Iterated integrals and the large noise limit of SDEs

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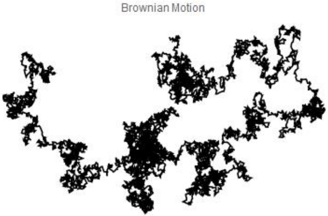
Brownian motion as a two-dimensional object

Signature as a transform on rough paths:

$$Sig(X) = (1, \int_0^T \mathrm{d}X_{t_1}, \int_0^T \int_0^{t_2} \mathrm{d}X_{t_1} \otimes \mathrm{d}X_{t_2}, \ldots).$$



Motivation

Theorem: (Hambly, Lyons, Geng, Yang, B.) If

$$Sig(X|_{[0,T]}) = Sig(\tilde{X}|_{[0,T]}),$$

then for all smooth V,

$$\begin{aligned} \mathrm{d}Y_t &= V(Y_t)\mathrm{d}X_t,\ Y_0 = y \\ \mathrm{d}\tilde{Y}_t &= V(\tilde{Y}_t)\mathrm{d}\tilde{X}_t,\ \tilde{Y}_0 = y \end{aligned}$$

we have $Y_T = \tilde{Y}_T$.

Applicaion: Chinese handwriting recognition.

Inversion problem

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- **Hambly-Lyons Theorem** : $X \in C^1, X' \neq 0$,

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- Non-commutativity of $\otimes$ important!

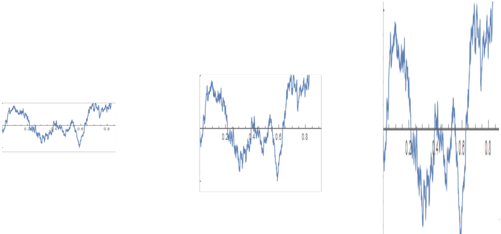
Lower bound for iterated integrals

- Let $(M_i)_{i=1}^{d}$ be constant matrices such that

$$\sup_{\|x\|\leq 1, \|y\|\leq 1} \|\sum_{i=1}^{d} M_i x^i y\| \leq 1, \; \|Y_0\| \leq 1,$$

$$\text{and} \quad \mathrm{d}Y_t^{\lambda} = \lambda \sum_{i=1}^{d} M_i Y_t^{\lambda} \mathrm{d}X_t^i, \; \text{then}$$

$$\limsup_{\lambda\to\infty} \frac{\log \|Y_T^{\lambda}\|}{\lambda} \leq \limsup_{n\to\infty} \|n! \int_0^T \cdots \int_0^{t_1} \mathrm{d}X_{t_1} \otimes \ldots \otimes \mathrm{d}X_{t_n}\|^{\frac{1}{n}}.$$



Hyperbolic development

- Hambly-Lyons considers "Hyperbolic development":

$$\mathrm{d}Y_t^\lambda = \lambda \begin{pmatrix} 0 & \dots & 0 & \mathrm{d}X_t^1 \\ \vdots & \ddots & 0 & \vdots \\ 0 & \dots & 0 & \mathrm{d}X_t^d \\ \mathrm{d}X^1 & \dots & \mathrm{d}X_t^d & 0 \end{pmatrix} Y_t^\lambda, \quad Y_0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

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- Dynamical system argument $\Longrightarrow$

$$\begin{aligned} & \mathrm{length}(X|_{[0,T]}) \\ = \quad & \limsup_{\lambda\to\infty} \frac{\log \|Y_T^\lambda\|}{\lambda} \\ \leq \quad & \limsup_{n\to\infty} \|n! \int_0^T \dots \int_0^{t_1} \mathrm{d}X_{t_1} \otimes \dots \otimes \mathrm{d}X_{t_n}\|^{\frac{1}{n}}. \end{aligned}$$

Large noise asymptotics

- **Generally**, if X is a p-rough path, $(M_i)_{i=1}^{d}$ constant matrices,

$$\mathrm{d}Y_t^{\lambda} = \lambda \sum_{i=1}^{d} M_i Y_t^{\lambda} \mathrm{d}X_t^i$$

$$\|Y_0\| \leq 1, \quad \sup_{\|x\|\leq 1, \|y\|\leq 1} \|\sum_{i=1}^{d} M_i x^i y\| \leq 1$$

$$\limsup_{\lambda\to\infty} \frac{\log \|Y_T^{\lambda}\|}{\lambda^p} \leq \limsup_{n\to\infty} \|\frac{n}{p}! \int_0^T \ldots \int_0^{t_1} \mathrm{d}X_{t_1} \ldots \mathrm{d}X_{t_n}\|^{\frac{p}{n}}.$$

