Small dispersion limit of the Kadomtsev Petviashvili equation

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Nonlinear waves

Nonlinear waves are described by partial differential equations that have terms that contain

- nonlinearity
- dissipation
- dispersion

Solutions

- nonlinearity $\longrightarrow$ solutions develop singularity in finite time (shock wave)
- nonlinearity $+$ small dispersion $\longrightarrow$ dispersive shock waves
- nonlinearity $+$ small dissipation $\longrightarrow$ dissipative shock waves

Goals of the talk

- To give a quantitative description of the formation of dispersive shock waves at the onset of the oscillations and at later times in a 2-dimensional model.

Joint work with Boris Dubrovin (SISSA), Jens Eggers (Bristol), Christian Klein (Dijon) and Giuseppe Pitton (SISSA)

- J. Eggers, T. Grava, C. Klein, Shock formation in the dispersion less Kadomtsev Petviashvili equation, Nonlinearity 2016
- B.Dubrovin, T.Grava, C. Klein, On critical behaviour generalised KP equation to appear in Physica D 2016
- T. Grava, C. Klein and G. Pitton, Development of dispersive shock waves in the solution of the KPI equation, in preparation.

2D-model: the KP equation (1970)

Let us consider the equation for the scalar function $u = u(x, y, t; \epsilon)$

$$(u_t + uu_x + \epsilon^2 u_{xxx})_x = \sigma u_{yy}, \quad \sigma = \pm 1, \ \ \epsilon > 0.$$

- Kadomtsev-Petviashvili (KP) equations I or II for $\sigma = \pm 1$.
- The solutions model long weakly dispersive waves which propagate essentially in one direction with weak transverse effects.
- for $\sigma = -1$ weak surface tension compare to gravitational force, $\sigma = 1$ strong surface tension.

For $\epsilon = 0$ one has the dKP equation or Zabolotskaya-Khokhlov equation (1969)

$$(u_t + uu_x)_x = \sigma u_{yy}.$$

Nonlocal hyperbolic PDE: generic solution develops shock in finite time.
Goal: study the formation of dispersive shock waves, namely solutions of the KP when $\epsilon \to 0$.

General features of KP and dKP equations

The KP equation $$(u_t + uu_x + \epsilon^2 u_{xxx})_x = \pm u_{yy}$$

- is integrable via inverse scattering (M.Ablowitz, P.Clarkson, J.Villarroel, A.Fokas, L.Sung, M.Boiti, F.Pempinelli, B.Prinari...)
- for $\epsilon > 0$ the Cauchy problem is well posed in H^s for all $t > 0$. For $s \geq 4$ classical solutions (J. Bourgain, Y.Liu. L. Molinet, J.C. Saut, N. Tzvetkov, ...).

The dKP equation $$(u_t + uu_x)_x = \pm u_{yy},$$

- integrable via inverse scattering (S.Manakov, P.Santini)
- particular solutions have been obtained with various techniques:
  - Einstein-Weil geometry *M. Dunajski, L. Mason, and P. Tod,*
  - $\bar{\partial}$-approach *B. Konopelchenko, L. Martinez Alonso, and O. Ragnisco,*
  - Hydrodynamic reductions *J. Gibbons, S. Tsarev*
  - Conformal maps *J. Gibbons and Y. Kodama,*

The dKP equation $\qquad (u_t + uu_x)_x = \pm u_{yy},$

- is a hyperbolic PDE;
- Cauchy problem is well posed in H^s for $0 < t < t_c$ (A.Rozanova). Here t_c is the time where the gradients of $u(x, y, t)$ first diverge (Shock formation).

Comparison of solutions of the KP and dKP equation

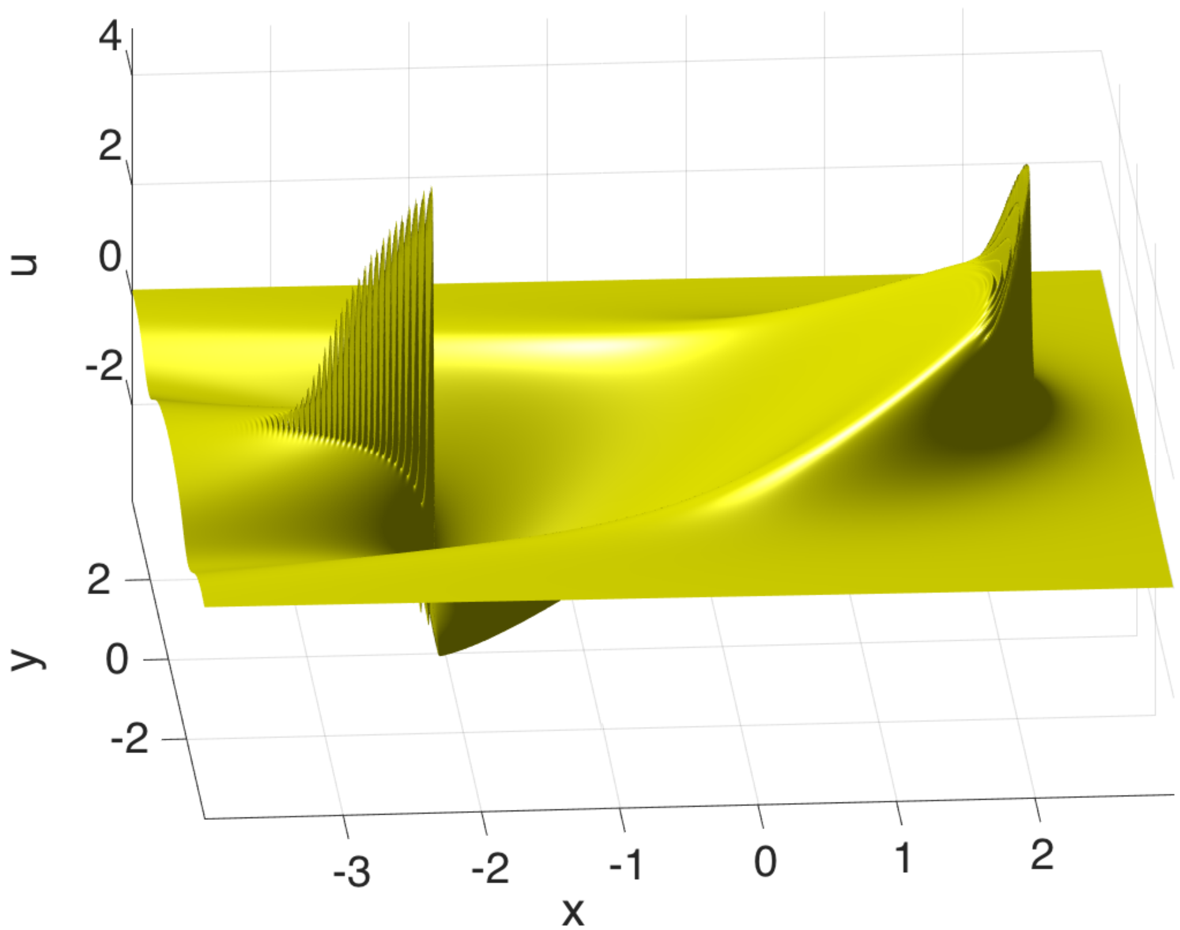
$$(u_t + uu_x + \epsilon^2 u_{xxx})_x = \pm u_{yy}, \qquad (u_t + uu_x)_x = \pm u_{yy}$$

Three regimes are present

- $t < t_c$. The gradients are bounded and the solution of the KP equation is expected to be closed to the dKP solution in the limit $\epsilon \to 0$
- $t \simeq t_c$. Universal behaviour, independent from the initial data.
- $t > t_c$. the KP solution develops oscillations (dispersive shocks). The KPI solutions generically has a second caustic zone where very high lumps start to appear.

