Discrete Integrable Equations in $3D$

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Joint project with E. Ferapontov, I. Roustemoglou

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Integrability tests for discrete systems

1. Multidimensional consistency (Adler, Bobenko, Suris, Nijhoff, ...)
2. Symmetry approach (Mikhailov, Shabat, Yamilov, Wang, ...)
3. Algebraic entropy, singularity confinement,... (Vialet, Hulburd, Hone,...)

Programme of classification of $2+1$-dimensional integrable systems

1. Dispersive deformations:
   Given a dispersionless integrable system

   $$A(u)u_t + B(u)u_x + C(u)u_y = 0$$

   can we construct a dispersive integrable deformations?

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Approach in $2+1$D

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Take an integrable $2+1$D dispersionless system. It can be decoupled into a pair of $1+1$D equations infinitely many ways (hydrodynamic reductions).

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STEP 3

Require that *all* hydrodynamic reductions can be deformed into reductions of the perturbed system by adding a suitable formal series of dispersive terms

Outline

1. Dispersionless 3D systems: method of hydrodynamic reductions.

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4. Classification results.
5. Semi-discrete systems.

Consider a system of quasilinear equations

$$A(\mathbf{u})\mathbf{u}_x + B(\mathbf{u})\mathbf{u}_y + C(\mathbf{u})\mathbf{u}_t = 0.$$

Let us seek a multiphase solution $\mathbf{u}(R^1, \dots, R^N)$, where $R^i = R^i(x, y, t)$ satisfy a pair of commuting $1+1$-dimensional equations

$$R^i_y = \mu^i(R)R^i_x, \qquad R^i_t = \lambda^i(R)R^i_x$$

Definition

[Ferapontov-Khusnutdinova] A quasilinear system is said to be integrable if for any number of phases N it possesses infinitely many N-phase solutions parametrised by N arbitrary functions of one variable.

Example: dKP equation

$$u_t = uu_x + w_y, \quad u_y = w_x$$

N-phase solutions: $u = u(R^1, ..., R^N)$, $w = w(R^1, ..., R^N)$ where

$$R^i_y = \mu^i(R) R^i_x, \qquad R^i_t = \lambda^i(R) R^i_x$$

Then

$$\partial_i w = \mu^i \partial_i u, \qquad \lambda^i = u + (\mu^i)^2$$

Functions $u(R)$ and $\mu^i(R)$ obey the Gibbons-Tsarev system

$$\partial_j \mu^i = \frac{\partial_j u}{\mu^j - \mu^i}, \qquad \partial_i \partial_j u = 2\frac{\partial_i u \partial_j u}{(\mu^j - \mu^i)^2}$$

Remark

The Gibbons-Tsarev system is **in involution!**

In particular in the case $N = 1$ we have

$$u = R, \quad w = W(R), \quad \mu(R) = W'(R),$$

$$R_y = W'(R) R_x, \quad R_t = \left(R + (W'(R))^2\right) R_x$$

Dispersive deformations

Consider the KP equation

$$u_t = uu_x + w_y + \epsilon^2 u_{xxx} \qquad\qquad u_y = w_x$$

Let us seek a formal 1-phase solution in the form

$$u = U(R) + \epsilon\kappa_1(R)R_x + \epsilon^2\left(\kappa_2(R)R_{xx} + \kappa_3(R)R_x^2\right) + \epsilon^3(\dots) + \dots$$
$$w = W(R) + \epsilon\rho_1(R)R_x + \epsilon^2\left(\rho_2(R)R_{xx} + \rho_3(R)R_x^2\right) + \epsilon^3(\dots) + \dots$$

and let us require

$$R_y = \mu(R)R_x + \epsilon(a_1(R)R_{xx} + a_2(R)R_x^2) + \epsilon^2(\dots) + \dots$$
$$R_t = (U(R) + \mu(R)^2)R_x + \epsilon(A_1(R)R_{xx} + A_2(R)R_x^2) + \epsilon^2(\dots) + \dots$$

