Painlevé monodromy varieties: geometry and quantisation

Volodya Roubtsov, ITEP Moscow
and LAREMA, UMR 6093 du CNRS, Université d'Angers.
Based on Chekhov-Mazzocco-R. *arXiv:1511.03851*

Talk at the
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Plan:

- Painlevé equations, Isomonodromy and Affine cubics;
- Decorated character varieties;
- Quantisation;
- Perspectives and output;

Painlevè list

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = 6w^2 + z \qquad \frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = 2w^3 + zw + \alpha$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \frac{1}{w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{\alpha w^2 + \beta}{z} + \gamma w^3 + \frac{\delta}{w}$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \frac{1}{2w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w}$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \frac{3w-1}{2w(w-1)}w_z^2 - \frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{\gamma w}{z} + \frac{(w-1)^2}{z^2}\frac{\alpha w^2 + \beta}{w} + \frac{\delta w(w+1)}{w-1}$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z}\right)w_z^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z}\right)w_z + \\ + \frac{w(w-1)(w-z)}{z^2(z-1)^2}\left[\alpha + \beta\frac{z}{w^2} + \gamma\frac{z-1}{(w-1)^2} + \delta\frac{z(z-1)}{(w-z)^2}\right]$$

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- Recently: P_{II} - has a genuine fully NC analogue (V. Retakh-V.R.)
- More recently: P_{IV} - has a (non genuine) fully NC analogue (M. Cafasso-M. Iglesias)

Confluences of the Painlevé equations-1

$$\begin{tikzcd}
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P_{VI} \arrow[r] & P_V \arrow[ur] \arrow[r] \arrow[dr] & P_V^{deg} \arrow[ur] \arrow[dr] & P_{II}^{JM} \arrow[r] & P_I \\
 & & P_{IV} \arrow[r] \arrow[ur] & P_{II}^{FN} \arrow[ur] &
\end{tikzcd}$$

Confluences of the Painlevé equations-2

Example

Take $w(z) = \epsilon \tilde{w}(\tilde{z}) + \frac{1}{\epsilon^5}, \quad z = \epsilon^2 \tilde{z} - \frac{6}{\epsilon^{10}}, \quad \alpha = \frac{4}{\epsilon^{15}}$ then PII

$$\frac{\mathrm{d}^2 w}{\mathrm{d} z^2} = 2w^3 + zw + \alpha$$

becomes

$$\frac{\mathrm{d}^2 \tilde{w}}{\mathrm{d} \tilde{z}^2} = 6\tilde{w}^2 + \tilde{z} + \epsilon^2 (2\tilde{w}^3 + \tilde{z}\tilde{w}),$$

that for $\epsilon \to 0$ is PI.

All Painlevé equations are **isomonodromic deformation equations** (Jimbo-Miwa1980)

$$\frac{\mathrm{d}B}{\mathrm{d}\lambda} - \frac{\mathrm{d}A}{\mathrm{d}z} = [A, B]$$

$A = A(\lambda; z, w, w_z),\ B = B(\lambda; z, w, w_z) \in \mathfrak{sl}_2.$

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The monodromy data are encoded in an affine cubic surfaces called *monodromy varieties*.

Painlevé monodromy manifolds Saito and van der Put

$$M_\varphi := \mathrm{Spec}(\mathbb{C}[\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3]/ < \varphi = 0 >)$$

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PIV $\quad x_1x_2x_3 + x_1^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_2 x_3 + 1 = \omega_4$

PIII $\quad x_1x_2x_3 + x_1^2 + x_2^2 + \omega_1 x_1 + \omega_2 x_2 = \omega_1 - 1$

PII $\quad x_1x_2x_3 + x_1 + x_2 + x_3 = \omega_4$

PI $\quad x_1x_2x_3 + x_1 + x_2 + 1 = 0$

PVI as isomonodromic deformation

Painlevé sixth equation

- The Painlevé VI equation describes the isomonodromic deformations of the rank 2 meromorphic connections on $\mathbb{P}^1$ with simple poles.

$$\frac{\mathrm{d}Y}{\mathrm{d}\lambda} = \left(\frac{A_1(z)}{\lambda} + \frac{A_2(z)}{\lambda - t} + \frac{A_3(z)}{\lambda - 1},\right) Y, \ \lambda \in \mathbb{C} \setminus \{0, t, 1\} \quad (1)$$

where $A_1, A_2, A_3 \in \mathfrak{sl}_2(\mathbb{C})$, $A_1 + A_2 + A_3 = -A_\infty$, diagonal.

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- Monodromy matrices $\gamma_j(Y_\infty) = Y_\infty M_j$
- Describes by generators of the fundamental group under the anti-isomorphism

$$\rho : \pi_1\left(\mathbb{P}^1 \setminus \{0, t, 1, \infty\}, \lambda_1\right) \to \mathrm{SL}_2(\mathbb{C}).$$

PVI as isomonodromic deformation

- $\mathrm{eigen}(M_j) = \mathrm{eigen}(\exp(2\pi i A_j))$
- We fix the base point λ_1 at infinity and the generators of the fundamental group to be $\gamma_1, \gamma_2, \gamma_3$ such that γ_j encircles only the pole i once and are oriented in such a way that

$$M_1 M_2 M_3 M_\infty = \mathbb{I}, \qquad M_\infty = \exp(2\pi i A_\infty). \tag{2}$$

- Eigenvalues of A_j are $(\theta_j, -\theta_j)$, $j = 0, t, 1, \infty$.
-

$$\alpha := (\theta_\infty - 1/2)^2; \qquad \beta := -\theta_0^2;$$
$$\gamma := \theta_1^2; \qquad \delta := (1/4 - \theta_t)^2.$$

PVI as isomonodromic deformation

Let:

$$G_j := \mathrm{Tr}(M_j) = 2\cos(\pi\theta_j), \quad j = 0, t, 1, \infty,$$

The Riemann-Hilbert correspondence

$$\mathcal{F}(\theta_0, \theta_t, \theta_1, \theta_\infty)/\mathcal{G} \to \mathcal{M}(G_1, G_2, G_3, G_\infty)/\mathrm{SL}_2(\mathbb{C}),$$

where $\mathcal{G}$ is the gauge group, is defined by associating to each Fuchsian system its monodromy representation class. The representation space $\mathcal{M}(G_1, G_2, G_3, G_\infty)$ is realised as an affine cubic surface (Jimbo)

$$x_1 x_2 x_3 + x_1^2 + x_2^2 + x_3^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4 = 0, \qquad (3)$$

where:

PVI as isomonodromic deformation

$$x_1 = \mathrm{Tr}\,(M_2 M_3)\,, \quad x_2 = \mathrm{Tr}\,(M_1 M_3)\,, \quad x_3 = \mathrm{Tr}\,(M_1 M_2)\,.$$

and

$$-\omega_i := G_k G_j + G_i G_\infty, i \neq k, j,$$

$$\omega_\infty = G_1^2 + G_2^2 + G_3^2 + G_\infty^2 + G_1 G_2 G_3 G_\infty - 4.$$

Iwasaki proved that the triple (x_1, x_2, x_3) satisfying the cubic relation (3) provides a set of coordinates on a large open subset

$$S \subset \mathcal{M}(G_1, G_2, G_3, G_\infty).$$

In what follows, we restrict to such open set.

General Affine Cubic

The main object studied in this talk is the affine irreducible cubic surface $M_\varphi := \mathbb{C}[x_1, x_2, x_3]/_{\langle \varphi=0\rangle}$ where

$$\varphi = x_1x_2x_3+\epsilon_1^{(d)}x_1^2+\epsilon_2^{(d)}x_2^2+\epsilon_3^{(d)}x_3^2+\omega_1^{(d)}x_1+\omega_2^{(d)}x_2+\omega_3^{(d)}x_3+\omega_4^{(d)} = 0, \tag{4}$$

According to Saito and Van der Put, all monodromy manifolds $\mathcal{M}^{(d)}$ have the form of M_φ for φ form the list above.