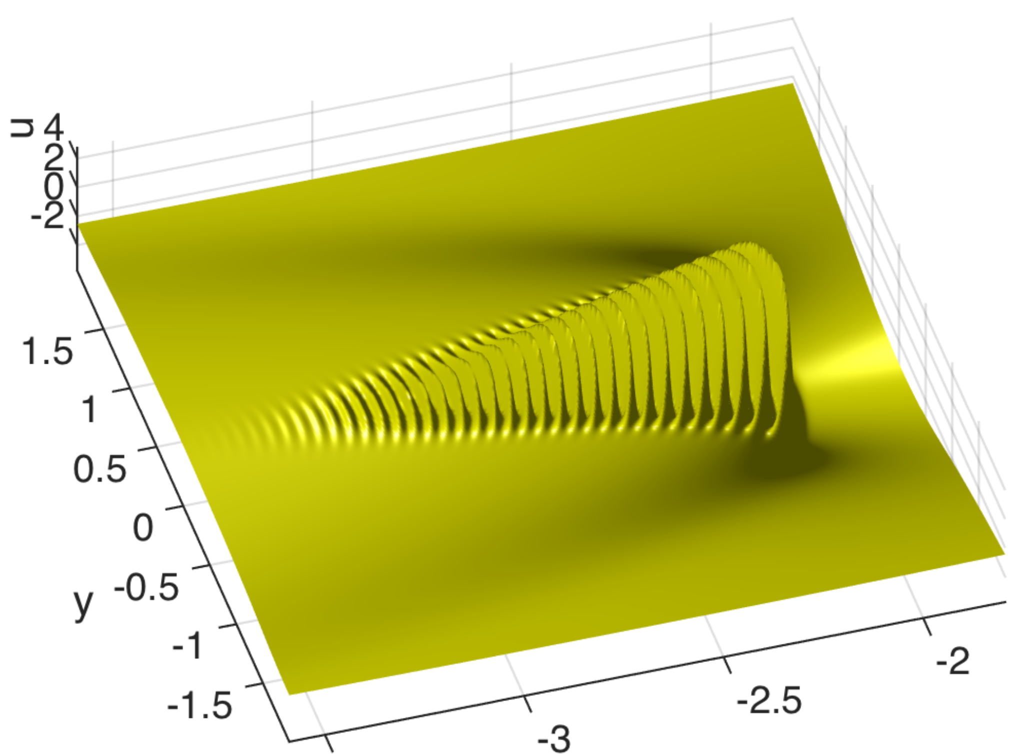
  For $t > t_c$ the dispersive shocks of the KPII solution have been recently been described by M. Ablowitz, A. Demirci, Yi-Ping Ma for one initial data, i.e. a step of parabolic form $x = cy^2$ reducing the problem to a one-dimensional problem (cylindrical KdV equation).

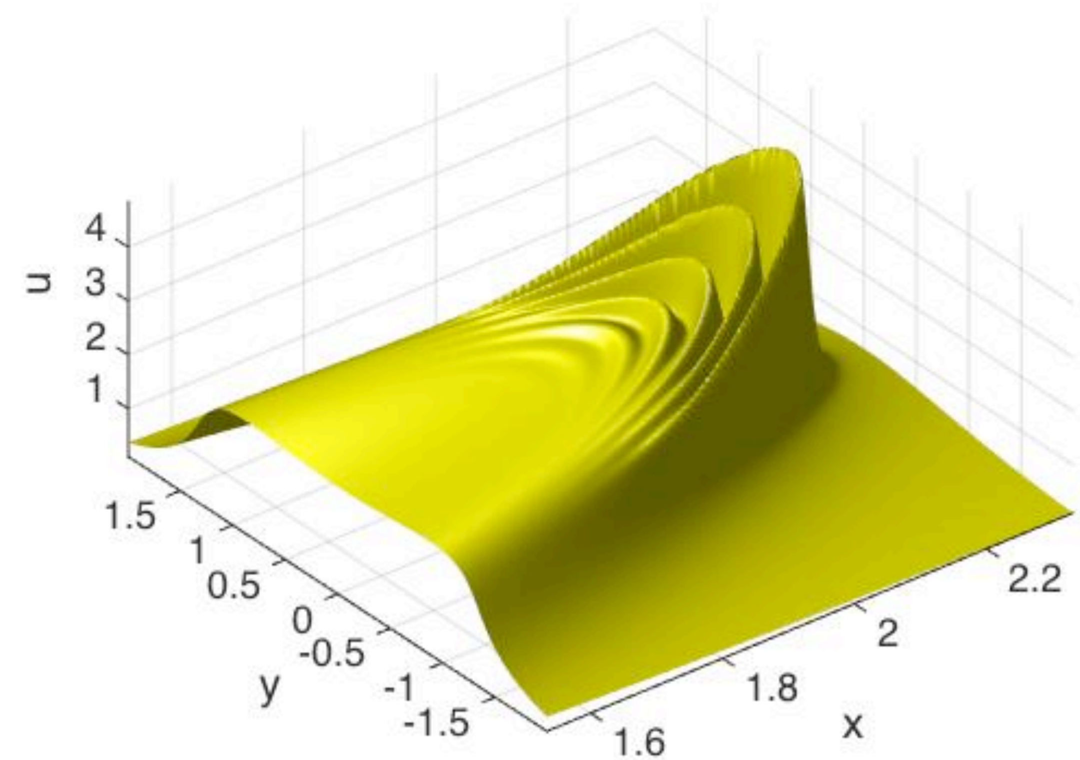
Numerical solutions



KPII solution

$$u_0(x,y) = -6\partial_x \mathrm{sech}^2(x^2+y^2)$$
$$\epsilon = 10^{-2}, \quad t = 0.4$$





Solution to the dKP equation and singularity formation

The solution of the dKP equation can be obtained by a *deformation* of the method of characteristics (after Manakov-Santini)

$$\begin{cases} u(x,y,t) = F(\xi,y,t) \\ x = tF(\xi,y,t) + \xi \end{cases}$$

with $F(\xi,y,0) = u_0(\xi,y)$ the initial data, and the function $F(\xi,y,t)$ satisfies

$$\pm F_t = \partial_\xi^{-1} F_{yy} + t(F_\xi \partial_\xi^{-1} F_{yy} - F_y^2)$$
$$F(\xi,y,0) = u_0(\xi,y).$$

Remark: if the initial data $u_0(x,y)$ is y independent, the dKP equation reduces to the Hopf or inviscid Burgers equation $u_t + uu_x = 0$, $F_t = F_y = 0$ and $F(\xi,y,t) = u_0(\xi)$:

$$\begin{cases} u(x,t) = F(\xi) \\ x = tF(\xi) + \xi. \end{cases}$$

Solution of dKP and of the equation for the function $F(\xi, y, t)$

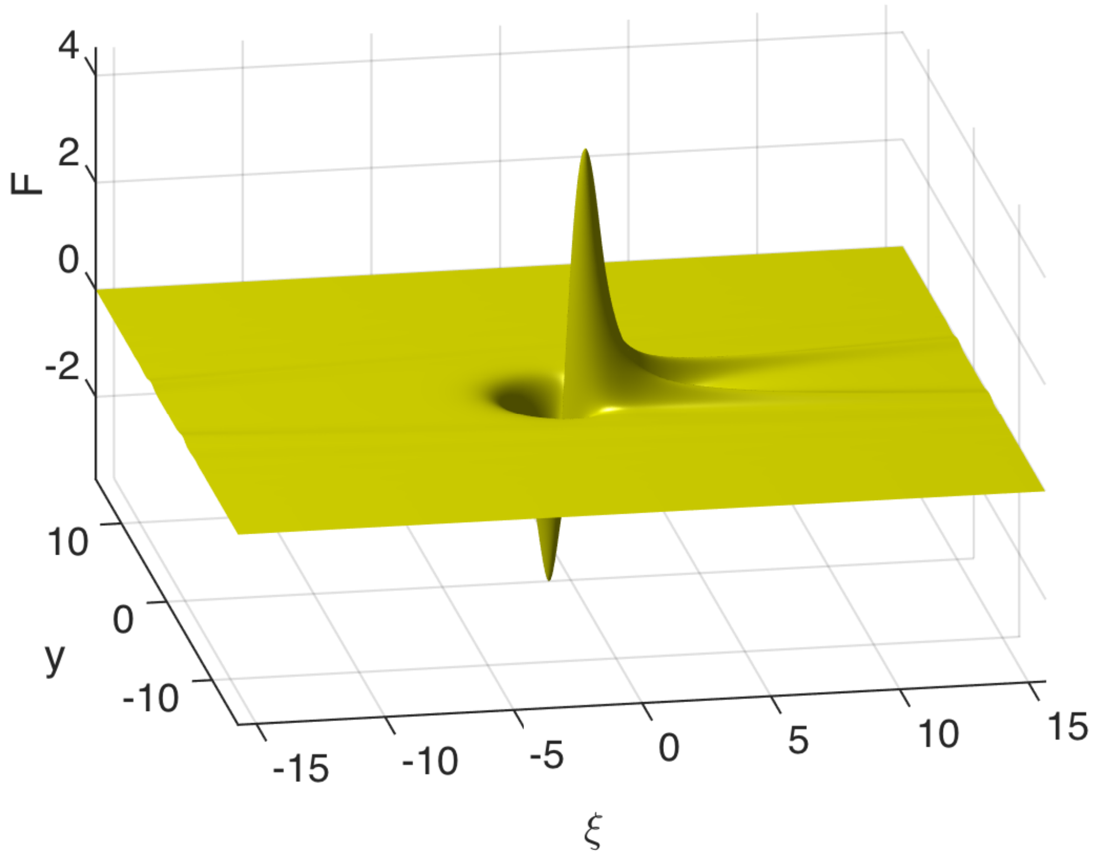
$$\begin{cases} \quad u(x,y,t) = F(\xi,y,t) \\ \quad x = tF(\xi,y,t) + \xi \end{cases}$$

