}K_{b)} = 0,$$

where

$$\tilde{\Gamma}^1_{11} = \partial_1 \ln A - 2B, \quad \tilde{\Gamma}^2_{22} = \partial_2 \ln B - 2A,$$
$$\tilde{\Gamma}^1_{12} = -\left(\frac{1}{2}\partial_2 \ln A + A\right), \quad \tilde{\Gamma}^2_{12} = -\left(\frac{1}{2}\partial_1 \ln B + B\right),$$

and $K_1 = Af$, $K_2 = Bk$.

Claim: The above overdetermined system of PDEs is equivalent to the Killing equations

$$\tilde{\nabla}_{(a}K_{b)} = 0,$$

where

$$\tilde{\Gamma}^1_{11} = \partial_1 \ln A - 2B, \quad \tilde{\Gamma}^2_{22} = \partial_2 \ln B - 2A,$$
$$\tilde{\Gamma}^1_{12} = -\Big(\frac{1}{2}\partial_2 \ln A + A\Big), \quad \tilde{\Gamma}^2_{12} = -\Big(\frac{1}{2}\partial_1 \ln B + B\Big),$$

and $K_1 = Af, \; K_2 = Bk$.

Recall: A and B are determined by your HT system.

- Killing forms of the above connection are in $1-1$ correspondence with Hamiltonians of the HT system.

- Killing forms of the above connection are in $1-1$ correspondence with Hamiltonians of the HT system.

Theorem

A HT system admits 1, 2 *or* 3 *Hamiltonian formulations iff its associated connection defined above admits* 1, 2 *or* 3 *independent linear first integrals respectively.*

- Killing forms of the above connection are in $1-1$ correspondence with Hamiltonians of the HT system.

Theorem

A HT system admits 1, 2 *or* 3 *Hamiltonian formulations iff its associated connection defined above admits* 1, 2 *or* 3 *independent linear first integrals respectively.*

- This connection defines the following projective structure

$$Y'' = (\partial_X \ln (AB))Y' - (\partial_Y \ln (AB))(Y')^2.$$

- Killing forms of the above connection are in $1-1$ correspondence with Hamiltonians of the HT system.

Theorem

A HT system admits 1, 2 or 3 Hamiltonian formulations iff its associated connection defined above admits 1, 2 or 3 independent linear first integrals respectively.

- This connection defines the following projective structure

$$Y'' = (\partial_X \ln(AB))\, Y' - (\partial_Y \ln(AB))(Y')^2.$$

Theorem

This projective structure is metrisable by the Lorentzian metric

$$A\, B\, d\left(R^1\right) d\left(R^2\right).$$

Corollary

A HT sytem is trihamiltonian iff its associated connection has symmetric Ricci tensor and $(AB)^{-1}\partial_1\partial_2 \ln(AB) = const$ *(metric of constant curvature).*

Corollary

A HT sytem is trihamiltonian iff its associated connection has symmetric Ricci tensor and $(AB)^{-1}\partial_1\partial_2 \ln(AB) = const$ *(metric of constant curvature).*

- A two-dimensional Frobenius manifold $(M, \eta, e, E, \circ)$ (η=flat metric, e=identity, E=Euler vector field) with flat coordinates (t^1, t^2) defines a HT system

$$\frac{\partial t^i}{\partial t} = e^j c^i_{jl} \frac{\partial t^l}{\partial x},$$

where $\partial_i \circ \partial_j = c^k_{ij}\partial_k$ and $e \circ \partial_j = \partial_j$.

Corollary

A HT sytem is trihamiltonian iff its associated connection has symmetric Ricci tensor and $(AB)^{-1}\partial_1\partial_2 \ln(AB) = const$ *(metric of constant curvature).*

- A two-dimensional Frobenius manifold $(M, \eta, e, E, \circ)$ (η=flat metric, e=identity, E=Euler vector field) with flat coordinates (t^1, t^2) defines a HT system

$$\frac{\partial t^i}{\partial t} = e^j c^i_{jl} \frac{\partial t^l}{\partial x},$$

where $\partial_i \circ \partial_j = c^k_{ij}\partial_k$ and $e \circ \partial_j = \partial_j$.

Theorem

HT systems arising from Frobenius manifolds are trihamiltonian.

Corollary

A HT sytem is trihamiltonian iff its associated connection has symmetric Ricci tensor and $(AB)^{-1}\partial_1\partial_2 \ln(AB) = const$ (metric of constant curvature).

- A two-dimensional Frobenius manifold $(M, \eta, e, E, \circ)$ (η=flat metric, e=identity, E=Euler vector field) with flat coordinates (t^1, t^2) defines a HT system
$$\frac{\partial t^i}{\partial t} = e^j c^i_{jl} \frac{\partial t^l}{\partial x},$$
where $\partial_i \circ \partial_j = c^k_{ij}\partial_k$ and $e \circ \partial_j = \partial_j$.

Theorem

HT systems arising from Frobenius manifolds are trihamiltonian.

- The flat metrics determining the Poisson brackets are η^{ij} (metric), $h^{ij} = E^k \eta^{ik} c^j_{kl}$ (intersection form) and $h^{ik} h^{jl} \eta_{kl}$ (whatever).

Outlook

- Statement of the problem of R. Liouville

Outlook

- Statement of the problem of R. Liouville
- Metrisability of the Painlevé equations and derivation of their first integrals

Outlook

- Statement of the problem of R. Liouville
- Metrisability of the Painlevé equations and derivation of their first integrals

Outlook

- Statement of the problem of R. Liouville
- Metrisability of the Painlevé equations and derivation of their first integrals
- Killing forms of affine connections

Outlook

- Statement of the problem of R. Liouville
- Metrisability of the Painlevé equations and derivation of their first integrals
- Killing forms of affine connections
- Hamiltonian structures of HT systems

Outlook

- Statement of the problem of R. Liouville
- Metrisability of the Painlevé equations and derivation of their first integrals
- Killing forms of affine connections
- Hamiltonian structures of HT systems

Thank you!