Here d is an index running on the list of the extended Dynkin diagrams $\widetilde{D}_4, \widetilde{D}_5, \widetilde{D}_6, \widetilde{D}_7, \widetilde{D}_8, \widetilde{E}_6, \widetilde{E}_7^*, \widetilde{E}_7^{**}, \widetilde{E}_8$. The coefficients $\omega^{(d)}$ are defined by:

$$\begin{aligned}
\omega_1^{(d)} &= -G_1^{(d)} G_\infty^{(d)} - \epsilon_1^{(d)} G_2^{(d)} G_3^{(d)},\\
\omega_2^{(d)} &= -G_2^{(d)} G_\infty^{(d)} - \epsilon_2^{(d)} G_1^{(d)} G_3^{(d)},\\
\omega_3^{(d)} &= -G_3^{(d)} G_\infty^{(d)} - \epsilon_3^{(d)} G_1^{(d)} G_2^{(d)},\\
\omega_4^{(d)} &= \epsilon_2^{(d)}\epsilon_3^{(d)} \left(G_1^{(d)}\right)^2 + \epsilon_1^{(d)}\epsilon_3^{(d)} \left(G_2^{(d)}\right)^2 + \epsilon_1^{(d)}\epsilon_2^{(d)} \left(G_3^{(d)}\right)^2 +\\
&\left(G_\infty^{(d)}\right)^2 + G_1^{(d)} G_2^{(d)} G_3^{(d)} G_\infty^{(d)} - 4\epsilon_1^{(d)}\epsilon_2^{(d)}\epsilon_3^{(d)}./
\end{aligned} \tag{5}$$

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- In algebraic geometry - projective completion:

$$\overline{M}_{\widetilde{\varphi}} := \{(u,v,w,t) \in \mathbb{P}^3 \,|x_1^2 t + x_2^2 t + x_3^2 t - x_1 x_2 x_3 +$$

$$+\omega_3 x_1 t^2 + \omega_2 x_2 t^2 + \omega_3 x_3 t^2 + \omega_4 t^3 = 0\}$$

is a del Pezzo surface of degree three and differs from it by three smooth lines at infinity forming a triangle [Oblomkov]
$t = 0, \qquad x_1 x_2 x_3 = 0.$

This family of cubics is a variety
$M_\varphi = \{(\bar{x},\bar{\omega}) \in \mathbb{C}^3 \times \Omega) : \varphi(\bar{x},\bar{\omega}) = 0\}$ where
$\bar{x} = (x_1, x_2, x_3), \quad \bar{\omega} = (\omega_1, \omega_2, \omega_3, \omega_4)$ and the "$\bar{x}-$forgetful" projection $\pi : M_\varphi \to \Omega : \pi(\bar{x},\bar{\omega}) = \bar{\omega}$. This projection defines a family of affine cubics with generically non–singular fibres $\pi^{-1}(\bar{\omega})$
The cubic surface M_φ has a volume form $\vartheta_{\bar{\omega}}$ given by the Poincaré residue formulae:

$$\vartheta_{\bar{\omega}} = \frac{dx_1 \wedge dx_2}{(\partial\varphi_{\bar{\omega}})/(\partial x_3)} = \frac{dx_2 \wedge dx_3}{(\partial\varphi_{\bar{\omega}})/(\partial x_1)} = \frac{dx_3 \wedge dx_1}{(\partial\varphi_{\bar{\omega}})/(\partial x_2)}. \tag{6}$$

The volume form is a holomorphic 2-form on the non-singular part of M_φ and it has singularities along the singular locus. This form defines the Poisson brackets on the surface in the usual way as

$$\{x_1, x_2\}_{\bar\omega} = \frac{\partial \varphi_{\bar\omega}}{\partial x_3} \tag{7}$$

The other brackets are defined by circular transposition of x_1, x_2, x_3. For $(i, j, k) = (1, 2, 3)$:

$$\{x_i, x_j\}_{\bar\omega} = \frac{\partial \varphi_{\bar\omega}}{\partial x_k} = x_i x_j + 2\epsilon_i^d x_k + \omega_i^d \tag{8}$$

and the volume form (6) reads as

$$\vartheta_{\bar\omega} = \frac{dx_i \wedge dx_j}{(\partial \varphi_{\bar\omega}/\partial x_k)} = \frac{dx_i \wedge dx_j}{(x_i x_j + 2\epsilon_i^d x_k + \omega_i^d)}. \tag{9}$$

Observe that for any $\varphi \in \mathbb{C}[x_1, x_2, x_3]$ the following formulae define a Poisson bracket on $\mathbb{C}[x_1, x_2, x_3]$:

$$\{x_i, x_{i+1}\} = \frac{\partial \varphi}{\partial x_{i+2}}, \qquad x_{i+3} = x_i, \tag{10}$$

and φ itself is a central element for this bracket, so that the quotient space

$$M_\varphi := \mathbb{C}[x_1, x_2, x_3]/_{\langle \varphi = 0 \rangle}$$

inherits the Poisson algebra structure $[\mathrm{Nambu} \sim 70]$.
Today we are going to re-parametrize and quantize it.

Affine Cubic as it is -2:

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- PVI ($\tilde{D}_4$) cubic with only $\omega_4 \neq 0$ admits the log-canonical symplectic structure $\bar{\vartheta} = \frac{du \wedge dv}{uv}$ under isomorphism $\mathbb{C}^* \times \mathbb{C}^*/\imath \to M_\varphi$ by

$$(u, v) \to (x_1 = -(u + \frac{1}{u}), x_2 = -(v + \frac{1}{v}), x_3 = -(uv + \frac{1}{uv})$$

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  and $\imath : \mathbb{C}^* \to \mathbb{C}^*$ is the involution $\imath(u) = \frac{1}{u}$, $\imath(v) = \frac{1}{v}$.
- The family (3) can be "uniformize" by some analogues of theta-functions related to toric mirror data on log-Calabi-Yau surfaces (M. Gross, P. Hacking and S.Keel (see Example 5.12 of "Mirror symmetry for log-Calabi-Yau varieties I, arXiv:1106.4977).

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Use the confluence scheme of the Painlevé equations.

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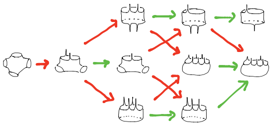

Basic ideas

- The character variety of a Riemann sphere with 4 holes $\mathrm{Hom}(\pi_1(\mathbb{P}^1 \setminus \{0, \mathrm{t}, 1, \infty\}); \mathrm{SL}_2(\mathbb{C}))/\mathrm{SL}_2(\mathbb{C})$ is the monodromy cubic of the Painlevé VI $(\mathrm{Goldman\text{-}Toledo})$.

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- The confluent Painlevé monodromy manifolds are "decorated character varieties" $(\mathrm{Chekhov\text{-}Mazzocco\ \text{-}R.2015})$.

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- The shear coordinates on the Teichmüller space can be complexified) $\Rightarrow$ coordinate description for the character variety.

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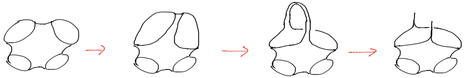
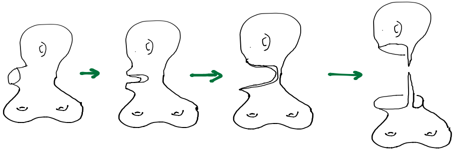
- The shear coordinates on the Teichmüller space can be complexified) $\Rightarrow$ coordinate description for the character variety.
- To visualize the confluence and the "decoration" we shall introduce two moves correspond to certain asymptotics in the (complexified) shear coordinates.

Two mouves

- **Hooking holes:**



- **Pinching two sides of the same hole:**



Teichmüller space

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Idea:

- Teichmüller theory for a Riemann surfaces with holes is well understood.

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Idea:

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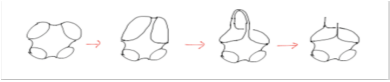
- Take confluences of holes to obtain cusps.



Teichmüller space

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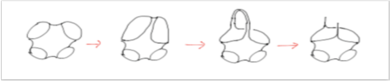

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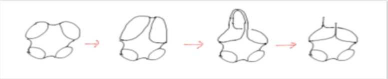

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This will provide a tcrcluster algebra of geometric type

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where Δ is a *Fuchsian group,* i.e. a discrete sub-group of $\mathbb{P}SL_2(\mathbb{R})$.