The Brownian motion case

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- If

$$\mathrm{d}Y_t^\lambda = \lambda \begin{pmatrix} 0 & \dots & 0 & \circ\mathrm{d}B_t^1 \\ \vdots & \ddots & 0 & \vdots \\ 0 & \dots & 0 & \circ\mathrm{d}B_t^d \\ \circ\mathrm{d}B^1 & \dots & \circ\mathrm{d}B_t^d & 0 \end{pmatrix} Y_t^\lambda, \quad Y_0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

then

$$\begin{aligned} \frac{(d-1)T}{2} &\leq \limsup_{\lambda\to\infty} \frac{\log \|Y_T^\lambda\|}{\lambda^2} \\ &\leq \limsup_{n\to\infty} \left\| \frac{n}{2}! \int_0^T \dots \int_0^{t_1} \circ\mathrm{d}B_{t_1} \dots \circ \mathrm{d}B_{t_n} \right\|^{\frac{2}{n}} \end{aligned}$$

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- **Key step**: Use martingale method to find for $\mu < 0$

$$\mathbb{E}(\|Y_T^\lambda\|^\mu).$$

The Itô case

- If $\bullet$ is Itô, consider instead

$$\begin{aligned} \mathrm{d}Y_t^\lambda &= -\lambda X_t^\lambda \sum_{i=1}^d \bullet \mathrm{d}B_t^i, \ Y_0^\lambda = 1 \\ \mathrm{d}X_t^\lambda &= \lambda Y_t^\lambda \sum_{i=1}^d \bullet \mathrm{d}B_t^i, \ X_0^\lambda = 0. \end{aligned}$$

then

$$\begin{aligned} \frac{dT}{2} &\leq \limsup_{\lambda\to\infty} \frac{\log \|(X_T^\lambda, Y_T^\lambda)\|}{\lambda^2} \\ &\leq \limsup_{n\to\infty} \left\| \frac{n}{2}! \int_0^T \dots \int_0^{t_1} \bullet \mathrm{d}B_{t_1} \otimes \dots \otimes \bullet \mathrm{d}B_{t_n} \right\|^{\frac{2}{n}} \end{aligned}$$

Estimating the iterated integrals

Theorem:
For Stratonovitch iterated integral, a.s.

$$\frac{d-1}{2}T \leq \limsup_{n} \| \left(\frac{n}{2}\right)! \int_0^T \int_0^{t_n} \ldots \int_0^{t_2} \circ \mathrm{d}B_{t_1} \otimes \ldots \otimes \circ \mathrm{d}B_{t_n} \|^{\frac{2}{n}}$$

and for Itô,

$$\frac{d}{2}T \leq \limsup_{n} \| \left(\frac{n}{2}\right)! \int_0^T \int_0^{t_n} \ldots \int_0^{t_2} \bullet \mathrm{d}B_{t_1} \otimes \ldots \otimes \bullet \mathrm{d}B_{t_n} \|^{\frac{2}{n}}.$$

Question: What about upper bound?

Upper bound

- G. Ben Arous's bounds:

$$\mathbb{E}\Big[\sum_{i_1,\ldots,i_n}\Big(\int_0^T \cdots \int_0^{t_2} \bullet \mathrm{d}B_{t_1}^{i_1} \ldots \bullet \mathrm{d}B_{t_n}^{i_n}\Big)^2\Big] = \frac{d^n T^n}{n!}$$

$$\mathbb{E}\Big[\sum_{i_1,\ldots,i_n}\Big(\int_0^T \cdots \int_0^{t_2} \circ \mathrm{d}B_{t_1}^{i_1} \ldots \circ \mathrm{d}B_{t_n}^{i_n}\Big)^2\Big] \leq \frac{5^n d^n T^n}{2^n n!}$$

implies a.s.

$$\limsup_n \Big\| \frac{n}{2}! \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} \bullet \mathrm{d}B_{t_1} \otimes \ldots \otimes \bullet \mathrm{d}B_{t_n} \Big\|^{\frac{2}{n}} \leq \frac{d^2}{2} T$$

$$\limsup_n \Big\| \frac{n}{2}! \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} \circ \mathrm{d}B_{t_1} \otimes \ldots \otimes \circ \mathrm{d}B_{t_n} \Big\|^{\frac{2}{n}} \leq \frac{1}{2}\frac{5^2}{2^2} d^2 T$$

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$$\leq \frac{1}{2}\frac{5^2}{2^2} d^2 T$$

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$$\leq \frac{d^2}{2} T.$$

- **Question**: Is the limsup in fact *deterministic*?