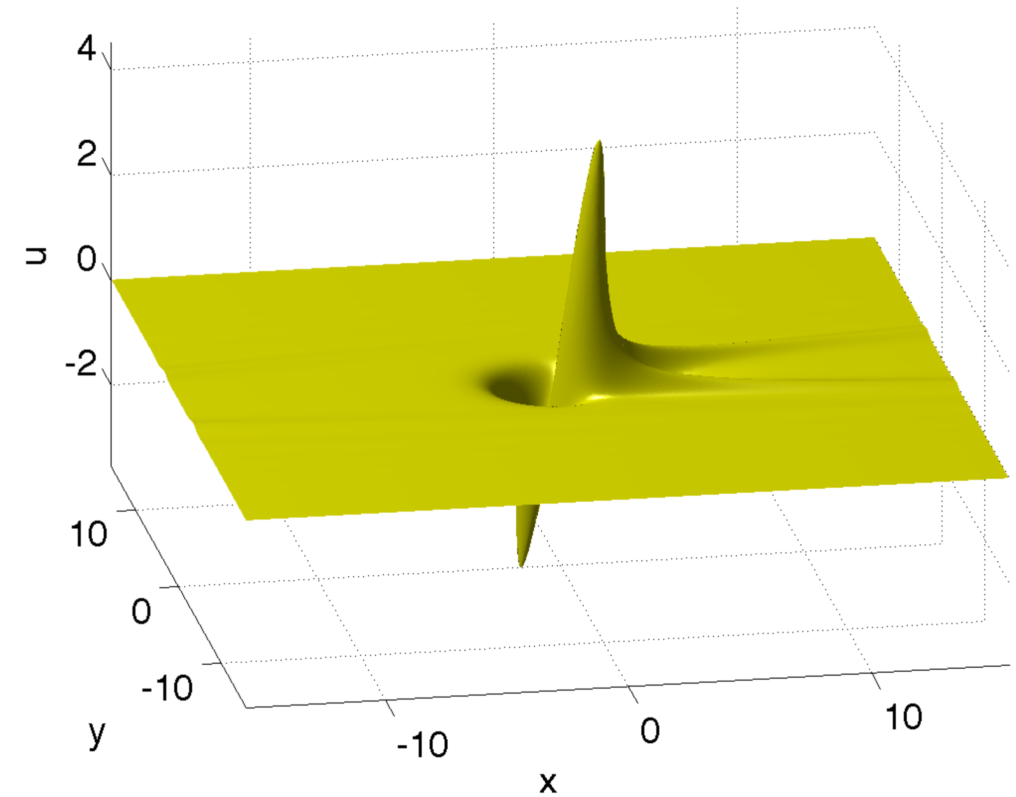
Shock formation: the solution $u(x,y,t)$ has a singularity (blow up of gradients) when the map $x = tF(\xi,y,t) + \xi$ is not invertible for $\xi = \xi(x,y,t)$ while the function $F(\xi,y,t)$ is still smooth.
Remark:the equation for $F(\xi,y,t)$

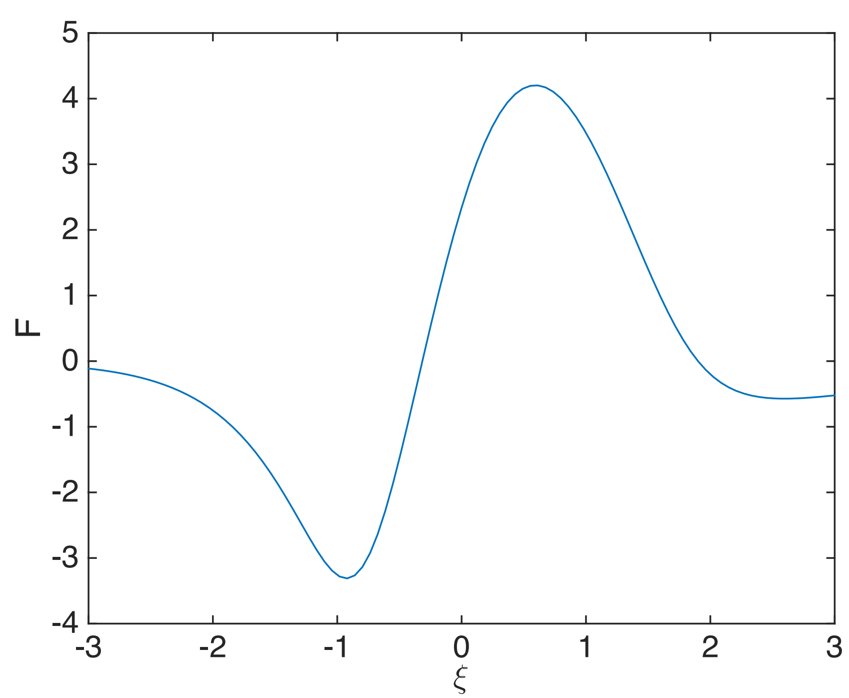
$$\pm F_t = \partial_\xi^{-1} F_{yy} + t(F_\xi \partial_\xi^{-1} F_{yy} - F_y^2)$$

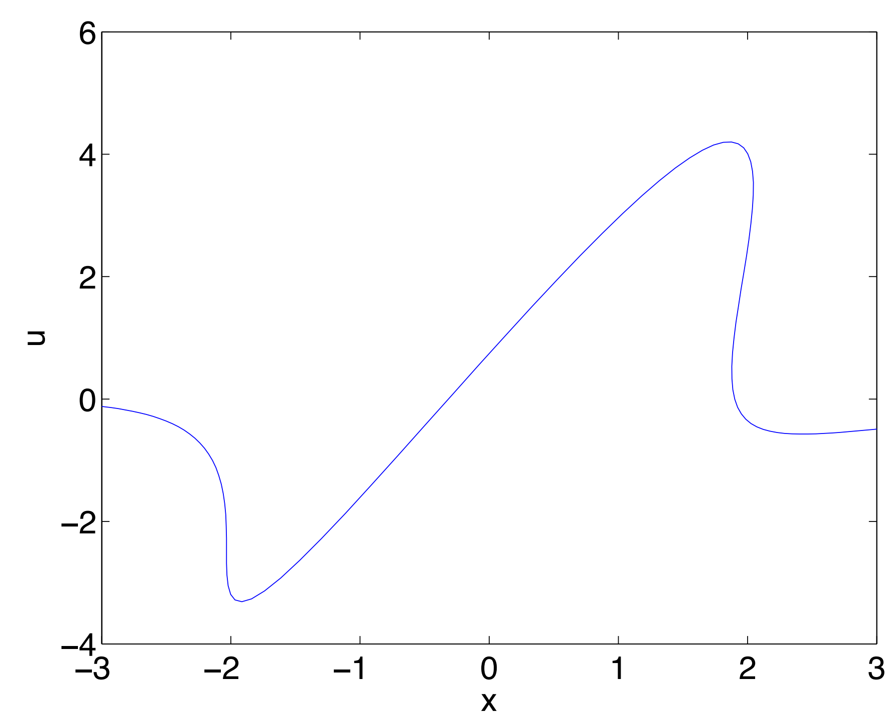
is "less nonlinear" then the dKP equation $(u_t + uu_x)_x = \pm u_{yy}$, so the solution $F(\xi,y,t)$ exist for longer times then the dKP solution $u(x,y,t)$, at least numerically.

Numerical solution : $u_0(x,y) = \partial_x \mathrm{sech}^2(x^2+y^2)$









Singularity formation (Manakov-Santini)

The solution of the dKP equation becomes singular when the characteristics equations

$$\begin{cases} \quad u(x,y,t) = F(\xi,y,t) \\ \quad x = tF(\xi,y,t) + \xi \end{cases}$$

is not invertible as a single valued function of x and y. This happens when

$$tF_\xi + 1 = 0, \quad F_{\xi\xi} = 0, \quad F_{\xi y} = 0.$$

The singularity is generic if the third derivatives with respect to ξ and y are not zeros:

$$F_{\xi\xi\xi} \neq 0, \quad F_{\xi yy} \neq 0, \quad F_{\xi yy} \neq 0.$$

Behaviour of solutions near the singular point (x_c, y_c, t_c)

Let $\bar{x} = x - x_c$, $\bar{y} = y - y_c$, $\bar{t} = t - t_c$, and $\bar{\xi} = \xi - \xi_c$ be the rescaled variables near the critical values and $u_c = u(x_c, y_{,c}, t_c)$. In the coordinate system

$$X = \bar{x} + a_1 \bar{t} + a_2 \bar{t}\bar{y} + P_3(\bar{y}),$$

$$T = \bar{t} + P_2(\bar{y}), \quad \zeta = F^c_{\xi}\left(\bar{\xi} + \frac{F^c_{\xi\xi y}}{F^c_{\xi\xi\xi}}\bar{y}\right),$$

with P_2 and P_3 polynomials of degree two and three in $\bar{y}$, the characteristic equation near the singular points takes the normal form

$$u(x, y, t) = F(\xi, y, t) \simeq u_c + \zeta + \beta\bar{y}$$

$$X \simeq -k\zeta^3 + T\zeta.$$

In the X, T and ζ variables same singularity as the Hopf solution! Typical scaling $\bar{u} \sim \bar{t}^{1/2}$, $\bar{x} \sim \bar{t}^{3/2}$, and $\bar{y} \sim \bar{t}^{1/2}$.