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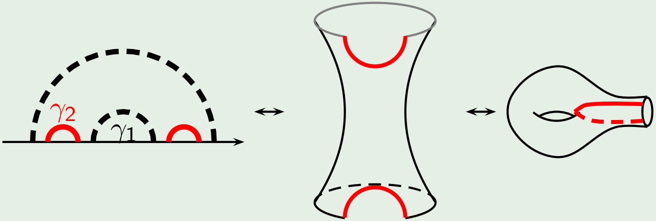

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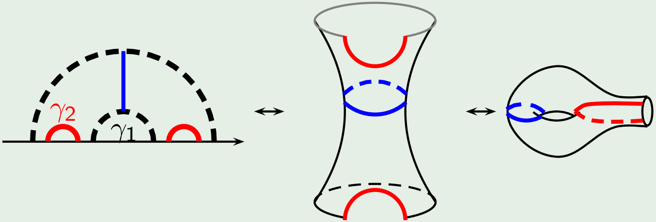

Theorem

Elements in $\pi_1(\Sigma_{g,s})$ are in 1-1 correspondence with conjugacy classes of closed geodesics.

Coordinates: geodesic lengths

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The geodesic length functions form an algebra with multiplication:

$$G_\gamma G_{\tilde{\gamma}} = G_{\gamma\tilde{\gamma}} + G_{\gamma\tilde{\gamma}^{-1}}.$$

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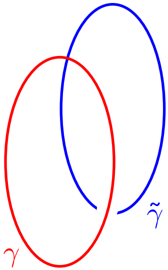

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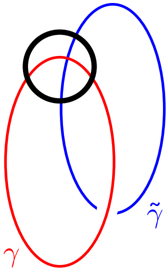

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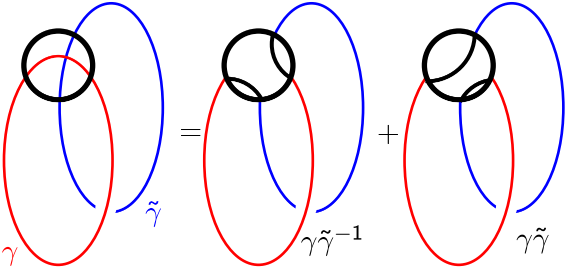
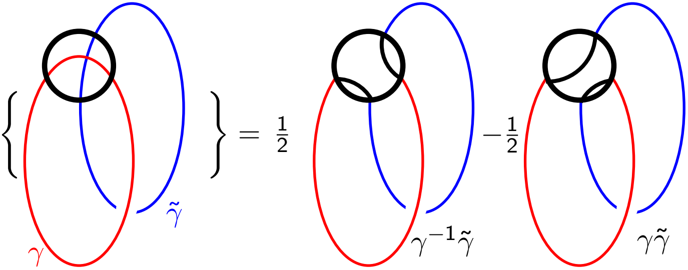
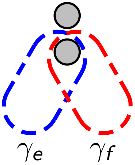


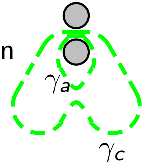
Poisson structure

$$\{G_\gamma, G_{\tilde\gamma}\} = \frac{1}{2} G_{\gamma\tilde\gamma} - \frac{1}{2} G_{\gamma\tilde\gamma^{-1}}.$$

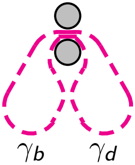




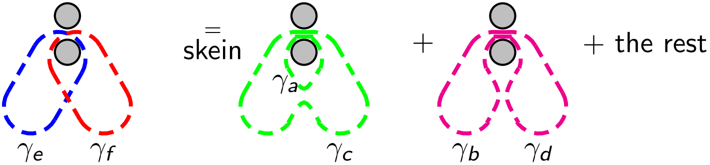
$\overset{=}{\text{skein}}$



$+$



$+$ the rest


γ_e
γ_f
$\stackrel{=}{\text{skein}}$
γ_a
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$+$
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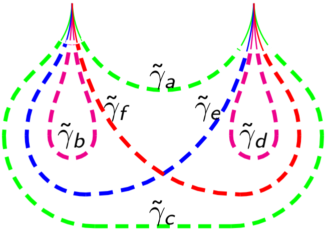
$\tilde{\gamma}_a$
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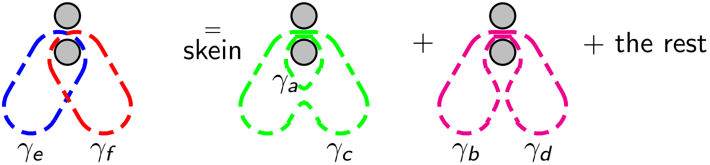
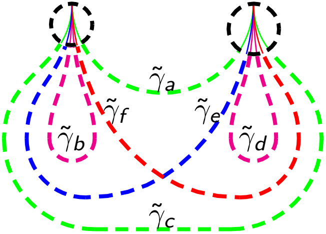
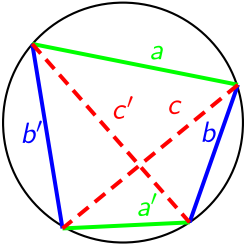




$$G_{\tilde\gamma_e} G_{\tilde\gamma_f} = G_{\tilde\gamma_a} G_{\tilde\gamma_c} + G_{\tilde\gamma_b} G_{\tilde\gamma_d}$$

Ptolemy Relation

$$aa' + bb' = cc'$$



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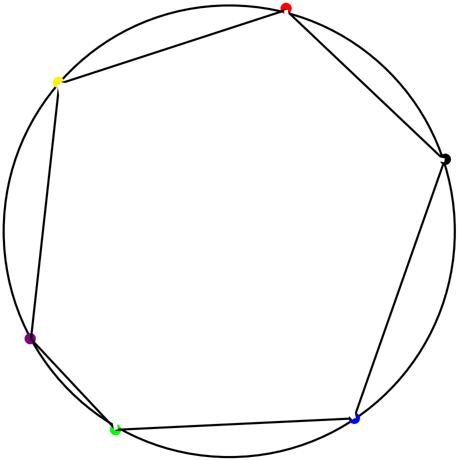

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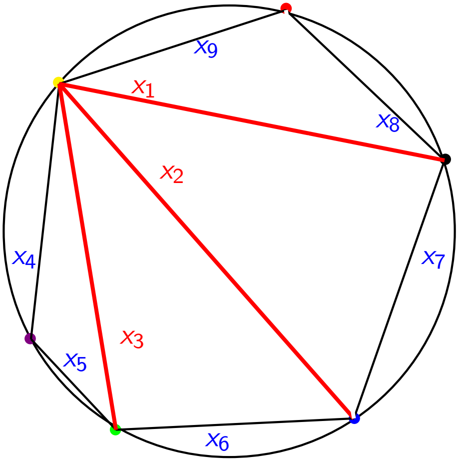

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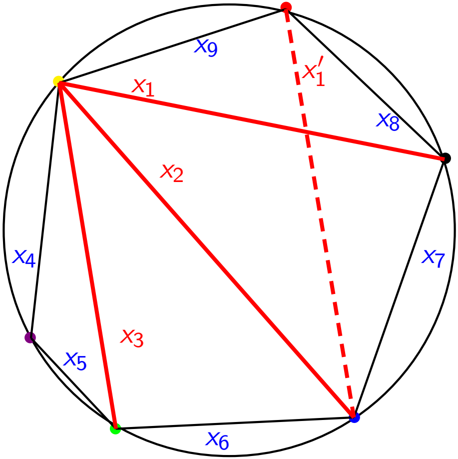

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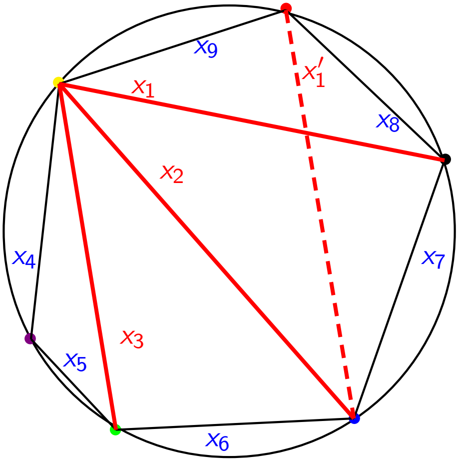
$$x_1 x_1' = x_9 x_7 + x_8 x_2$$



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$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) \to (x_1', x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9)$$