Generalised Miura transfromations

$$R \to \phi(R) + \epsilon\phi_1(R)R_x + \cdots$$

Up to the Miura transformation we can seek a 1-phase solution in the form

$$u = R$$
$$w = W(R) + \epsilon\rho_1(R)R_x + \epsilon^2\left(\rho_2 R_{xx} + \rho_3 R_x^2\right) + \epsilon^2(\dots) + \dots$$

$$R_y = \mu(R)R_x + \epsilon(a_1(R)R_{xx} + a_2(R)R_x^2) + \epsilon^2(\dots) + \dots$$
$$R_t = (R + \mu(R)^2)R_x + \epsilon(A_1(R)R_{xx} + A_2(R)R_x^2) + \epsilon^2(\dots) + \dots$$

Requiring that this is a formal solution of the KP equation we obtain

$$u = R,\ w = W(R) + \epsilon^2 \left(W'' R_{xx} + \frac{1}{2}(W''' - (W'')^3) R_x^2 \right) + O(\epsilon^4)$$

$$R_y = W' R_x + \epsilon^2 \left(W'' R_{xx} + \frac{1}{2}(W''' - W''^3) R_x^2 \right)_x + O(\epsilon^4)$$

$$R_t = (R + W'^2) R_x + \epsilon^2 \left((2W'W'' + 1) R_{xx} + (W'W''' - W'W''^3 + \frac{W''^2}{2}) R_x^2 \right.$$

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NOTE

- Procedure is entirely algebraic;
- Similar results can be (and are) obtained for two, three (and so on) phase solutions.

Reconstruction of dispersive corrections

Consider the dKP equation

$$u_t = uu_x + w_y, \quad w_x = u_y.$$

Let us add all possible dispersive corrections which are differential polynomials in u, w with coefficients being functions in u, w:

$$u_t = uu_x + w_y + \epsilon\left(\alpha_1 u_{xx} + \alpha_2 u_{xy} + \alpha_3 u_{yy} + \alpha_4 w_{yy} + \cdots\right) + \epsilon^2() + \cdots$$

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Generalised Miura group

$$u \to \phi(u) + \epsilon(\phi_1(u)u_x + \phi_2(u)u_y) + \epsilon^2() + \cdots$$

$$w \to \psi(w) + \epsilon(\psi_1(u,w)u_x + \psi_2(u,w)u_y) + \epsilon^2() + \cdots$$

Now we seek the deformed hydrodynamic reductions for this equation and obtain:

$$u_t = uu_x + w_y +$$

$$+\epsilon^2(h_1 u_{xxx} + h_2(2uu_y u_{yy} + u_x u_y^2 + w_y u_{yy}) + h_3(\frac{3}{2}u_y u_{yy} - \frac{1}{2}u_x w_{yy}))+$$

$$+O(\epsilon^4),$$

and h_1, h_2, h_3 are arbitrary *constants*. Note that h_1 corresponds to KP.

Reconstruction of dispersive corrections

1. We have computed the dispersive corrections up to order ϵ^4. The moduli space of the corrections (up to Miura group) is finite dimensional, i.e. correction coefficients are constants but not functions in u, w (unlike the situation in 1+1-dimensions).

Reconstruction of dispersive corrections

1. We have computed the dispersive corrections up to order ϵ^4. The moduli space of the corrections (up to Miura group) is finite dimensional, i.e. correction coefficients are constants but not functions in u, w (unlike the situation in 1+1-dimensions).
2. We conjecture that any (non-linearly degenerate) integrable dispersionless $2+1$-dimensional equation can be deformed in this way and the moduli space of the corrections will be finite dimensional.

Definition

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A $2 + 1$-dimensional system is said to be integrable if all hydrodynamic reductions of its dispersionless limit (which is supposed to be linearly non-degenerate) can be deformed into reductions of the corresponding dispersive counterpart.

Classification strategy

- We first classify quasilinear systems which may potentially occur as dispersionless limits of integrable equations.
- We reconstruct dispersive terms requiring the inheritance of hydrodynamic reductions of the dispersionless limit by the full dispersive equation.