Double scaling limit

We are looking for a solution of $u(x, y, t; \epsilon)$ of the KP equation near the point of gradient blow-up (x_c, y_c, t_c) for the dKP equation in the form

$$u(x, y, t; \epsilon) = u_c + h(X, T; \epsilon) + \beta \bar{y}$$

with X and T the rescaled variables. Let

$$h(X, T; \epsilon) = \lambda^{\frac{1}{3}} H(\mathcal{X}, \mathcal{T}; \bar{\epsilon}) + \mathcal{O}(\lambda)$$

$$X = \lambda \mathcal{X}, \quad T = \lambda^{\frac{2}{3}} \mathcal{T}, \quad \epsilon = \lambda^{\frac{7}{6}} \bar{\epsilon}, \quad \bar{y} = \lambda^{\frac{1}{3}} \mathcal{Y}.$$

and suppose that the limit

$$H(\mathcal{X}, \mathcal{T}; \bar{\epsilon}) = \lim_{\lambda \to 0} \lambda^{-\frac{1}{3}} h(\lambda \mathcal{X}, \lambda^{\frac{2}{3}} \mathcal{T}; \lambda^{\frac{7}{6}} \bar{\epsilon})$$

exists. Then the function $H(\mathcal{X}, \mathcal{T}; \bar{\epsilon})$ satisfies the KdV equation

$$H_{\mathcal{T}} + H H_{\mathcal{X}} + \bar{\epsilon}^2 H_{\mathcal{X}\mathcal{X}\mathcal{X}} = 0.$$

Choosing $\lambda = \epsilon^{6/7}$ one has $\bar{\epsilon} = 1$ and

$$H_{\mathcal{T}} + HH_{\mathcal{X}} + H_{\mathcal{XXX}} = 0.$$

The matching with the outer solution implies that $H(\mathcal{X}, \mathcal{T})$ behaves like the root of the cubic equation for

$$\mathcal{X} = H\mathcal{T} - H^3$$

for large negative $\mathcal{T}$ or for large $|\mathcal{X}|$. The particular smooth solution of the KdV equation that satisfies this requirement is the solution of Painlevé I-2 equation.

Double scaling limit of the KP solution

Conjecture: the solution of the KP equation in the limit $\epsilon \to 0$ behaves near the critical point (x_c, y_c) as

$$u(x,y,t;\epsilon) \simeq u_c + (\epsilon\tilde{\gamma})^{\frac{2}{7}} U\left(\frac{X}{k(\epsilon\tilde{\gamma})^{6/7}}, \frac{T}{k(\epsilon\tilde{\gamma})^{4/7}}\right) + O(\epsilon^{\frac{4}{7}})$$

where $X = \bar{x} - u_c(\bar{t} + c_1\bar{y}) + P_3(\bar{y}),\ T = \bar{t} + P_2(\bar{y}^2)$ and $U(\mathcal{X}, \mathcal{T})$ solves the Painlevél-2 equation

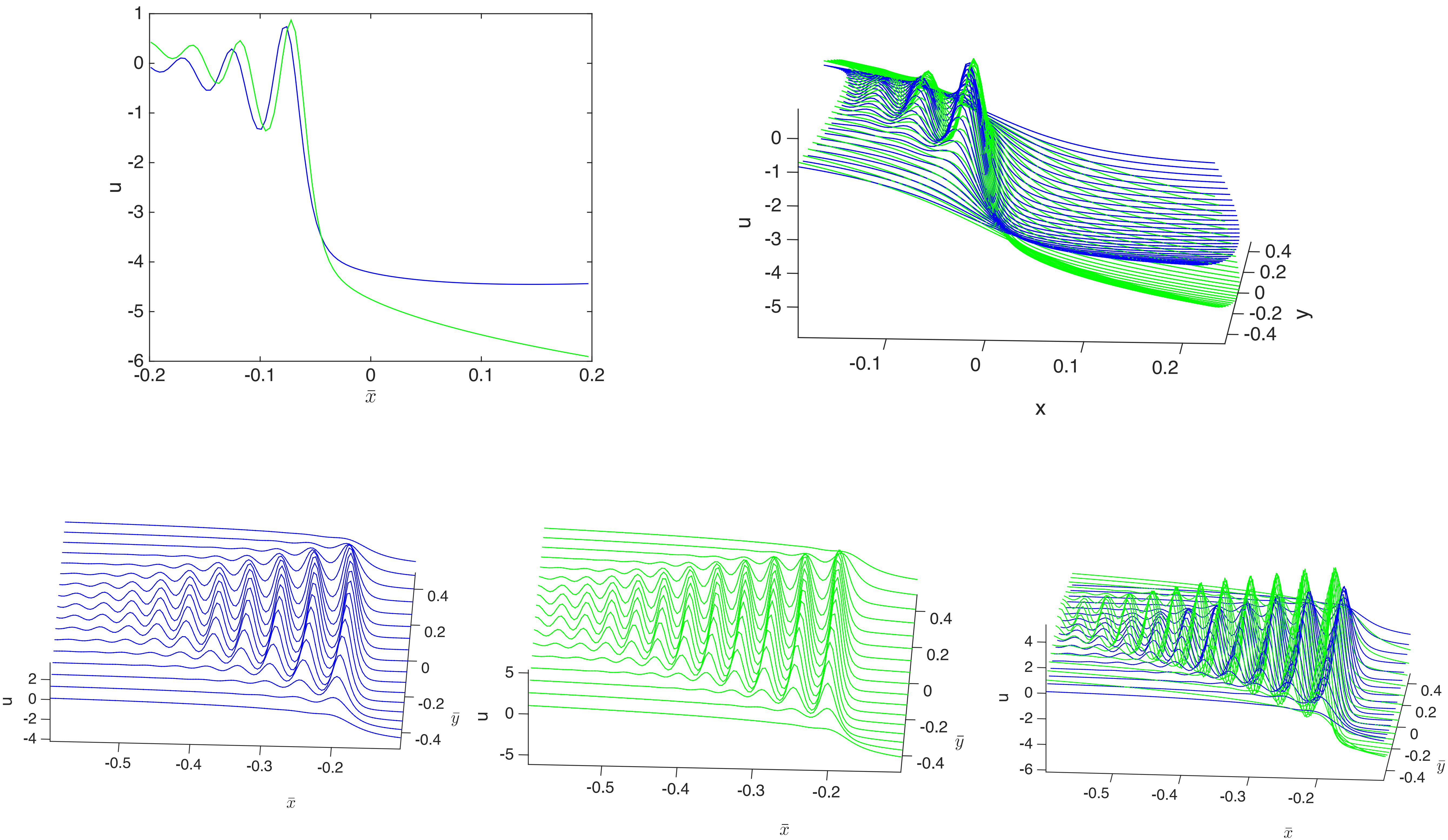
$$\mathcal{X} = \mathcal{T}\, U - \left[U^3 + \frac{1}{2}(U_{\mathcal{X}}^2 + 2U\, U_{\mathcal{X}\mathcal{X}}) + \frac{1}{10} U_{\mathcal{X}\mathcal{X}\mathcal{X}\mathcal{X}}\right],$$

with asymptotic behavior given by

$$U(\mathcal{X}, \mathcal{T}) = \mp(|\mathcal{X}|)^{1/3} \mp \frac{1}{3}\mathcal{T}|\mathcal{X}|^{-1/3} + O(|\mathcal{X}|^{-1}), \qquad \text{as } \mathcal{X} \to \pm\infty.$$

The function $U(\mathcal{X}, \mathcal{T})$ solves also the KdV equation. For the KdV equation a similar conjecture was formulated by Dubrovin (2006) and proved by TG and T. Claeys 2008.

