# Brownian Manifolds, Negative Type and Geotemporal Covariances

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# Overview

# 1. Gaussian Processes

A Gaussian process $X$ *on (parametrised by)* $I$: $X = \{X_t, t \in I\}$, on $(\Omega, \mathcal{F}, P)$, say, taking values $X_t \in \mathbb{R}$ (or a metric space $M$). *No structure* is needed on the index set $I$. All that is needed is a mean function $\mu$ on $I$ ($\mu = 0$ below unless otherwise stated), and a *covariance function* $c$ on $I \times I$,

$$c(s,t) := \mathrm{cov}(X_s, X_t),$$

which is *positive definite (pd)* ('non-negative definite'):

$$\sum_{i,j=1}^{n} c(t_i, t_j) u_i u_j \geq 0$$

$$(n \in \mathbb{N}, t_i \in I, u_i \in \mathbb{R}). \qquad (PD)$$

NB. We distinguish this from a Gaussian process *in (taking values in)* a space $M$ ('m for metric', or 'm for manifold'). Here we need structure on $M$ in order to define the covariance operator.

# 2. Brownian Motion

$BM(\mathbb{R})$: here $I = \mathbb{R}_+$,

$$c(s,t) = \frac{1}{2}(|s| + |t| - |s - t|) = \min(s,t).$$

$BM(\mathbb{R}^2)$: planar BM: $I = \mathbb{R}_+$, $M = \mathbb{R}^2$: *time* non-negative, *values* in the plane. Similarly for $BM(\mathbb{R}^d)$: BM taking values in $d$-space.

*Spatial processes*
With $I = \mathbb{R}_+$, or $\mathbb{R}$, we have a random (stochastic) process *unfolding with time (totally ordered)*.
From Einstein and Relativity, we know that we may/should think of *time and space together*.
With $I$ the plane $\mathbb{R}^2$, space $\mathbb{R}^3$ or $d$-space $\mathbb{R}^d$, we have a *spatial process*.
With $I = \mathbb{R}^d \times \mathbb{R}_+$, we have a *spatio-temporal* process.

# 3. Lévy's BM

In his book *Processus stochastiques et mouvement brownien (PSMB)* (1948/1965), Lévy introduced such *spatial BM* (or BM with *multidimensional time*) with $I = \mathbb{R}^d$ as the real-valued centred Gaussian process $B = (B_t : t \in \mathbb{R}^d)$ with $B_0 = 0$ and *incremental variance* (also called the *variogram*)

$$i(s,t) := E[(B_t - B_s)^2] = \|B_t - B_s\|^2 = |t - s|,$$

regarding $B : t \mapsto B_t$ as a map from $\Omega$ to the Hilbert space $H := L^2(\Omega, \mathcal{F}, P)$. Then

$$i(s,t) = c(s,s) + c(t,t) - 2c(s,t), \qquad (i \mapsto c)$$

$$c(s,t) = \frac{1}{2}(i(s,0) + i(t,0) - i(s,t)). \qquad (c \mapsto i)$$

So $c$, $i$ are equivalent; $c$ is more common; $i$ is more convenient here.

# 4. Negative Type

A kernel $k : M \times M \to \mathbb{R}_+$ is of *negative type (nt)* if

$$\sum_{i,j=1}^{n} k(x_i, x_j) u_i u_j \le 0$$

for all $n = 2, 3, \cdots$, all points $x_i \in M$ and all real $u_i$ with $\sum u_i = 0$, and of *positive type* (or positive definite, pd) if

$$\sum_{i,j=1}^{n} k(x_i, x_j) u_i u_j \ge 0$$

for all $n = 2, 3, \cdots$ and all points $x_i \in M$; similarly for *strictly positive type*.

Covariances $c$ are of positive type. So, incremental variances $i$ are of negative type: the first two terms on the right of $(i \mapsto c)$ contribute 0 to the relevant summation, as $\sum u_i = 0$, so the sum is $\le 0$ as $c$ is of positive type.

For negative type on locally compact groups, see e.g. the book by Berg and Forst (1975).