Let us illustrate our approach by classifying integrable discrete wave-type equations of the form

$$\triangle_{t\bar{t}}\, u - \triangle_{x\bar{x}}\, f(u) - \triangle_{y\bar{y}}\, g(u) = 0,$$

where f and g are functions to be determined and

$$\triangle_x = \frac{T_x - 1}{\epsilon}, \quad \triangle_{\bar{x}} = \frac{1 - T_x^{-1}}{\epsilon}, \ldots,$$

$$T_x = e^{\epsilon \partial_x}, \quad T_y = e^{\epsilon \partial_y}, \quad T_z = e^{\epsilon \partial_z}$$

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Using expansions of the form

$$\triangle_{t\bar{t}} = \frac{(e^{\epsilon \partial_t} - 1)(1 - e^{-\epsilon \partial_t})}{\epsilon^2} = \partial_t^2 + \frac{\epsilon^2}{12}\partial_t^4 + \dots,$$

we can rewrite the above equation as an infinite series in ϵ,

$$u_{tt} - f(u)_{xx} - g(u)_{yy} + \frac{\epsilon^2}{12}[u_{tttt} - f(u)_{xxxx} - g(u)_{yyyy}] + \dots = 0.$$

Dispersionless limit

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The resulting integrable systems are:

$$u_{tt} - (u - \ln(1 - e^u))_{xx} - (\ln(1 - e^u))_{yy} = 0,$$

$$u_{tt} - u_{xx} - (e^u)_{yy} = 0.$$

One-component reductions:

$$u = R(x, y, t), \quad R_t = \lambda(R) R_x, \quad R_y = \mu(R) R_x,$$
$$\lambda^2 = f' + g' \mu^2.$$

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Deformation:

$$R_y = \mu(R) R_x + \epsilon(a_1(R) R_{xx} + a_2(R) R_x^2) + \epsilon^2 (b_1(R) R_{xxx} + \dots) + \dots,$$

$$R_t = \lambda(R) R_x + \epsilon(A_1(R) R_{xx} + A_2(R) R_x^2) + \epsilon^2 (B_1(R) R_{xxx} + \dots) + \dots,$$

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Order ϵ^2:

$$f'' + g'' = 0, \quad g''(1 + f') - g'f'' = 0, \quad f''^2(1 + 2f') - f'(f' + 1)f''' = 0.$$

Notice that these are *second order* conditions in addition to *third order* dispersionless integrability conditions

$$f'g'f''' = f''(f''g' + g''f'), \quad f'g'g''' = g''(f''g' + g''f')$$

The solution is $f(u) = u - \ln(e^u + 1),\ g(u) = \ln(e^u + 1)$, resulting in the difference equation

$$\triangle_{t\bar{t}}\, u - \triangle_{x\bar{x}}\, [u - \ln(e^u + 1)] - \triangle_{y\bar{y}}\, [\ln(e^u + 1)] = 0,$$

which is an equivalent form of the Hirota equation, known as the 'gauge-invariant form' , or the 'Y-system'.

The first nontrivial term for expansions of R_y, R_t is

$$R_y = \mu(R)\ R_x + \epsilon^2(b_1 R_{xxx} + b_2 R_{xx} R_x + b_3 R_x^3) + O(\epsilon^4),$$

$$R_t = \lambda(R)\ R_x + \epsilon^2(B_1 R_{xxx} + B_2 R_{xx} R_x + B_3 R_x^3) + O(\epsilon^4),$$

where

$$b_1 = \frac{1}{12}\left(\mu^2 - 1\right)\mu',$$
$$B_1 = \frac{\left(\mu^2 - 1\right) e^R \left(\mu^2 + 2\mu\mu' e^R + 2\mu\mu' - 1\right)}{24\left(e^R + 1\right)^2 \lambda},$$