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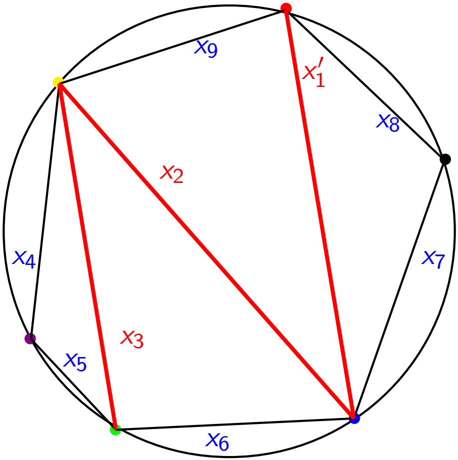

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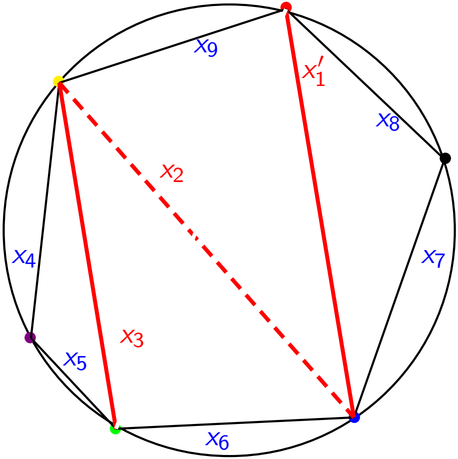

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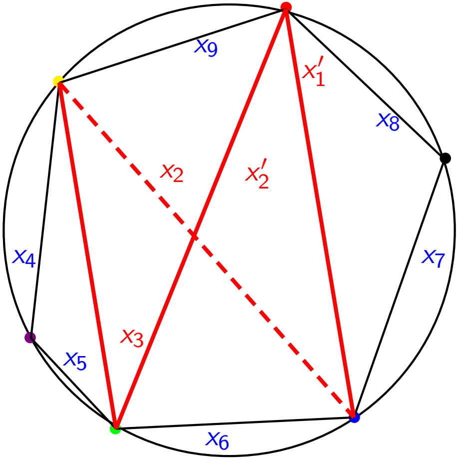

Cluster algebra

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$$x_i x_i' = \prod_{j:b_{ij}>0} x_j^{b_{ij}} + \prod_{j:b_{ij}<0} x_j^{-b_{ij}}, \quad x_j' = x_j \,\forall j\neq i.$$

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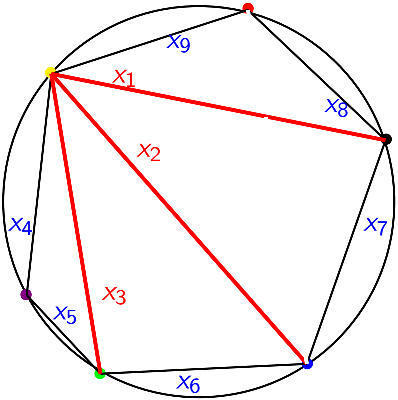
Definition

A cluster algebra of rank n is a set of all seeds $(x_1, \dots, x_n, B)$ related to one another by sequences of mutations $\mu_1, \dots, \mu_k$. The cluster variables $x_1, \dots, x_k$ are called exchangeable, while $x_{k+1}, \dots, x_n$ are called frozen. [Fomin-Zelevnsky 2002].

Example

Cluster algebra of rank 9 with 3 exchangeable variables x_1, x_2, x_3 and 6 frozen ones $x_4, \ldots, x_9$.



Outline

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All Berenstein-Zelevinsky cluster algebras are geometric

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- Introduce cusped laminations
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Poisson structure

Theorem

The Poisson algebra of the λ-lengths of a complete cusped lamination is a Poisson cluster algebra [Chekhov-Mazzocco. *ArXiv:1509.07044*].

$$\{g_{s_i,t_j}, g_{p_r,q_l}\} = g_{s_i,t_j} g_{p_r,q_l} \mathcal{I}_{s_i,t_j,p_r,q_l}$$

$$\mathcal{I}_{s_i,t_j,p_r,q_l} = \frac{\epsilon_{i-r}\delta_{s,p} + \epsilon_{j-r}\delta_{t,p} + \epsilon_{i-l}\delta_{s,q} + \epsilon_{j-l}\delta_{t,q}}{4}$$

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- Replace tr by two characters: tr and tr_K.

Cusped fat graphs

Cusped fat graph (a graph with the prescribed cyclic ordering of edges entering each vertex) $\mathcal{G}_{g,s,n}$ a *spine of the Riemann surface* $\Sigma_{g,s,n}$ with g handles, s holes and $n>0$ decorated bordered cusps if

(a) this graph can be embedded without self-intersections in $\Sigma_{g,s,n}$;

(b) all vertices of $\mathcal{G}_{g,s,n}$ are three-valent except exactly n one-valent vertices (endpoints of the open edges), which are placed at the corresponding bordered cusps;

(c) upon cutting along all *non-open* edges of $\mathcal{G}_{g,s,n}$ the Riemann surface $\Sigma_{g,s,n}$ splits into s polygons each containing exactly one hole and being simply connected upon contracting this hole.

Geometric laminations

We call geometric *cusped geodesic lamination* (CGL) on a bordered cusped Riemann surface a set of nondirected curves up to a homotopy equivalence such that

(a) these curves are either closed curves (γ) or *arcs* ($\mathfrak{a}$) that start and terminate at bordered cusps (which can be the same cusp);

(b) these curves have no (self)intersections inside the Riemann surface (but can be incident to the same bordered cusp);

(c) these curves are not empty loops or empty loops starting and terminating at the same cusp.

Decorated character variety-2

We introduce *the fundamental groupoid of arcs* $\mathcal{G}$ as the set of all directed paths $\gamma_{ij} : [0,1] \to \tilde{\Sigma}_{g,s,n}$ such that $\gamma_{ij}(0) = m_i$ and $\gamma_{ij}(1) = m_j$ modulo homotopy. The groupoid structure is dictated by the usual path–composition rules.
For each m_j, $j = 1, \ldots, n$, the isotopy group

$$\Pi_j = \{\gamma_{jj} | \gamma_{jj} : [0,1] \to \tilde{\Sigma}_{g,s,n},\ \gamma_{jj}(0) = m_j,\ \gamma_{jj}(1) = m_j\} / \{\mathrm{homotopy}\}$$

is isomorphic to the usual fundamental group and $\Pi_j = \gamma_{ij}^{-1} \Pi_i \gamma_{ij}$ for any arc $\gamma_{ij} \in \mathcal{G}$.
The decoration assigns to each arc γ_{ij} a matrix $M_{ij} \in SL_2(\mathbb{R})$, for example $M_{ij} = X(k_j) L X(z_n) R \cdots L X(z_1) R X(k_i)$.

Decorated character variety-3

To associate a matrix in $SL_2(\mathbb{C})$ - complexify the coordinates. The decorated character variety is:

$$\mathrm{Hom}\left(\mathcal{G}, SL_2(\mathbb{C})\right) /_{\prod_{j=1}^{n} B_j},$$

where B_j is the (unipotent) Borel subgroup in $SL_2(\mathbb{C})$ (one Borel subgroup for each cusp) with the characters:

$$\begin{aligned} \mathrm{Tr}_K: \quad & SL_2(\mathbb{C}) \to \mathbb{C} \\ & M \mapsto \mathrm{Tr}(MK), \qquad \text{where} \quad K = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}. \end{aligned}$$

PVI

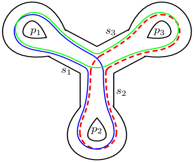
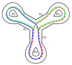


FIGURE 4. The fat graph of the 4 holed Riemann sphere. The dashed geodesic is x_1; the solid geodesics are x_2 and x_3.