# 5. Lévy's BM on the Sphere

Lévy also showed that one can take $I = \mathbb{S}^d$ (the $d$-sphere: a $d$-dimensional manifold embedded in $\mathbb{R}^{d+1}$), with the North Pole $O$ playing the role of the origin above and *geodesic distance* $d$ on the sphere in place of Euclidean distance:

$$i(s,t) = \|B_t - B_s\|^2 = d(s,t):$$

$$\sqrt{d}(s,t) = \|B_s - B_t\|. \qquad (*)$$

So the map $B$ (BM) gives $\sqrt{d}$ as a *Hilbert distance*. The question of *existence* of BM on $I$ is thus a *geometric* one, involving *embedding in Hilbert space*.

*White noise*

White-noise integrals for Lévy's BM were used by Chentsov (1957, TPA 2), and Lévy himself (Neyman Festschrift, 1966). McKean (1963, TPA 8) treated BM with $I = \mathbb{S}^d$ and $\mathbb{R}^{d+1}$ together in this way.

# 6. Spaces of Constant Curvature

Riemannian manifolds of constant curvature $\kappa$ come (of course!) in three kinds:
(i) $\kappa > 0$: *Spheres* $\mathbb{S}^d$ (we take the radius as 1);
(ii) $\kappa = 0$: *Euclidean space* $\mathbb{R}^d$;
(iii) $\kappa < 0$: *hyperbolic space* $\mathbb{H}^d$. See e.g.
[Wol1] Joseph A. Wolf, *Spaces of constant curvature*, AMS, 1967 (6th ed. 2011).
One can extend Lévy's results on BM on $\mathbb{R}^d$ and $\mathbb{S}^d$ to $\mathbb{H}^d$: J. Faraut and K. Harzallah, 1974, AIF – or, by white noise, Takenaka, Kubo and Urakawa, 1981, Nagoya Math. J.
We summarise this by saying that $\mathbb{S}^d, \mathbb{R}^d, \mathbb{H}^d$ are *Brownian manifolds*: they can be index spaces for Brownian motion.
These three families are the main examples of Riemannian *symmetric spaces* (below) of *rank one*. By contrast, the other examples are not Brownian; see below.

# 7. Symmetric spaces

A *symmetric space* (Helgason [Hel1,2,3,4], Wolf [Wol1,2]) is a Riemannian manifold $M$ whose curvature tensor is invariant under parallel translation. These are the spaces where at each point $x$ the *geodesic symmetry* exists: this fixes $x$ and reverses the (direction of) geodesics through $x$ and $O$, an involutive automorphism [Wol2, Ch. 11]. Then $M$ is a Riemannian homogeneous space $M = G/K$, with $G$ a closed subgroup of the isometry group of $M$ containing the transvections, and $K$ the isotropy subgroup of $G$ fixing the base-point $O$; $(G, K)$ is called a Riemannian symmetric pair. The Banach algebra $L_1(K\backslash G/K)$ of (Haar) integrable functions on $G$ bi-invariant under $K$ is *commutative*. Such pairs are called *Gelfand pairs*, and such Banach algebras *commutative spaces* [Wol2]. We restrict attention to the *isotropic* case, of pd functions of $x := \cos d(\mathbf{x}, \mathbf{y}) \in [-1, 1]$.

# 8. Spherical functions

For harmonic analysis here, one needs (cf. the Fourier transform in Euclidean space and the Gelfand transform for Banach algebras) spherical measures, spherical functions and the spherical transform [Wol2, Ch. 8, 9]. For $(G, K)$ a Gelfand pair, a spherical measure $m$ is a $K$-bi-invariant multiplicative linear functional on $C_c(K\backslash G/K)$; a *spherical function* is a continuous function $\omega : G \to \mathbb{C}$ with the measure $m_\omega(f) := \int_G f(x)\omega(x^{-1})d\mu_G(x)$ spherical. The *spherical transform* for $(G, K)$ is the map

$$f \mapsto \hat{f}(\omega) := m_\omega(f) = \int_G f(x)\omega(x^{-1})d\mu_G(x).$$

The *positive definite* spherical functions $\phi$ on $(G, K)$ are in bijection with the irreducible unitary representations $\pi$ of $G$ with a $K$-fixed unit vector $u$ via

$$\phi(g) = \langle u, \pi(g)u\rangle \qquad (GNS)$$

(the *Gelfand-Naimark-Segal* construction). These form the *spherical dual*, $\Lambda$.