Consider the following examples of integrable 3D dispersionless equations

$$(u_1 - u_2)u_{12} + (u_3 - u_1)u_{13} + (u_2 - u_3)u_{23} = 0,$$

$$\partial_1 \left(\ln \frac{u_3}{u_2} \right) + \partial_2 \left(\ln \frac{u_1}{u_3} \right) + \partial_3 \left(\ln \frac{u_2}{u_1} \right) = 0.$$

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These are dispersionless (continuum) limits of the lattice KP

$$(\triangle_1 u - \triangle_2 u)\triangle_{12} u + (\triangle_3 u - \triangle_1 u)\triangle_{13} u + (\triangle_2 u - \triangle_3 u)\triangle_{23} u = 0.$$

and lattice Schwarzian KP equations

$$\triangle_1 \left(\ln \frac{\triangle_3 u}{\triangle_2 u} \right) + \triangle_2 \left(\ln \frac{\triangle_1 u}{\triangle_3 u} \right) + \triangle_3 \left(\ln \frac{\triangle_2 u}{\triangle_1 u} \right) = 0.$$

Our main result provides a classification of integrable conservative equations of the form

$$\triangle_1 f + \triangle_2 g + \triangle_3 h = 0,$$

$$f = f(\triangle_1 u, \triangle_2 u, \triangle_3 u),\ \ g = g(\triangle_1 u, \triangle_2 u, \triangle_3 u),\ \ h = h(\triangle_1 u, \triangle_2 u, \triangle_3 u)$$

The corresponding dispersionless limits are scalar conservation laws of the form

$$\partial_1 f(u_1, u_2, u_3) + \partial_2 g(u_1, u_2, u_3) + \partial_3 h(u_1, u_2, u_3) = 0.$$

Discrete conservation laws

The classification is performed modulo elementary transformations of the form $u \to \alpha u + \alpha_i x^i$, as well as permutations of the independent variables x^i. We show that any integrable equation of such a form arises as a conservation law of a certain discrete integrable equation of octahedron type,

$$F(T_1 u,\ T_2 u,\ T_3 u,\ T_{12} u,\ T_{13} u,\ T_{23} u) = 0.$$

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$$F(T_1u,\ T_2u,\ T_3u,\ T_{12}u,\ T_{13}u,\ T_{23}u) = 0.$$

Theorem

Integrable discrete conservation laws are naturally grouped into 7 three-parameter families,

$$\alpha I + \beta J + \gamma K = 0,$$

where α, β, γ are free parameters and I, J, K denote left hand sides of three linearly independent discrete conservation laws of seven octahedron equations.

Classification Results: octahedron equations

Octahedron equations (Adler, Bobenko, Suris)

$$\frac{T_2\tau - T_{12}\tau}{T_{23}\tau} = T_1\tau\left(\frac{1}{T_{13}\tau} - \frac{1}{T_3\tau}\right), \tag{1}$$

$$T_{12}uT_{13}u + T_2uT_{23}u + T_1uT_3u = T_{12}uT_{23}u + T_1uT_{13}u + T_2uT_3u, \tag{2}$$

$$\frac{T_{23}\tau}{T_3\tau} + \frac{T_{12}\tau}{T_2\tau} + \alpha\frac{T_{12}\tau + T_{23}\tau}{T_2\tau + T_3\tau} = \frac{T_{12}\tau}{T_1\tau} + \frac{T_{13}\tau}{T_3\tau} + \alpha\frac{T_{12}\tau + T_{13}\tau}{T_1\tau + T_3\tau}, \tag{3}$$

$$(T_1u - T_2u)T_{12}u + (T_3u - T_1u)T_{13}u + (T_2u - T_3u)T_{23}u = 0, \tag{4}$$

$$\frac{T_{13}\tau - T_{12}\tau}{T_1\tau} + \frac{T_{12}\tau - T_{23}\tau}{T_2\tau} + \frac{T_{23}\tau - T_{13}\tau}{T_3\tau} = 0, \tag{5}$$

$$(T_2\triangle_1u)(T_3\triangle_2u)(T_1\triangle_3u) = (T_2\triangle_3u)(T_3\triangle_1u)(T_1\triangle_2u), \tag{6}$$