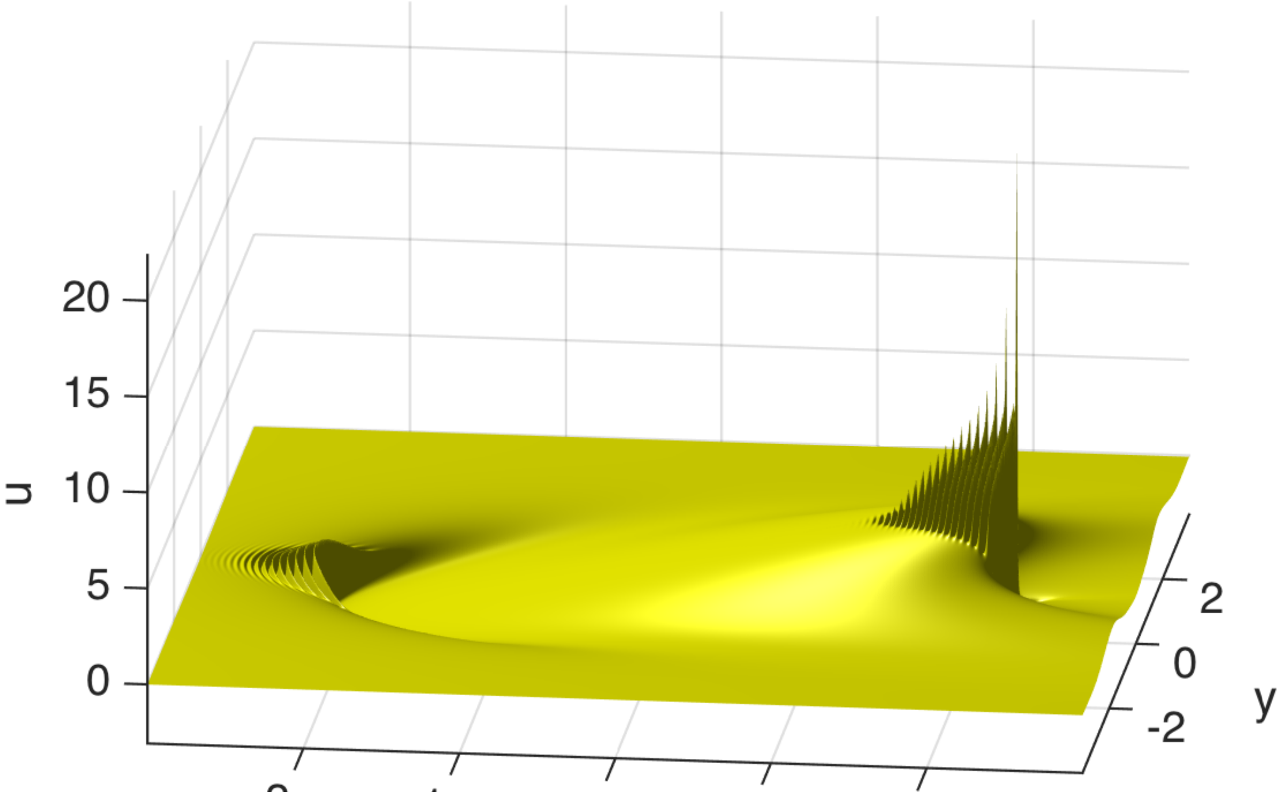
Numerical solutions of KPII and PI2 approximation

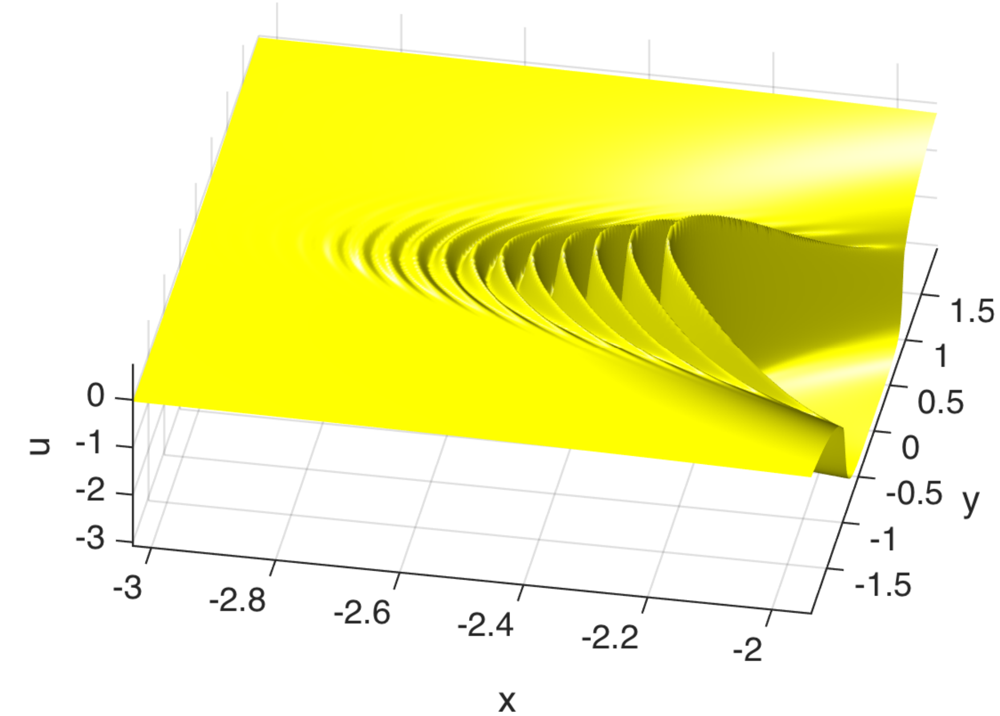


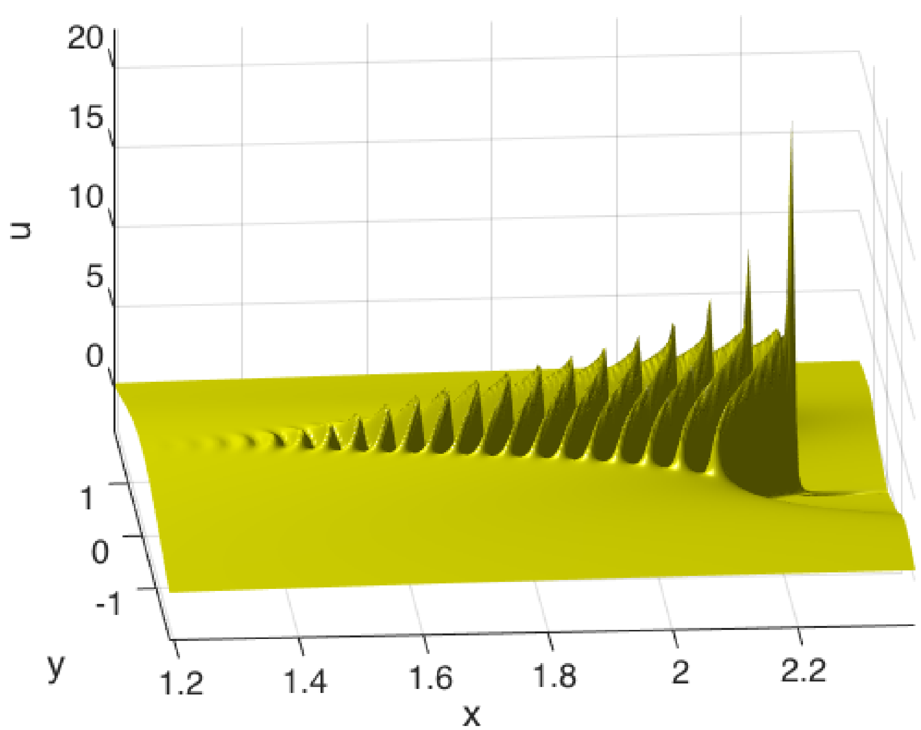
Numerical solutionof KPI

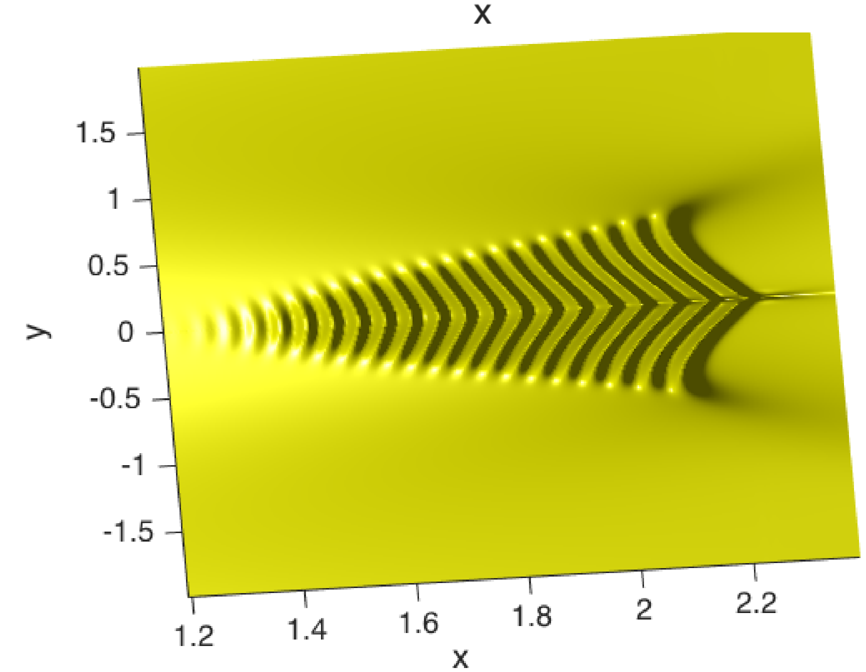
$$u_0(x,y) = -6\partial_x \operatorname{sech}^2(x^2+y^2) \qquad \epsilon = 10^{-2}, \quad t = 0.4$$

Oscillations start at t=0.22









Numerical solutions

- The numerical method used is the Fourier pseudospectral method.
- For the time evolution we have used the composite Runge Kutta method introduced by Driscoll (fourth order method).
- Fourier modes: 2^{15} in x and y for $\epsilon \in [0.02, 0.05]$ and 2^{14} in x and y for $\epsilon \in [0.06, 0.1]$.
- Time step: $4 * 10^{-5}$ for $\epsilon = 0.02, 0.03$, 10^{-4} for $\epsilon = 0.05, 0.06, 0.08$, and $2 * 10^{-4}$ for $\epsilon = 0.07, 0.09, 0.10$ and $t \simeq 1.1$
- Domain in x and y is $[-5\pi, 5\pi]$.
- Note that $2^{15}x2^{15}x4x10^{5} \simeq 4 * 10^{14}$ namely each simulation gives a massive file.