# 9. Compact Lie groups

With $G$ compact, all spherical functions are pd (from ($GNS$)); this simplifies dealing with the spherical dual $\Lambda$.

*Weights*

When $G$ is compact, the $\pi$ here are in bijection with the *dominant weights*, in the sense of the Cartan-Weyl theory of weights; see e.g. Applebaum [App1, Ch. 2], or Wolf [Wol2, 6.3]. In the rank-one case, the dominant weights are a subset $\Lambda \subset \mathbb{R}$; here $\Lambda$ is specified by the *Cartan-Helgason theorem* ([Wol2, 11.4B], [Hel3, V.1.1, 534-538, 550]).

*Spheres* $\mathbb{S}^d$

Here we need the *Gegenbauer* (or *ultraspherical*) *polynomials*, of index $\lambda$,

$$P_n(x), \quad x = \cos d(\mathbf{x}, \mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in \mathbb{S}^d, \quad \lambda = \frac{1}{2}(d-1).$$

For the 2-sphere in 3-space (i.e. the Earth), these reduce ($\lambda = \frac{1}{2}$) to the familiar *Legendre polynomials*, $P_n(x)$, and for the circle (1-sphere) in the plane to the *Tchebycheff polynomials*.

# 10. The Kazhdan property

The geometrical property of being Brownian has an algebraic interpretation in the case $M = G/K$ of a symmetric space. Kazhdan (1967) defined a locally compact group to have Property $(T)$, now called the *Kazhdan property*, if the unit representation is isolated in the space of unitary representations. Such *Kazhdan groups* have been much studied; see the book by Bekka, de la Harpe and Valette (2008). In the rank-one case, the spherical dual can be identified with a set $\Lambda \subset \mathbb{R}$, where if $M$ is compact $\Lambda$ is a discrete set tending to infinity, while if $M$ is Euclidean, or is real or complex hyperbolic space, $\Lambda = [0, \infty)$, so $M$ is Brownian but not Kazhdan (0 corresponding to the unit representation). But if $M$ is quaternionic hyperbolic space, or the octonion (Cayley) projective plane, $\Lambda = \{0\} \cup [\lambda_0, \infty)$, where $\lambda_0 > 0$ (Kostant, BAMS 1969; Faraut & Harzallah 1974). So here $M$ is Kazhdan but not Brownian. So we have examples of Brownian and of non-Brownian manifolds.

# 11. The Bochner-Godement theorem

In its modern formulation, this very useful result is as follows:

**Bochner-Godement theorem**.
The general isotropic positive definite function $\psi$ on a symmetric space is given (to within scale) by a mixture of positive definite spherical functions $\phi_\lambda$ over the spherical dual $\Lambda$ by a probability measure $\mu$:

$$\psi(x) = c \int_\Lambda \phi_\lambda(x)\mu(d\lambda). \qquad (BG)$$

We give five special cases below: two classical, one from my early work, two recent. For background and details, see e.g. [Wol1, Th. 9.3.4], Faraut [Far2, Th. 1.2]; cf. Faraut and Harzallah [FarH, Th. 3.1], Askey and Bingham [AskB].

**Bochner's theorem (1933)**. The general positive definite function $\psi : \mathbb{R} \to \mathbb{R}$ is a multiple of a characteristic function (Fourier-Stieltjes transform of a probability measure $F$):

$$\psi(t) = c \int_{\mathbb{R}} e^{itx} dF(x).$$

This is the Euclidean case $G = \mathbb{R}, K = \{e\}$; similarly for locally compact abelian groups (Weil).

**Bochner-Schoenberg theorem (1940-42)**. For $\mathbb{S}^d$, the general isotropic pd function is (to within scale) a *mixture* of Gegenbauer polynomials $P_n^\lambda(x)$: for $\mathbf{x}, \mathbf{y} \in \mathbb{S}^d$, $x = \cos d(\mathbf{x}, \mathbf{y}) \in [-1, 1]$,

$$c \sum_{n=0}^{\infty} a_n P_n^\lambda(x), \;\; a_n \geq 0, \;\; \sum_{n=0}^{\infty} a_n = 1. \qquad (BS)$$

This is the Bochner-Godement theorem for spheres; cf. Bingham (1973), Faraut (1973).