$$\left(\frac{T_{12}\tau}{T_2\tau} - 1\right)\left(\frac{T_{13}\tau}{T_1\tau} - 1\right)\left(\frac{T_{23}\tau}{T_3\tau} - 1\right) = \left(\frac{T_{12}\tau}{T_1\tau} - 1\right)\left(\frac{T_{13}\tau}{T_3\tau} - 1\right)\left(\frac{T_{23}\tau}{T_2\tau} - 1\right) \tag{7}$$

here $\tau = e^{\lambda u/\epsilon}$, $\lambda =$ *const* which is specific for each case.

One of the seven cases mentioned above is the octahedron equation

$$(T_2\triangle_1 u)(T_3\triangle_2 u)(T_1\triangle_3 u) = (T_2\triangle_3 u)(T_3\triangle_1 u)(T_1\triangle_2 u),$$

known as the Schwarzian KP equation in its standard form. It possesses three conservation laws

$$I = \triangle_2 \ln\left(1 - \frac{\triangle_3 u}{\triangle_1 u}\right) - \triangle_3 \ln\left(\frac{\triangle_2 u}{\triangle_1 u} - 1\right) = 0,$$

$$J = \triangle_3 \ln\left(1 - \frac{\triangle_1 u}{\triangle_2 u}\right) - \triangle_1 \ln\left(\frac{\triangle_3 u}{\triangle_2 u} - 1\right) = 0,$$

$$K = \triangle_1 \ln\left(1 - \frac{\triangle_2 u}{\triangle_3 u}\right) - \triangle_2 \ln\left(\frac{\triangle_1 u}{\triangle_3 u} - 1\right) = 0,$$

The linear combination $I + J + K = 0$ coincides with the Schwarzian KP.

Discrete conservation laws: the idea of the proof

The dispersionless limit is of the form

$$\partial_1 f + \partial_2 g + \partial_3 h = 0,$$

we denote

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$$f = f(a, b, c),\ g = g(a, b, c),\ h = h(a, b, c).$$

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Necessary integrability conditions at **order** ϵ^1 give:

$$f_a = g_b = h_c = 0, \quad f_b + g_a + f_c + h_a + g_c + h_b = 0.$$

Thus the equation $\partial_1 f + \partial_2 g + \partial_3 h = 0$ becomes

$$Fu_{12} + Gu_{13} + Hu_{23} = 0, \quad F = f_b + g_a,\ G = f_c + h_a,\ H = g_c + h_b$$

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Any integrable equation of this type is equivalent to

$$[p(u_1) - q(u_2)]u_{12} + [r(u_3) - p(u_1)]u_{13} + [q(u_2) - r(u_3)]u_{23} = 0,$$

and functions p, q, r satisfy the integrability conditions

$$p'' = p'\left(\frac{p'-q'}{p-q} + \frac{p'-r'}{p-r} - \frac{q'-r'}{q-r}\right),$$

$$q'' = q'\left(\frac{q'-p'}{q-p} + \frac{q'-r'}{q-r} - \frac{p'-r'}{p-r}\right),$$

$$r'' = r'\left(\frac{r'-p'}{r-p} + \frac{r'-q'}{r-q} - \frac{p'-q'}{p-q}\right).$$

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3. Taking linear combinations of the four conservation laws in each of the above seven cases, and replacing partial derivatives u_1, u_2, u_3 by discrete derivatives $\triangle_1 u, \triangle_2 u, \triangle_3 u$, we obtain discrete equations of our form which, at this stage, are the *candidates* for integrability.

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4. Applying the ϵ^2-integrability test to the above linear combinations, we obtain constraints on the coefficients, and the final classification result.