Shear coordinates in the Teichmüller space

Fatgraph:



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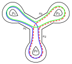
Decompose each hyperbolic element in Right, Left and Edge matrices Fock, Thurston

$$R := \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad L := \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix},$$

$$X_y := \begin{pmatrix} 0 & -\exp\left(\frac{y}{2}\right) \\ \exp\left(-\frac{y}{2}\right) & 0 \end{pmatrix}.$$



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The three geodesic lengths: $x_i = \mathrm{Tr}(\gamma_{jk})$

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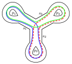

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$$\{x_1, x_2\} = x_1x_2 + 2x_3 + \omega_3, \quad \{x_2, x_3\} = x_2x_3 + 2x_1 + \omega_1, \quad \{x_3, x_1\} = x_3x_1 + 2x_2 + \omega_2.$$

The confluence from the cubic associated to PVI to the one associated to PV is realized by

$$p_3 \to p_3 - 2\log[\epsilon],$$

in the limit $\epsilon \to 0$. We obtain the following shear coordinate description for the PV cubic:

$$\begin{aligned}
x_1 &= -e^{s_2+s_3+\frac{p_2}{2}+\frac{p_3}{2}} - G_3 e^{s_2+\frac{p_2}{2}},\\
x_2 &= -e^{s_3+s_1+\frac{p_3}{2}+\frac{p_1}{2}} - e^{s_3-s_1+\frac{p_3}{2}-\frac{p_1}{2}} - G_3 e^{-s_1-\frac{p_1}{2}} - G_1 e^{s_3+\frac{p_3}{2}},\\
x_3 &= -e^{s_1+s_2+\frac{p_1}{2}+\frac{p_2}{2}} - e^{-s_1-s_2-\frac{p_1}{2}-\frac{p_2}{2}} - e^{s_1-s_2+\frac{p_1}{2}-\frac{p_2}{2}} - G_1 e^{-s_2-\frac{p_2}{2}} - G
\end{aligned}$$

where

$$G_i = e^{\frac{p_i}{2}}+e^{-\frac{p_i}{2}},\quad i=1,2,\quad G_3 = e^{\frac{p_3}{2}},\quad G_\infty = e^{s_1+s_2+s_3+\frac{p_1}{2}+\frac{p_2}{2}+\frac{p_3}{2}}.$$

These coordinates satisfy the following cubic relation:

$$x_1x_2x_3+x_1^2+x_2^2-(G_1G_\infty+G_2G_3)x_1-(G_2G_\infty+G_1G_3)x_2-\\ -G_3G_\infty x_3+G_\infty^2+G_3^2+G_1G_2G_3G_\infty=0. \qquad (12)$$

Note that the parameter p_3 is now redundant, we can eliminate it by rescaling. To obtain the correct PV- cubic, we need to pick $p_3=-p_1-p_2-2s_1-2s_2-2s_3$ so that $G_\infty=1$.

These coordinates satisfy the following cubic relation:

$$
\begin{aligned}
&x_1x_2x_3 + x_1^2 + x_2^2 - (G_1G_\infty + G_2G_3)x_1 - (G_2G_\infty + G_1G_3)x_2 - \\
&-G_3G_\infty x_3 + G_\infty^2 + G_3^2 + G_1G_2G_3G_\infty = 0.
\end{aligned} \tag{12}
$$

Note that the parameter p_3 is now redundant, we can eliminate it by rescaling. To obtain the correct PV- cubic, we need to pick $p_3 = -p_1 - p_2 - 2s_1 - 2s_2 - 2s_3$ so that $G_\infty = 1$.
$\{x_1, x_2\} = x_1x_2 - G_3G_\infty, \quad \{x_2, x_3\} = x_2x_3 + 2x_1 - (G_1G_\infty + G_2G_3), \quad \{x_3, x_1\} = x_3x_1 + 2x_2 - (G_2G_\infty + G_1G_3)$.

Geometrically speaking, sending the perimeter p_3 to infinity means that we are performing a chewing-gum move:
two holes, one of perimeter p_3 and the other of perimeter $s_1+s_2+s_3+\frac{p_1}{2}+\frac{p_2}{2}+\frac{p_3}{2}$, become infinite, but the area between them remains finite.
This is leading to a Riemann sphere with three holes and two cusps on one of them. In terms of the fat-graph, this is represented by Figure 2.
The geodesic x_3 corresponds to the closed loop obtained going around p_1 and p_2 (green and red loops), while x_1 and x_2 are "asymptotic geodesics" starting at one cusp, going around p_1 and p_2 respectively, and coming back to the other cusp.

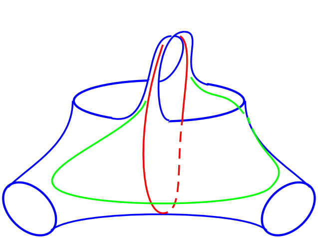

Figure: The process of confluence of two holes on the Riemann sphere with four holes. Chewing-gum move: hook two holes together and stretch to infinity by keeping the area between them finite (see Fig.). As a result we obtain a Riemann sphere with one less hole, but with two new cusps on the boundary of this hole. The red geodesic line which was initially closed becomes infinite, therefore two horocycles (the green dashed circles) must be introduced in order to measure its length.

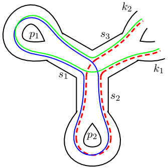

FIGURE 5. The fat graph corresponding to PV. The geodesic x_3 remains closed, while x_1 (the dashed line) and x_2 become arcs.

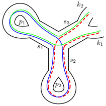

FIGURE 5. The fat graph corresponding to PV. The geodesic x_3 remains closed, while x_1 (the dashed line) and x_2 become arcs.

$\gamma_b =$
$X(k_1)RX(s_3)RX(s_2)RX(p_2)RX(s_2)LX(s_3)LX(k_1)$- BUT its length is $b = \mathrm{tr}_K(\gamma_b) = \mathrm{tr}(bK)$, $K = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$

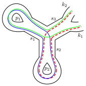

Figure 5. The fat graph corresponding to PV. The geodesic x_3 remains closed, while x_1 (the dashed line) and x_2 become arcs.

$$\{g_{s_i,t_j}, g_{p_r,q_l}\} = g_{s_i,t_j} g_{p_r,q_l} \frac{\epsilon_{i-r}\delta_{s,p}+\epsilon_{j-r}\delta_{t,p}+\epsilon_{i-l}\delta_{s,q}+\epsilon_{j-l}\delta_{t,q}}{4}$$

$$\begin{aligned} \{b, d\} &= \{g_{1_3,1_4}, g_{2_1,1_8}\} \\ &= g_{1_3,1_4} g_{2_1,1_8} \frac{\epsilon_{3-1}\delta_{1,2} + \epsilon_{4-1}\delta_{1,2} + \epsilon_{3-8}\delta_{1,1} + \epsilon_{4-8}\delta_{1,1}}{4} \\ &= -bd\frac{1}{2} \end{aligned}$$

The character variety of a Riemann sphere with three holes and two cusps on one boundary is 7-dimensional (rather than 2-dimensional like in PVI case). The fat-graph admits a complete cusped lamination as displayed in the figure below. A full set of coordinates on the character variety is given by the five elements in the lamination and the two parameters G_1 and G_2 which determine the perimeter of the two non-cusped holes.

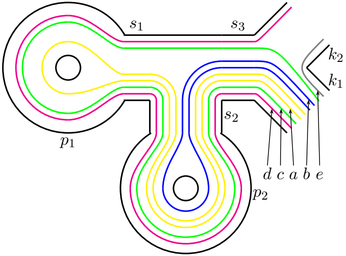

FIGURE 6. The system of arcs for PV.

Notice that there are two shear coordinates associated to the two cusps, they are denoted by k_1 and k_2, their sum corresponds to what we call p_3 above.
These shear coordinates do not commute with the other ones, they satisfy the following relations:

$$\{s_3, k_1\} = \{k_1, k_2\} = \{k_2, s_3\} = 1.$$

As a consequence in the character variety, the elements G_3 and G_∞ are not Casimirs.
In terms of shear coordinates, the elements in the lamination are expressed as follows:

$$\begin{aligned} a &= e^{k_1+s_1+2s_2+s_3+\frac{p_1}{2}+p_2}, \qquad b = e^{k_1+s_2+s_3+\frac{p_2}{2}}, \qquad e = e^{\frac{k_1}{2}+\frac{k_2}{2}}, \\ c &= e^{k_1+s_1+s_2+s_3+\frac{p_1}{2}+\frac{p_2}{2}}, \qquad d = e^{\frac{k_1}{2}+\frac{k_2}{2}+s_1+s_2+s_3+\frac{p_1}{2}+\frac{p_2}{2}}. \end{aligned} \tag{13}$$

They satisfy the following Poisson relations:

$$a \qquad \{a,b\} = ab, \quad \{a,c\} = 0, \quad \{a,d\} = -\frac{1}{2}ad, \quad \{a,e\} = \frac{1}{2}ae, \tag{14}$$

$$\{b,c\} = 0, \quad \{b,d\} = -\frac{1}{2}bd, \quad \{b,e\} = \frac{1}{2}be, \tag{15}$$

$$\{c,d\} = -\frac{1}{2}cd, \quad \{c,e\} = \frac{1}{2}ce, \quad \{d,e\} = 0, \tag{16}$$

so that the element $G_3 G_\infty = de$ is a Casimir.