Similarly for compact symmetric spaces of rank one *(Askey-Bingham theorem*, ZfW 1976):

$$c \sum_{n=0}^{\infty} a_n \phi_n(x), \; a_n \geq 0, \; \sum_{n=0}^{\infty} a_n = 1, \qquad (AB)$$

with $\phi_n$ the spherical functions (countably many, all pd).

**Guella-Menegatto-Peron theorem (2016)**. The general isotropic pd function on $\mathbb{S}^{d_1} \times \mathbb{S}^{d_2}$ is

$$c \sum_{m,n=0}^{\infty} a_{mn} P_m^{\lambda_1}(x_1) P_n^{\lambda_2}(x_2), \; a_{mn} \geq 0, \; \sum a_{mn} = 1,$$

$$x_i = \cos d(\mathbf{x}_i, \mathbf{y}_i), \quad \mathbf{x}_i, \mathbf{y}_i \in \mathbb{S}^{d_i}, \quad \lambda_i = \frac{1}{2}(d_i - 1).$$

This is immediate from the Bochner-Godement theorem, as the direct product of symmetric spaces is a symmetric space, with spherical dual the direct product of the spherical duals.

The case of 'sphere cross line', $M = \mathbb{S}^d \times \mathbb{R}$, gives us the 'geotemporal' of our title (taking Planet Earth to have unit radius!). Write $\Lambda, \Lambda_1, \Lambda_2$ for the spherical duals. The next result answers a question raised in Bingham, Mijatović and Symons [BinMS] in 2016:

**Berg-Porcu Theorem (2017)**. The class of isotropic stationary sphere-cross-line covariances coincides with the class of mixtures of products of Gegengauer polynomials $P_n^\lambda(x)$ and characteristic functions $\phi_n(t)$ on the line:

$$c \sum_{n=0}^{\infty} a_n P_n^\lambda(x) \phi_n(t), \ a_n \geq 0, \ \sum a_n = 1. \qquad (BP)$$

# Proof

Here $\Lambda_1$ is the Gegenbauer polynomials $P_n$, $\Lambda_2$ is the set of characters, which can also be identified with the line:

$$t \leftrightarrow e^{it\cdot} = (x \mapsto e^{itx}).$$

Use disintegration in $(BG)$ (Fubini's theorem extended beyond product measures: see e.g. Kallenberg [Kal, Th. 6.4]), integrating $\mu$ on $\Lambda$ first over the $x$-variable above for fixed $n$. This gives a probability measure, $\mu_n$ say; integrating the character $e^{itx}$ over $\mu_n(dx)$ gives its CF $\phi_n(t)$, the second factor in $(BP)$; the remaining integration, a summation over $n$, gives the first factor $P_n^\lambda$ in $(BP)$. Equivalently, take $\lambda = (\lambda_1, \lambda_2)$ in $(BG)$ as random with law $\mu$, condition on its second coordinate, and use the Conditional Mean Formula (tower property). $\square$

## 12. Remarks.

This result, very recent, resolves a long-standing question in the geostatistical community. Here one needs to work on the sphere (Planet Earth), and take time into account (e.g. for Numerical Weather Prediction, NWP). A range of parametric models were known, but these did not suffice for practical purposes. Researchers were reduced to using 'covariances' which they knew were not positive definite! Thanks to taking on a research student (Symons) under the Mathematics of Planet Earth CDT, I became interested in this area, for the first time since my thesis work (1966-69), papers in the early 70s, and Askey-Bingham (1976). Askey-Bingham makes explicit use of the Bochner-Godement theorem, but I still did not make the link, despite [BinMS] being motivated by questions of this sort (but then, nor did anyone else). The moral (apart from human error, failings of memory, and communication failings between subject areas) is how precious theory in general, and the Bochner-Godement theorem in particular, is.

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