Further illustration

One of the cases correspond to a situation when $q = const$, $r = const$. Setting $q = -r = 1$ we obtain a single equation on p:

$$p'' = \frac{2p(p')^2}{p^2 - 1}.$$

The solution is $p = \frac{1+e^{u_1}}{1-e^{u_1}}$, which leads to the equation

$$e^{u_1} u_{12} - u_{13} + (1 - e^{u_1}) u_{23} = 0.$$

The four conservation laws are

$$\partial_1 e^{u_1} + \partial_3 (e^{u_2 - u_1} - e^{u_2}) = 0, \quad \partial_1 e^{-u_3} + \partial_2 (e^{u_1 - u_3} - e^{-u_3}) = 0,$$

$$\partial_2 (u_3 - \ln(1 - e^{u_1})) + \partial_3 (\ln(1 - e^{u_1}) - u_1) = 0,$$

$$\partial_1 \frac{u_2 u_3}{2} - \partial_2 \left(\frac{u_1 u_3}{2} - u_1 \ln(1 - e^{u_1}) - \mathrm{Li}_2(e^{u_1}) \right)$$

$$+\partial_3 \left(\frac{u_1^2}{2} - \frac{u_1 u_2}{2} - u_1 \ln(1 - e^{u_1}) - \mathrm{Li}_2(e^{u_1}) \right) = 0.$$

Further illustration

One of the cases correspond to a situation when $q = const$, $r = const$. Setting $q = -r = 1$ we obtain a single equation on p:

$$p'' = \frac{2p(p')^2}{p^2 - 1}.$$

The solution is $p = \frac{1+e^{u_1}}{1-e^{u_1}}$, which leads to the equation

$$e^{u_1} u_{12} - u_{13} + (1 - e^{u_1}) u_{23} = 0.$$

The four conservation laws are

$$\partial_1 e^{u_1} + \partial_3 (e^{u_2 - u_1} - e^{u_2}) = 0, \quad \partial_1 e^{-u_3} + \partial_2 (e^{u_1 - u_3} - e^{-u_3}) = 0,$$

$$\partial_2 (u_3 - \ln(1 - e^{u_1})) + \partial_3 (\ln(1 - e^{u_1}) - u_1) = 0,$$

$$\partial_1 \frac{u_2 u_3}{2} - \partial_2 \left(\frac{u_1 u_3}{2} - u_1 \ln(1 - e^{u_1}) - \mathrm{Li}_2(e^{u_1}) \right)$$

$$+ \partial_3 \left(\frac{u_1^2}{2} - \frac{u_1 u_2}{2} - u_1 \ln(1 - e^{u_1}) - \mathrm{Li}_2(e^{u_1}) \right) = 0.$$

Further illustration

Applying steps 3 and 4 we obtain the discrete equation:

$$e^{(T_1u - T_{13}u)/\epsilon} + e^{(T_{12}u - T_{23}u)/\epsilon} = e^{(T_1u - T_3u)/\epsilon} + e^{(T_2u - T_{23}u)/\epsilon}.$$

Finally, setting $\tau = e^{u/\epsilon}$ we find

$$\frac{T_2\tau - T_{12}\tau}{T_{23}\tau} = T_1\tau\left(\frac{1}{T_{13}\tau} - \frac{1}{T_3\tau}\right).$$

Discrete conservation laws: semi-discrete equations

Similar results can be proved in the case of semi-discrete equations. We have two possible cases:

- One continuous and two discrete variables;
- Two continuous and one discrete variable.

Discrete conservation laws: semi-discrete equations

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- Two continuous and one discrete variable.

Theorem

Integrable semi-discrete conservation laws

$$\Delta_1 f + \Delta_2 g + \partial_3 h = 0$$

are naturally grouped into 7 three-parameter families,

$$\alpha I + \beta J + \gamma K = 0,$$

where α, β, γ *are free parameters and* I, J, K *denote left hand sides of three linearly independent discrete conservation laws of seven semi-discrete octahedron equations.*