The symplectic leaves are determined by the values of the three Casimirs G_1, G_2 and $G_3 G_\infty$.
On each symplectic leaf, the PV monodromy manifold (12) is the subspace defined by those functions of a, b, c (and of the Casimir values $G_1,\ G_2,\ G_3 G_\infty$) which commute with $G_3 = e$. To see this, we can use relations (13) to determine the exponentiated shear coordinates in terms of a, b, c, d, e and then deduce he expressions of x_1, x_2, x_3 in terms of the lamination. We obtain the following expressions:

$$x_1 = -e\frac{a}{c} - d\frac{b}{c}, \qquad x_2 = -e\frac{b}{c} - G_1 d\frac{b}{a} - d\frac{b^2}{ac} - d\frac{c}{a}, \tag{17}$$

$$x_3 = -G_2\frac{c}{b} - G_1\frac{c}{a} - \frac{b}{a} - \frac{c^2}{ab} - \frac{a}{b}, \tag{18}$$

which automatically satisfy (12).

Due to the Poisson relations (14) the functions that commute with e are exactly the functions of $\frac{a}{b}, \frac{b}{c}, \frac{c}{a}$. Such functions may depend on the Casimir values G_1, G_2 and $G_3 G_\infty$ and e itself, so that $d = G_\infty$ becomes a parameter now. With this in mind, it is easy to prove that x_1, x_2, x_3 are algebraically independent functions of $\frac{a}{b}, \frac{b}{c}, \frac{c}{a}$ so that x_1, x_2, x_3 form a basis in the space of functions which commute with e.

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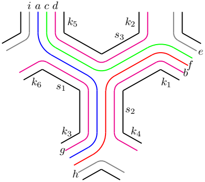
$$\pi : \mathrm{Spec}(\mathbb{C}[\mathrm{G}_1, \mathrm{G}_2, \mathrm{G}_3, \mathrm{G}_3^{-1}, \mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3]/\mathrm{x}_1\mathrm{x}_2\mathrm{x}_3 + \mathrm{x}_1^2 + \mathrm{x}_2^2 - \\ -(G_1 + G_2 G_3)x_1 - (G_2 + G_1 G_3)x_2 - G_3 x_3 + 1 + G_3^2 + G_1 G_2 G_3) - \\ \mapsto \mathrm{Spec}(\mathbb{C}[\mathrm{G}_1, \mathrm{G}_2, \mathrm{G}_3, \mathrm{G}_3^{-1}]). \tag{19}$$

The decorated character variety associated with PII^{JM} has 6 cusps on the boundary is 9-dimensional. The fat-graph admits a complete cusped lamination as displayed in the figure below.



"Confluented "Poisson algebras

$$
\begin{tikzcd}
 & & Poiss_{III}^{D_6} \arrow[dr, hook] & Poiss_{III}^{D_7} \arrow[l, hook'] \arrow[dr, hook] & Poiss_{III}^{D_8} \arrow[l, hook'] \\
Poiss_{VI} \arrow[r, hook] & Poiss_{V} \arrow[ur, hook] \arrow[dr, hook] & Poiss_{V}^{deg} \arrow[l, hook'] \arrow[ur, hook] \arrow[dr, hook] & Poiss_{II}^{JM} & Poiss_{I} \arrow[l, hook'] \\
 & & Poiss_{IV} \arrow[ur, hook] & Poiss_{II}^{FN} \arrow[l, hook'] \arrow[ur, hook] &
\end{tikzcd}
$$

Katz invariants and Stokes rays

Painlevé eqs	no. of cusps	Katz invariants	no. Stokes rays	pole–orders for φ
PVI	$(0,0,0,0)$	$(0,0,0,0)$	$(0,0,0,0)$	$(2,2,2,2)$
PV	$(0,0,2)$	$(0,0,1)$	$(0,0,2)$	$(2,2,4)$
PV_{deg}	$(0,0,1)$	$(0,0,1/2)$	$(0,0,1)$	$(2,2,3)$
PIV	$(0,4)$	$(0,2)$	$(0,4)$	$(2,6)$
$PIII^{D_6}$	$(0,2,2)$	$(0,1,1)$	$(0,2,2)$	$(2,4,4)$
$PIII^{D_7}$	$(0,1,2)$	$(0,1/2,1)$	$(0,1,2)$	$(2,3,4)$
$PIII^{D_8}$	$(0,1,1)$	$(0,1/2,1/2)$	$(0,1,1)$	$(2,3,3)$
PII^{FN}	$(0,3)$	$(0,3/2)$	$(0,3)$	$(2,5)$
PII^{MJ}	6	3	6	8
PI	5	$5/2$	5	7

Table: For each Painlevé isomonodromic problem, this table displays the number of cusps on each hole for the corresponding Riemann surface, the Katz invariants associated to the corresponding flat connection, the number of Stokes rays in the linear system defined by the flat connection and the number of poles of the quadratic differential φ defined by the linear system.

Notation: the fundamental matrix at an irregular singular point λ_k has the form

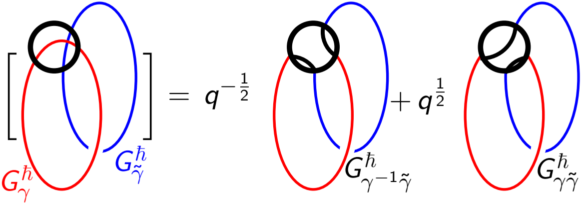
$$Y_k = G_k(\lambda)\lambda^{\Lambda_k}\begin{pmatrix} e^{Q_k(\lambda)} & 0 \\ 0 & e^{-Q_k(\lambda)} \end{pmatrix}$$

	$A(\lambda)$	Casimirs	extended exponents	$dim(\mathcal{C})$
PV	$\frac{A_0}{\lambda} + \frac{A_1}{\lambda-1} + A_\infty$	eigen(A_0),eigen(A_1),Λ_∞	$Q_\infty = \frac{t}{2}\lambda$	7
PIV	$\frac{A_0}{\lambda} + A_1 + A_\infty\lambda$	eigen(A_0),Λ_∞	$Q_\infty = \lambda^2 + \frac{t}{2}\lambda$	8
$PIII^{D_6}$	$\frac{A_0}{\lambda^2} + \frac{A_1}{\lambda} + A_\infty$	Λ_0, Λ_∞	$Q_\infty = \frac{t}{2}\lambda, Q_0 = \frac{t}{2}\frac{1}{\lambda}$	8
PII^{MJ}	$A_0 + A_1\lambda + A_\infty\lambda^2$	Λ_∞	$Q_\infty = \lambda^3 + \frac{t}{2}\lambda$	9

Table: Here $Q_k(\lambda)$ is polynomial in $(\lambda - \lambda_k)$ of order $n-1$ with n being the order of λ_k and Λ_k is the formal monodromy (diagonal). Expand $A(\lambda)$ near λ_k to calculate $Q_k(\lambda)$ and Λ_k, then diagonalize it using the gauge transformation $G_k(\lambda)$..