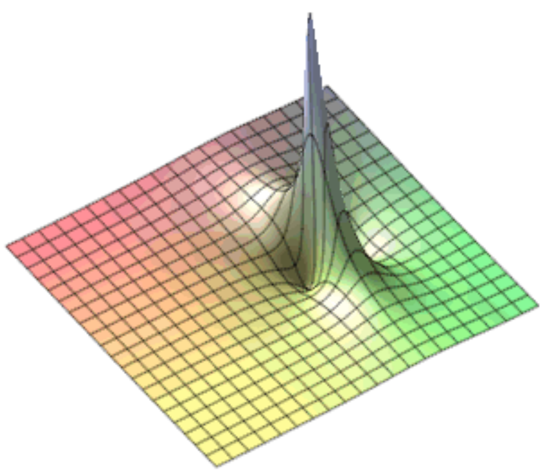
Lumps

The KPI equations has localised solutions called lumps that take the form in a suitable systems of coordinates

$$u(x,y,t;\epsilon) = 24 \frac{\frac{-(x-3b^2t)^2+3b^2y^2}{\epsilon^2} + 1/b^2}{\left[\frac{(x-3b^2t)^2+3b^2y^2}{\epsilon^2} + 1/b^2\right]^2}.$$

Observe that the maximum of the peak is $24b^2$ and moves in the positive x direction with speed $3b^2$ along the line $y = 0$.

$t = 0.$



Qualitative features of lump formation in the KPI solution

We consider the initial data

$$u_0(x,y) = -A\partial_x \operatorname{sech}^2(x^2+y^2).$$

- The solution of the KPI equation after the formation of dispersive shock waves, develops a region of lumps.
- Lumps correspond to discrete spectrum of the Schrödinger equation

  $$i\psi_y + \psi_{xx} + u\psi = 0.$$

  A.Fokas and L.Sung showed that if the initial data $u_0(x,y)$ is small (in a suitable norm) then there is no discrete spectrum. For the ϵ-dependent KP equation the data is always small, and the solution always develops into lumps.
- For fixed ϵ the maximum height u_{max} of the lump that is formed grows linearly with the maximum amplitude of the initial data u_{0max} as

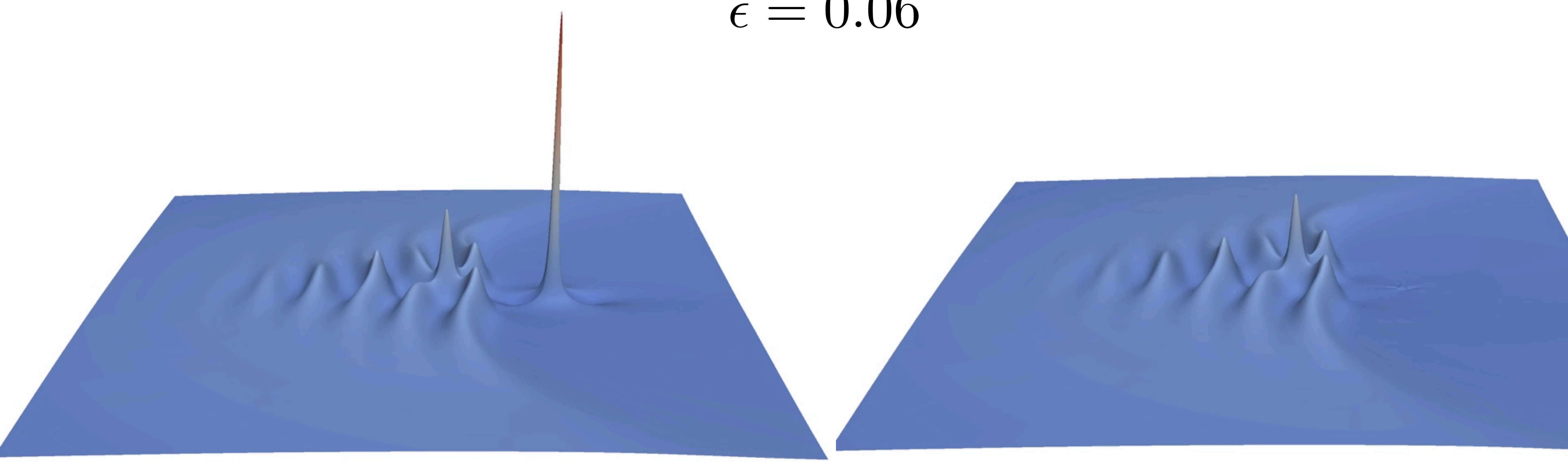
  $$u_{max} = c_0 + 7.6u_{0max}$$

Lump fitting

We consider the parametric fitting of the peak with a lump according to the formula

$$u(x,y,t;\epsilon) = 24\frac{(-(\frac{x-x_0}{\epsilon} - 3b^2\frac{t-t_0}{\epsilon})^2 + 3b^2\frac{(y-y_0)^2}{\epsilon^2} + 1/b^2)}{((\frac{x-x_0}{\epsilon} - 3b^2\frac{t-t_0}{\epsilon})^2 + 3b^2\frac{(y-y_0)^2}{\epsilon^2} + 1/b^2)^2}.$$