Semi-discrete octahedron equations (Adler-Bobenko-Suris)

$$\frac{T_{12}v}{T_2v}+\frac{T_1v_3}{T_1v}=\frac{T_1v}{v}+\frac{T_2v_3}{T_2v},$$
$$T_{12}v=\frac{T_1vT_2v}{v}+\frac{T_2vT_1v_3-T_1vT_2v_3}{T_2v-T_1v},$$
$$\frac{vT_{12}v}{T_1v}=\frac{T_1vT_2v_3}{T_1v_3},$$
$$(T_{12}u-T_2u)T_1u_3=(T_1u-u)T_2u_3,$$
$$v(T_{12}v-T_2v)T_1v_3=T_1v(T_1v-v)T_2v_3,$$
$$(T_2\triangle_1u)(\triangle_2u)T_1u_3=(T_1\triangle_2u)(\triangle_1u)T_2u_3,$$
$$(T_2\sinh\triangle_1u)(\sinh\triangle_2u)T_1u_3=(T_1\sinh\triangle_2u)(\sinh\triangle_1u)T_2u_3.$$

Discrete conservation laws: semi-discrete equations

Theorem

There are no integrable semi-discrete equations of the form

$$\partial_1 f + \partial_2 g + \Delta_3 h = 0$$

We compare numerical solutions for the discrete equation

$$\triangle_{t\bar{t}}\, u - \triangle_{x\bar{x}}\, [u - \ln(e^u + 1)] - \triangle_{y\bar{y}}\, [\ln(e^u + 1)] = 0,$$

and its dispersionless limit

$$u_{tt} - [u - \ln(e^u + 1)]_{xx} - [\ln(e^u + 1)]_{yy} = 0,$$

subject to the following Cauchy data:

Disrete equation: $u(x, y, 0) = 3e^{-(x^2+y^2)},\ u(x, y, -\epsilon) = 3e^{-(x^2+y^2)}$.

Dispersionless equation: $u(x, y, 0) = 3e^{-(x^2+y^2)},\ u_t(x, y, 0) = 0$.

Numerical Simulations: Dispersionless equation

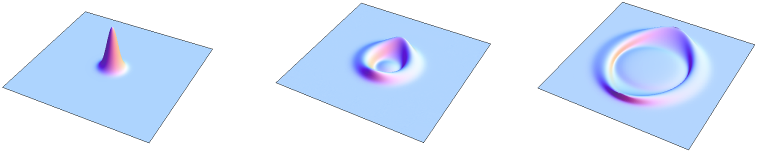

Figure: Numerical solution of the dispersionless equation for $t = 0, 4, 8$.

According to the general theory this solution is expected to break down in finite time [Klainerman].

The discrete equation can be equivalently written as

$$u(t+\epsilon) = -u(t-\epsilon) + (T_x + T_{\bar{x}})(u - \ln(e^u + 1)) + (T_y + T_{\bar{y}}) \ln(e^u + 1)$$

with

$$u(x,y,0) = 3e^{-(x^2+y^2)},\ u(x,y,-\epsilon) = 3e^{-(x^2+y^2)}.$$

No breakdown in this case.

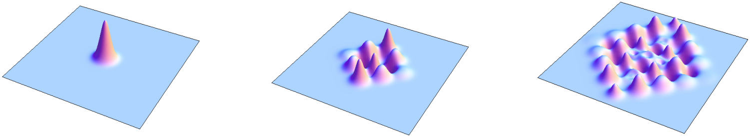

Figure: Numerical solution of the discrete equation for $\epsilon = 2$ and $t = 0, 4, 8$.

Numerical Simulations: Discrete equation

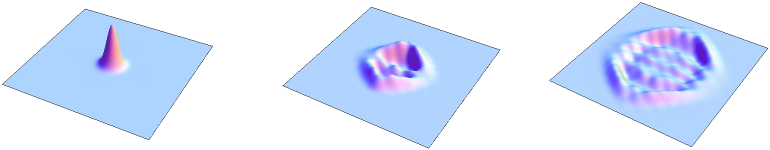

Figure: Numerical solution of the discrete equation for $\epsilon = 1$ and $t = 0, 4, 8$.