Quantisation

For standard geodesic lengths $G_\gamma \to G^{\hbar}_{\gamma}$ [Chekhov-Fock '99]:



$$[G^{\hbar}_{\gamma}, G^{\hbar}_{\tilde\gamma}] = q^{-\frac{1}{2}}\, G^{\hbar}_{\gamma^{-1}\tilde\gamma} + q^{\frac{1}{2}}\, G^{\hbar}_{\gamma\tilde\gamma}$$

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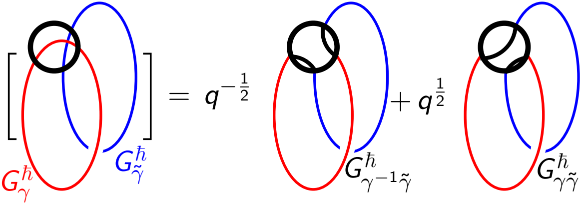

$$[G_\gamma^\hbar, G_{\tilde\gamma}^\hbar] = q^{-\frac{1}{2}} G_{\gamma^{-1}\tilde\gamma}^\hbar + q^{\frac{1}{2}} G_{\gamma\tilde\gamma}^\hbar$$

For arcs $g_{s_i,t_j} \to g_{s_i,t_j}^\hbar$:

$$q^{\mathcal{I}_{s_i,t_j,p_r,q_l}} g_{s_i,t_j}^\hbar g_{p_r,q_l}^\hbar = g_{p_r,q_l}^\hbar g_{s_i,t_j}^\hbar q^{\mathcal{I}_{p_r,q_l,s_i,t_j}}$$

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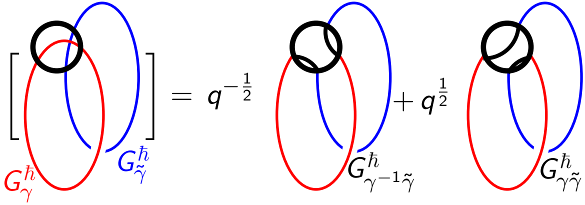

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This identifies the geometric basis of the quantum cluster algebras introduced by Berenstein - Zelevinsky.

Quantization-2

To produce the quantum Painlevé cubics, we introduce the Hermitian operators S_1, S_2, S_3 subject to the commutation inherited from the Poisson bracket of $\tilde{s}_i$:

$$[S_i, S_{i+1}] = i\pi\hbar\{\tilde{s}_i, \tilde{s}_{i+1}\} = i\pi\hbar, \quad i = 1, 2, 3,\ i+3 \equiv i.$$

Observe that thanks to this fact, the commutators $[S_i, S_j]$ are always numbers and therefore we have

$$\exp(aS_i)\exp(bS_j) = \exp\left(aS_i + bS_i + \frac{ab}{2}[S_i, S_j]\right),$$

for any two constants a, b. Therefore we have the Weyl ordering:

$$e^{S_1+S_2} = q^{\frac{1}{2}} e^{S_1} e^{S_2} = q^{-\frac{1}{2}} e^{S_2} e^{S_1}, \quad q \equiv e^{-i\pi\hbar}.$$

Quantization-2

Theorem

*(*L. Chekhov-M. Mazzocco-V.R*)*
Denote by X_1, X_2, X_3 the quantum Hermitian operators corresponding to x_1, x_2, x_3 as above. The quantum commutation relations are:

$$q^{-\frac{1}{2}} X_i X_{i+1} - q^{\frac{1}{2}} X_{i+1} X_i = \left(\frac{1}{q} - q\right) \epsilon_k^{(d)} X_k - (q^{-\frac{1}{2}} - q^{\frac{1}{2}}) \omega_k^{(d)} \quad (20)$$

where $\epsilon_i^{(d)}$ and $\omega_i^{(d)}$ are the same as in the classical case. The quantum operators satisfy the following quantum cubic relations:

$$q^{\frac{1}{2}} X_3 X_1 X_2 - q\epsilon_3^{(d)} X_3^2 - q^{-1}\epsilon_1^{(d)} X_1^2 - q\epsilon_2^{(d)} X_2^2 +$$

$$q^{\frac{1}{2}} \epsilon_3^{(d)} \omega_3 X_3 + q^{-\frac{1}{2}} \omega_1^{(d)} X_1 + q^{\frac{1}{2}} \omega_2^{(d)} X_2 - \omega_4^{(d)} = 0.$$

Quantization-2

The Hermitian operators X_1, X_2, X_3 corresponding to x_1, x_2, x_3 are introduced as follows: consider the classical expressions for x_1, x_2, x_3 in terms of s_1, s_2, s_3 and p_1, p_2, p_3. Write each product of exponential terms as the exponential of the sum of the exponents and replace those exponents by their quantum version. For example (the case $\tilde{D}_5$): the classical x_1 is

$$x_1 = -e^{s_2+s_3} - e^{-(\tilde{s}_2+\tilde{s}_3)} - G_2 e^{\tilde{s}_3} - G_3 e^{-\tilde{s}_2},$$

and its quantum version is defined as

$$X_1 = -e^{S_2} - (e^{p_2/2} + e^{-p_2/2})e^{S_3} - e^{S_3-S_2} - e^{S_3+S_2} =$$

$$-e^{S_2} - (e^{p_2/2} + e^{-p_2/2})e^{S_3} - q^{-1/2}e^{-S_2}e^{S_3} - q^{1/2}e^{S_2}e^{S_3}.$$

Quantization-2

- Our theorem and close results of Marta Mazzocco show that we can interpret the Cherednik algebra and their close "relatives" as a quantisation of the (group algebra of the) monodromy group of the Painlevé equations. Here $q := e^{-i\pi\hbar}$ and $q^n \neq 1$.

Quantization-2

- Our theorem and close results of Marta Mazzocco show that we can interpret the Cherednik algebra and their close "relatives" as a quantisation of the (group algebra of the) monodromy group of the Painlevé equations. Here $q := e^{-i\pi\hbar}$ and $q^n \neq 1$.
- The Askey-Wilson $AW(3)$ (or Zhedanov algebra) can be obtained from (20) for a special constant choice after a proper "rescaling".

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- $D-$brane world: live on $D3-$brane $\perp$ $6D-$affine variety $\mathcal{M}$. $1 + 3D-$world-volume with SUSY YM and product gauge group.

$D-$brane algebras and superpotentials. Basic principles:

- One can associate an algebra to the category of $D-$branes at a singular point p. In every known example, the collection of possible $D-$branes at p can be described as a collection of QFT with the same Lagrangian for each of the theories.

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- More precisely, one does specify the "matter representation" (as a collection of adjoint and bifundamental fields for the gauge groups G_i) and one specifies a superpotential $W-$ the trace of a polynomial in the matter fields.
- To such data one can assign a quiver whose vertices label the groups G_i and whose directed edges specify the bifundamental and adjoint fields in the matter representation.

Quiver Theory

- Action

$$\int d^4x[\int d^4\theta \Psi_i^\dagger e^V \Psi_i + (\frac{1}{4g^2}\int d^2\theta \mathrm{Tr} \mathcal{W}_\alpha \mathcal{W}^\alpha + \int \mathrm{d}^2\theta \mathrm{W}(\bar{\psi}) + \mathrm{h.c.})]$$

W = superpotential;
$V(\varphi_i;\bar{\varphi}_i) = \sum_i \mid \frac{\partial W}{\partial \varphi_i} \mid^2 + \frac{g^2}{4}(\sum_i q_i \mid \varphi_i \mid^2)^2$

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- Encode in a Quiver:
k nodes $\iff \mathcal{V}^{n_1},\dots \mathcal{V}^{n_k} \iff \prod_{j=1}^k U(n_j)$ gauge group;
Each arrow $i \to j \iff$ bifundamental fields X_{ij} of $U(n_i)\times U(n_j)$;
Each loop $i \to i \iff$ adjoint fields φ_i of $U(n_i)$;
Superpotential $W \iff$ linear combination of cycles: $\sum_i c_i$ gauge invariant operators;
Relations $\iff$ jacobian of $W(\varphi_i, X_{ij})$.
Vacuum:$\rightsquigarrow V(\varphi_i;\bar{\varphi}_i) = 0 \Rightarrow \frac{\partial W}{\partial \varphi_i} = 0$; $\sum_i q_i \mid \varphi_i \mid^2 = 0$.

Superpotential algebra

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- A universal feature of this family of theories is the relations in the path algebra determined by what are called "$F-$term constraints" in physics: $\frac{\partial W}{\partial \varphi_i} = 0$
- These are the algebra relations dictated by $\frac{\partial W}{\partial X_j}$. So, given a field theory description of the family of D-branes in the form above, the D-brane algebra is

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- This is called a superpotential algebra, which is a Calabi - Yau algebra.