$$\epsilon = 0.06$$



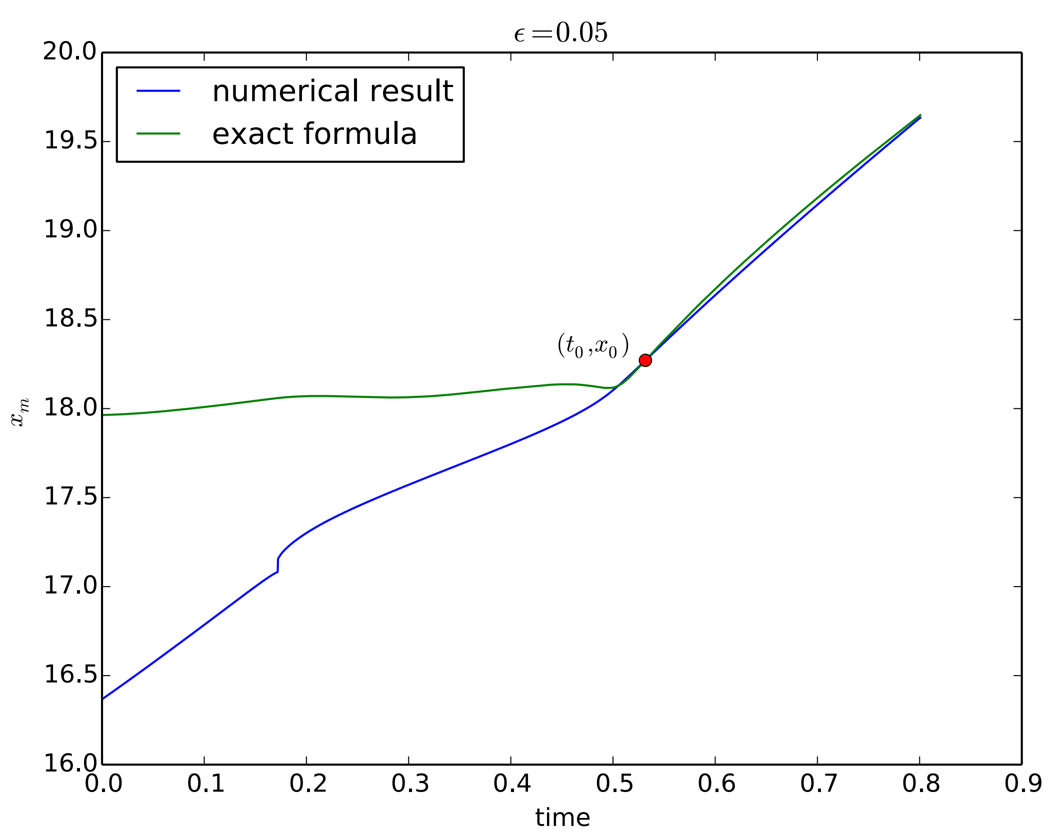
Velocity of the maximum of the peaks

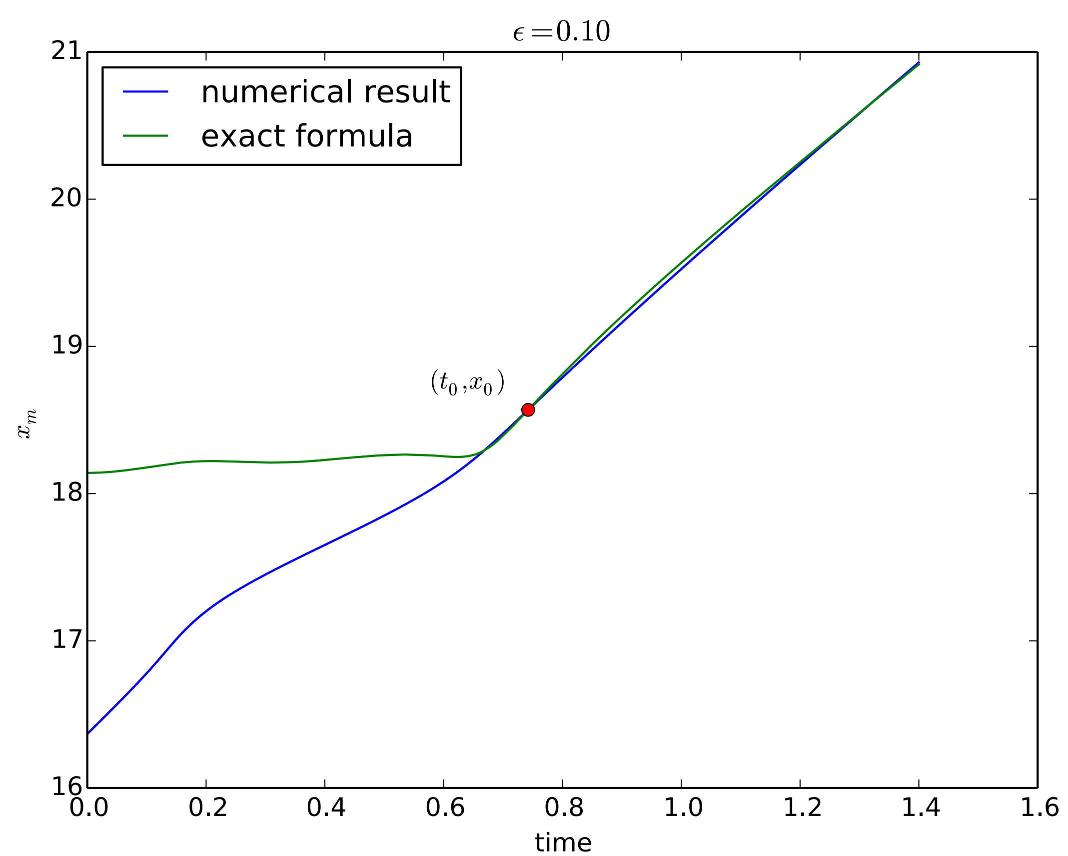
Given the lump solution

$$u(x,y,t;\epsilon) = 24\frac{\left(-\left(\frac{x-x_0}{\epsilon} - 3b^2\frac{t-t_0}{\epsilon}\right)^2 + 3b^2\frac{(y-y_0)^2}{\epsilon^2} + 1/b^2\right)}{\left(\left(\frac{x-x_0}{\epsilon} - 3b^2\frac{t-t_0}{\epsilon}\right)^2 + 3b^2\frac{(y-y_0)^2}{\epsilon^2} + 1/b^2\right)^2}.$$

then the velocity of the maximum of the peak is

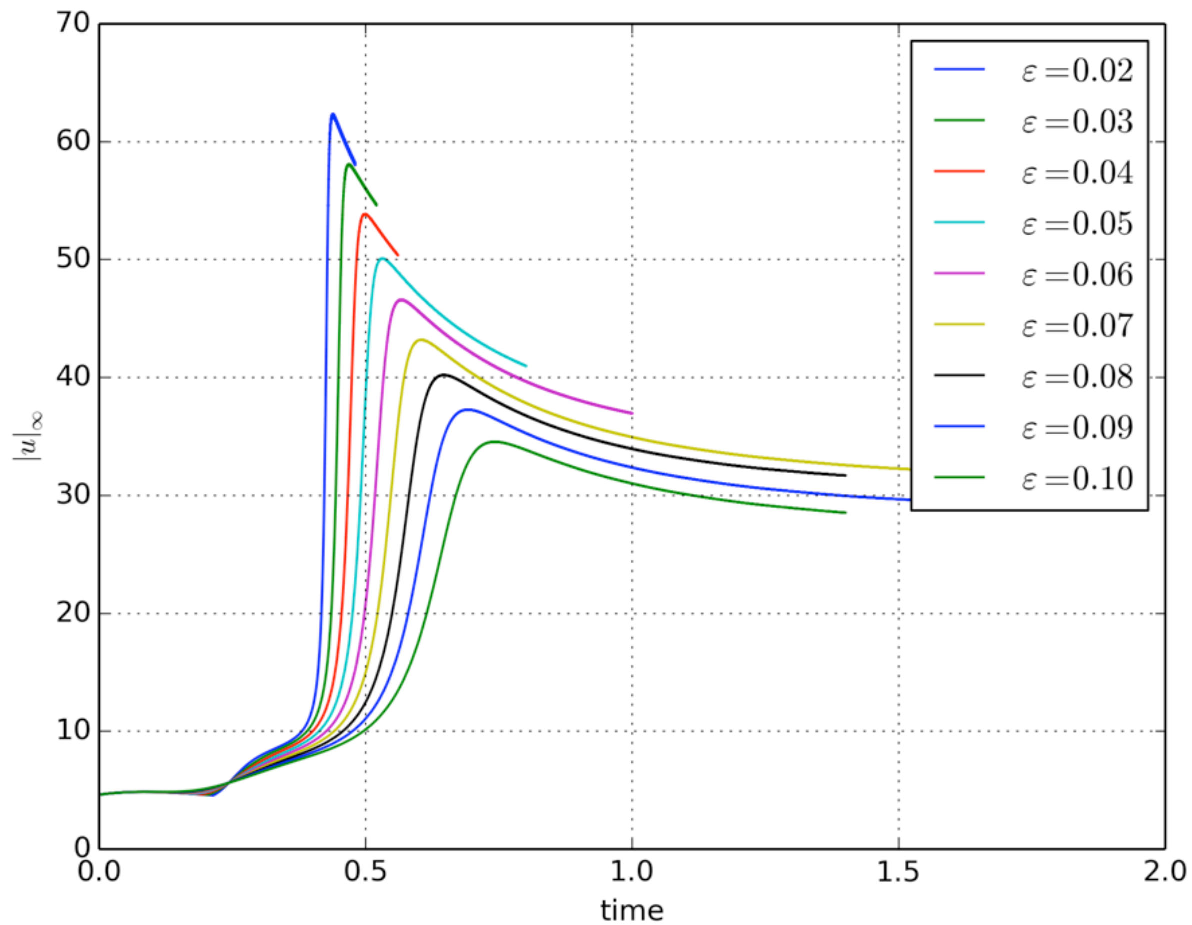
$$x_m(t) = x_0 + 24b(t)^2(t-t_0)/8$$

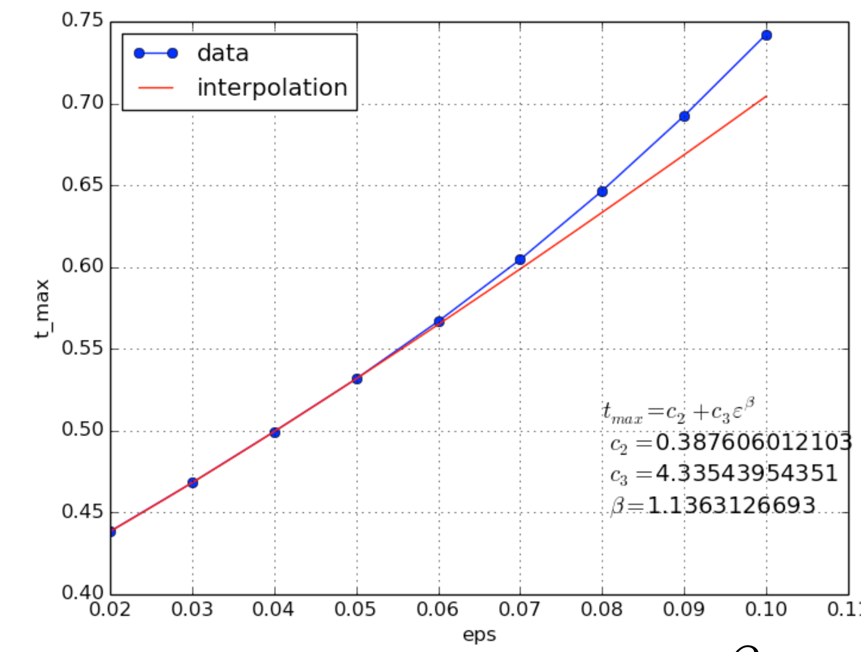




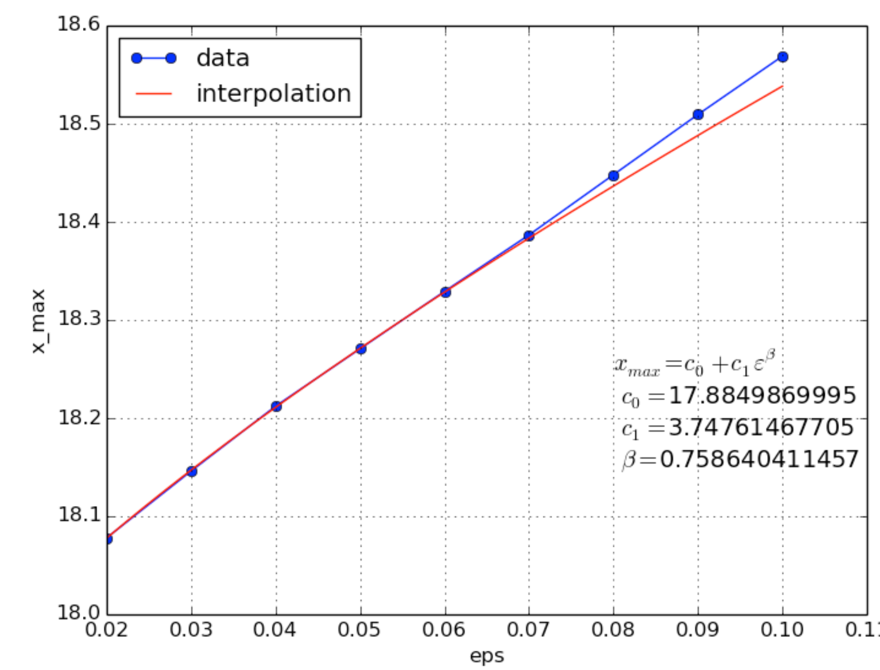
We study the time t_{max} and the position x_{max} of the appearance of the maximum peak as a function of ϵ.