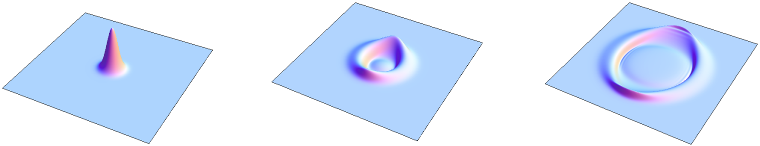

Figure: Numerical solution of the discrete equation for $\epsilon = 1/8$ and $t = 0, 4, 8$.

Numerical Simulations: Discrete equation

As $\epsilon \to 0$, solutions of the discrete equation tend to solutions of the dispersionless limit until the breakdown occurs.

At the breaking point

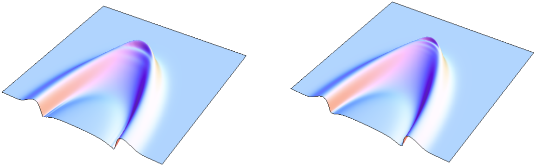

Figure: Formation of a dispersive shock wave in the numerical solution of the discrete equation for $\epsilon = 1/8$ (left) and $\epsilon = 1/16$ (right), at $t = 8$.

Other classes of semi-discrete equations

Consider the following equation

$$u_t = uu_y + w_y, \quad (T-1)w = \frac{\epsilon}{2}(T+1)u_y, \tag{8}$$

where T is a shift operator $T = e^{\epsilon D_x}$.
The continuous limit of this equation is

$$u_t = uu_y + w_y, \quad w_x = u_y.$$

Theorem

The following equations constitute a complete list of integrable equations of the form
$u_t = \phi u_x + \psi u_y + \eta w_x + \tau w_y + \epsilon() + \epsilon^2(),\ (T-1)w = \frac{\epsilon}{2}(T+1)u_y$:

$$u_t = uu_y + w_y,$$
$$u_t = (w + \alpha e^u)u_y + w_y,$$
$$u_t = u^2 u_y + (uw)_y + \frac{\epsilon^2}{12}u_{yyy},$$
$$u_t = u^2 u_y + (uw)_y + \frac{\epsilon^2}{12}(u_{yy} - \frac{3}{4}\frac{u_y^2}{u})_y,$$

$$u_t = \phi u_x + f \triangle_y^+ g + p \triangle_y^- q, \qquad (9)$$

where the non-locality w is defined as $w_x = \triangle_y^+ u$, and ϕ, f, g, p, q are functions of u and w.

Theorem

The following examples constitute a complete list of integrable equations of the form (9) with the non-locality of Toda type:

$$\begin{aligned} u_t &= u \triangle_y^- w, \\ u_t &= (\alpha u + \beta) \triangle_y^- e^w, \\ u_t &= e^w \sqrt{u} \triangle_y^+ \sqrt{u} + \sqrt{u} \triangle_y^- (e^w \sqrt{u}), \end{aligned}$$

here $\alpha, \beta = const.$

$$u_t = f\triangle_x^+ g + h\triangle_x^- k + p\triangle_y^+ q + r\triangle_y^- s,$$

where the non-locality w is defined as $\triangle_x^+ w = \triangle_y^+ u$, and the functions f, g, h, k, p, q, r, s depend on u and w.

Theorem

The following examples constitute a complete list of integrable equations of the above form with the fully discrete non-locality:

$$\begin{aligned}
u_t &= u\triangle_y^-(u-w),\\
u_t &= u(\triangle_x^+ + \triangle_y^-)w,\\
u_t &= \triangle_y^- e^{u-w},\\
u_t &= (\alpha e^{-u}+\beta)\triangle_y^- e^{u-w},\\
u_t &= (\alpha e^{u}+\beta)(\triangle_x^+ + \triangle_y^-)e^{w},\\
u_t &= \sqrt{\alpha-\beta e^{2u}}\left(e^{w-u}\triangle_y^+\sqrt{\alpha-\beta e^{2u}} + \triangle_y^-(e^{w-u}\sqrt{\alpha-\beta e^{2u}})\right),
\end{aligned}$$