Elementary example

- First example, we consider the case in which P is a smooth point. In physics language, the conformal fields theory is the $N = 4$ SUSY Yang-Mills theory, written in $N = 1$ language. The $N = 4$ gauge multiplet decomposes as an $N = 1$ gauge multiplet plus three complex scalar fields X, Y, Z each transforming in the adjoint representation of the group.

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- Thus, we find

$$\mathcal{A} = \mathbb{C}[X, Y, Z],$$

the (commutative) polynomial algebra in three variables.

Example 2. Sklyanin algebra-1

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- This algebra denotes by $Q_3(\mathcal{E}, a, b, c)$ where $(a, b, c) \in \mathbb{C}^3$ such that $Q_3(\mathcal{E}, a, b, c) = \mathbb{C} < X, Y, Z > / J_W$ with

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- The ideal J_W can be written as a non-commutative jacobian ideal $J_W =< \partial_X, \partial_Y, \partial_Z >\in \mathbb{C} < X, Y, Z >$ for superpotential

$$W = aXYZ + bYXZ + c(X^3 + Y^3 + Z^3)$$

Example 2. Sklyanin algebra-2

- Here we consider W as a cyclic word of three variables X, Y, Z, i.e. like an element of the quotient $A_\natural := \mathbb{C} < X, Y, Z > /[\mathbb{C} < X, Y, Z >, \mathbb{C} < X, Y, Z >]$ with

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- cyclic derivatives $\partial_X, \quad \partial_Y, \quad \partial_Z$ where

$$\partial_j : A_\natural \to \mathbb{C} < X, Y, Z >, j = X, Y, Z$$

defines for any cyclic word $\varphi \in A_\natural$ by

$$\partial_j \varphi := \sum_{k | i_k = j} X_{i_k+1} X_{i_k+2} \ldots X_{i_N} \ldots X_{i_1} X_{i_2} .. X_{i_k-1} \in \mathbb{C} < X, Y, Z >$$

Example 2. Sklyanin algebra-3

Etingof-Ginzburg:

- One can identify the Sklyanin algebra $Q_3(\mathcal{E}, 1, -q, \frac{c}{3})$ with the flat deformation of the Poisson algebra $(\mathbb{C}[x, y, z], \{-,-\}_\varphi)$ as above with $\varphi = \frac{1}{3}(x^3 + y^3 + z^3) + \tau xyz$ and $W = XYZ - qYXZ + \frac{c}{3}(X^3 + Y^3 + Z^3)$.

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- The coordinate ring $B_\varphi = \mathbb{C}[x, y, z]/\varphi\mathbb{C}[x, y, z]$ of the affine surface $\varphi = 0$ inherits a Poisson algebra structure.
- There is a degree 3 central element $\Phi \in Z(Q_3(\mathcal{E}, 1, -q, \frac{c}{3}))$ and the quotient of the Sklyanin 3-Calabi-Yau algebra by two-sided ideal $< \Phi >$ is a flat deformation of the Poisson algebra B_φ.

Superpotentials of marginal and relevant deformations-1

- There is a "physical interpretation" of the Sklyanin superpotential (Berenstein-Leigh) as a marginal deformation of the superpotential from the Example 1:

$$W + W_{marg} =$$

$$= g\,\mathrm{tr}(\mathrm{X}[\mathrm{Y},\mathrm{Z}]) + \mathrm{tr}(\mathrm{aXYZ} + \mathrm{bYXZ} + \frac{\mathrm{c}}{3}(\mathrm{X}^3 + \mathrm{Y}^3 + \mathrm{Z}^3)) \in \mathrm{A}_\natural.$$

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- The structure of the vacua of D-brane gauge theories relates to the Non-Commutative Geometry also via another superpotentials (relevant deformations) having the form

$$W_{rel} = \mathrm{tr}(\frac{m_1}{2}X^2 + \frac{m_2}{2}(Y^2 + Z^2) + e_1 X + e_2 Y + e_3 Z)$$

Superpotentials of marginal and relevant deformations-2

- The "vacua" of the theory with $W_{tot} = W + W_{marg} + W_{tel}$ superpotential corresponds to solutions of

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Superpotentials of marginal and relevant deformations-2

- The "vacua" of the theory with $W_{tot} = W + W_{marg} + W_{tel}$ superpotential corresponds to solutions of

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- The defining equations (for $a = 1, b = -q$):

$$\left\{ \begin{array}{rl} X_1X_2 - qX_2X_1 = & -cX_3^2 - m_2X_3 - e_3 \\ X_2X_3 - qX_3X_2 = & -cX_1^2 - m_1X_1 - e_1 \\ X_3X_1 - qX_1X_3 = & -cX_2^2 - m_2X_2 - e_2 \end{array} \right. \tag{21}$$

This relations contain our (20) (again, after a special constant choice and a "rescaling").

Etingof-Ginzburg ideology-1:

- Let $M = \mathbb{C}^3$ considering as the simplest Calabi-Yau manifold and $\varphi \in \mathcal{A} = \mathbb{C}[x_1, x_2, x_3]$ defines the Poisson bracket of jacobian type as above.

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- $M_\varphi : \varphi(x_1, x_2, x_3) = 0$ is an affine surface in M and the coordinate ring $\mathcal{B}_\varphi := \mathbb{C}[M_\varphi] = \mathcal{A}/(\varphi)$ is a commutative Poisson algebra with the structure induced by φ

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- Let
$\varphi^{\tau,\nu} = \tau x_1 x_2 x_3 + \frac{\nu}{3}(x_1^3 + x_2^3 + x_3^3) + P(x_1) + Q(x_2) + R(x_3) = 0$
be the family of affine surfaces containing the E_6 del Pezzo. Here $\mathrm{degP}, \mathrm{degQ}$ and $\mathrm{degQ} < 3$.

Etingof-Ginzburg ideology-2:

- Let $A = \mathbb{C} < X_1, X_2, X_3 >$ and $A_\natural$ be defined as above and $\Phi^{q,\nu}_{P,Q,R} = X_1X_2X_3 - qX_2X_1X_3 + \nu(X_1^3 + X_2^3 + X_3^3) + P(X_1) + Q(X_2) + R(X_3) \in A_\natural$

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- $\mathfrak{U}(\Phi^{q,\nu}_{P,Q,R})$ is a filtered algebra defined by three inhomogeneous "jacobian" relations:

$$X_iX_j - qX_jX_i = \nu X_k^2 + \frac{\mathrm{d}P(Q,R)}{\mathrm{d}X_k}, (i,j,k) = (1,2,3) \quad (22)$$

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- The superpotential $\Phi^{q,\nu}_{P,Q,R} = \Phi^{q,\nu} + \Phi_{P,Q,R}$ where $\Phi^{q,\nu} = X_1X_2X_3 - qX_2X_1X_3 + \nu(X_1^3 + X_2^3 + X_3^3) \in A_\natural^{(3)}$ and $\Phi_{P,Q,R} \in A_\natural^{(\leq 2)}$ is a 3-CY-superpotential (for generic parameters)

Etingof-Ginzburg ideology-3:

$$\begin{tikzcd} \mathcal{A}_\varphi \arrow[r, rightsquigarrow, "\text{fl. def.}"] \arrow[d] & \mathfrak{U}(\Phi^{q,\nu}_{P,Q,R}) \arrow[d] \\ \mathcal{B}_\varphi \arrow[r, rightsquigarrow] & B(\Phi^{q,\nu}_{P,Q,R}, \Psi) = \mathfrak{U}(\Phi^{q,\nu}_{P,Q,R})/(\Psi). \end{tikzcd}$$

In our case $\Phi^{q,0}_{P,Q,R} := X_1X_2X_3 - qX_2X_1X_3$

$$\Psi^{q,\epsilon,\omega} = X_1X_2X_3 - q^2X_2X_1X_3 + \epsilon_1^{(d)}\frac{q-1}{\sqrt{q}}X_1^2 + \epsilon_2^{(d)}q^{3/2}(q-1)X_2^2 + \epsilon_3^{(d)}\tfrac{q^3-1}{\sqrt{q}}X_3^2 - -\omega_1^{(d)}(q-1)X_1 - \omega_2^{(d)}q(q-1)X_2 - \omega_3^{(d)}(q^2-1)X_3 \tag{23}$$

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Many thanks for your attention!!!