$$|u(\epsilon = 0)|_\infty - |u(\epsilon)|_\infty = c\,\epsilon^{0.7}$$





$$t_{max} = c_2 + c_3 \epsilon^\beta$$



$$x_{max} = c_0 + c_1 \epsilon^{\frac{4}{5}}$$

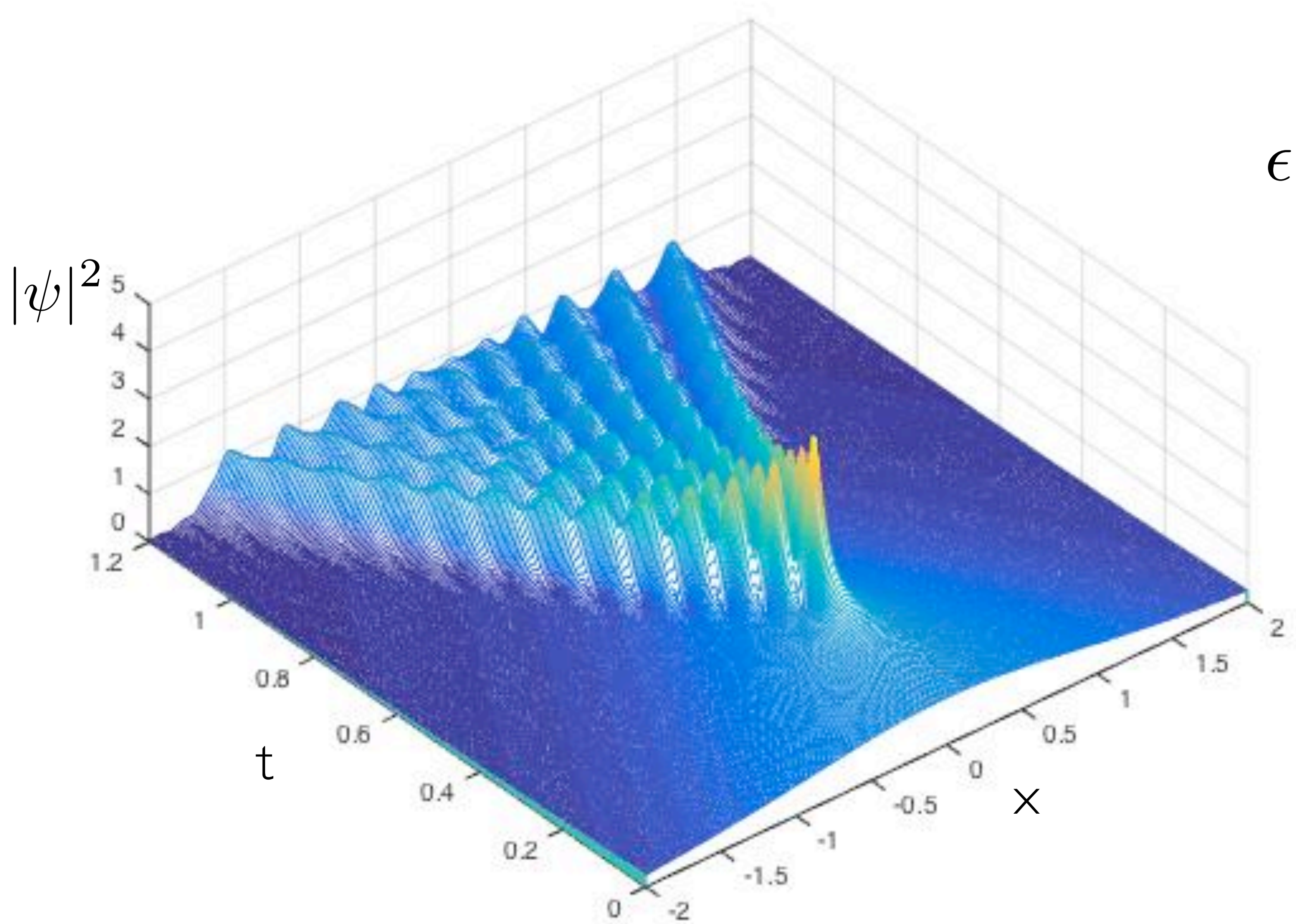
Same scaling as the semiclassical limit of the focusing NLS equation

Comparison with focusing NLS

$$i\epsilon\psi_t + \frac{\epsilon^2}{2}\psi_{xx} + |\psi|^2\psi = 0$$

$$\psi(x, t = 0, \epsilon) = A(x) exp^{\frac{i}{\epsilon}S(x)}$$

$\epsilon = 0.1$



Onset of oscillations Dubrovin-G.-Klein 2009, Bertola-Tovbis 2013, Dubrovin-G.Klein-Moro 2015

In a region of size $\epsilon^{\frac{4}{5}}$ around the critical point (x_c, t_c) and the critical value $\psi_c = \psi(x_c, t_c; \epsilon)$, the solution $\psi(x, t, \epsilon)$ is given by

$$|\psi(x,t,\epsilon)|^2 = |\psi_c|^2 + \epsilon^{\frac{2}{5}} \mathrm{Re} \left(\alpha \Omega \left(\frac{x - x_c + \beta(t - t_c)}{\gamma \epsilon^{\frac{4}{5}}} \right) \right) + O(\epsilon^{\frac{4}{5}})$$

where Ω solves the Painlevé equation $\Omega_{zz} = 6\Omega^2 - z$ with asymptotic behaviour $\Omega(z) = -\sqrt{\frac{z}{6}}$ as $|z| \to \infty$. The solution Ω has poles!. On the poles position (x_p, t_p) the solution is given by the Peregrine breather

$$|\psi(x,t,\epsilon)| = \left| Q_{Br} \left(\frac{x - x_p}{\epsilon}, \frac{t - t_p}{\epsilon} \right) \right| + O(\epsilon^{\frac{1}{5}})$$

where

$$|Q_{Br}(X,T)| = \sqrt{u_c} \left| \left(1 - 4 \frac{1 + 2iu_c T}{1 + 4u_c(X + 2Tv_c)^2 + 4u_c^2 T^2} \right) \right|$$

Peregrine breather

The maximum peak is approximated by the Peregrine breather (M. Bertola, A. Tovbis, CPAM 2013)

$$|\psi(x,t,\epsilon)| = \left| Q_P\left(\frac{x-x_p}{\epsilon}, \frac{t-t_p}{\epsilon}\right)\right| + O(\epsilon^{\frac{1}{5}})$$

where $|Q_P(X,T)| = b\left|\left(1 - 4\frac{1+2b^2T}{1+4b^2(X+2Ta^2)^2+4b^4T^2}\right)\right|$.
Note that $|\psi_{max}| = 3b + O(\epsilon^{\frac{1}{5}})$ where b is the maximum value at the critical point of the semiclassical limit. The position and the time of the maximum peak scale as

$$x_{max} = c_1 + c_2\epsilon^{\frac{4}{5}}, \quad t_{max} = c_3 + c_4\epsilon^{\frac{4}{5}}$$

Note that $|Q_P(X-3b^2T, Y)|^2 - b^2$ is the lump solution of the KPI equation up to scalings.

Conclusions

We have described the solution of the KP (I,II) equation in the small dispersion limit, in the region of development of dispersive shock waves. We showed that the solution $u(x, y, t; \epsilon)$ of the KP I equation in the limit $\epsilon \to 0$ has a second caustic region where lumps are formed.

- The maximum lump amplitude is proportional to the maximum initial data amplitude.
- The lump amplitude is slowly decreasing as a function of time.
- The position x_{max} of the first lump scales with ϵ like

$$x_{max} = c_1 + c_2 \epsilon^{\frac{4}{5}}.$$

- The time of the appearance of the first lump t_{max} scales with ϵ like

$$t_{max} = c_3 + c_4 \epsilon^{1.1}.$$

- the L^∞ norm scales like $|u(\epsilon = 0)|_\infty - |u(\epsilon)|_\infty = c_0\, \epsilon^{0.7}$.