The Brownian motion case

- One dimensional case: $\bullet$=Itô,

$$\limsup_n |(\frac{n}{2})! \int_0^t \int_0^{t_n} \cdots \int_0^{t_2} \mathrm{d}B_{t_1} \bullet \ldots \bullet \mathrm{d}B_{t_n}|^{\frac{2}{n}} = \frac{1}{2}t.$$

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- **Question**: limsup still deterministic for high dimensions?

Main result

Theorem: (B. and X. Geng)
Let B_t be a d-dim Brownian motion Then there exists **deterministic** $C < \infty$, a.s. for all t

$$\limsup_n \left\| \left(\frac{n}{2}\right)! \int_0^t \int_0^{t_n} \dots \int_0^{t_2} \circ \mathrm{d}B_{t_1} \otimes \dots \otimes \circ \mathrm{d}B_{t_n} \right\|^{\frac{2}{n}} = Ct.$$

Also true for Itô integrals, and adding any bounded drift to B_t.

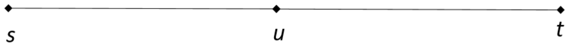
Key Lemma

- **Key Lemma**: If

$$A(s,t)=\limsup_{n}\|(\frac{n}{2})!\int_{s}^{t}\int_{s}^{t_n}\cdots\int_{s}^{t_2}\mathrm{d}B_{t_1}\otimes\ldots\otimes\mathrm{d}B_{t_n}\|^{\frac{2}{n}},$$

then

$$A(s,t)\leq A(s,u)+A(u,t).$$



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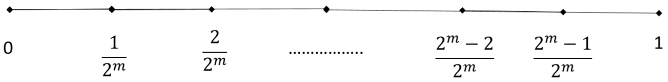

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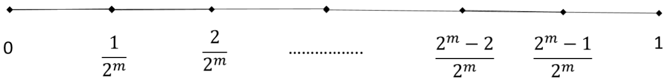
- 

- As $A(s,t) \leq A(s,u) + A(u,t)$,

$$\Longrightarrow A(0,1) \leq \lim_{m\to\infty} \frac{1}{2^m} \sum_{i=0}^{2^m-1} 2^m A\left(\frac{i}{2^m}, \frac{i+1}{2^m}\right).$$

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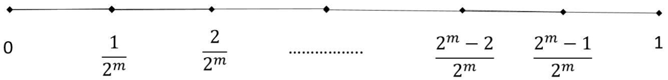
- 

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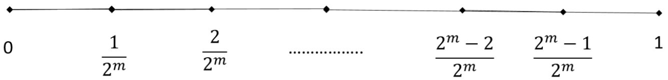
- 

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- $\{2^m A(\frac{i}{2^m}, \frac{i+1}{2^m})\}_{i=0}^{2^m}$ are i.i.d $\implies$ R.H.S. is $\mathbb{E}[A(0,1)]$.

$$A(0,1) \leq \mathbb{E}(A(0,1)) \implies A(0,1) \text{ deterministic.}$$

General rough paths

- **Question**: For p-rough path X, find

$$\limsup_n \|(\frac{n}{p})! \int_0^T \int_0^{t_n} \dots \int_0^{t_1} \mathrm{d}X_{t_1} \otimes \dots \otimes \mathrm{d}X_{t_n}\|^{\frac{p}{n}}?$$

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$$\omega(s,t) = \limsup_n \left\|\left(\frac{n}{p}\right)! \int_s^t \int_s^{t_n} \ldots \int_s^{t_1} \mathrm{d}X_{t_1} \otimes \ldots \otimes \mathrm{d}X_{t_n}\right\|^{\frac{p}{n}}$$

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  - Ignore "tree-like" path;
  - Cass-Litterer-Lyons integrability estimates.

Pure rough paths

If X is p-rough path,

$$X = \exp((t-s)(P_1 + P_2 + \ldots + P_m))$$

where P_i are Lie polynomial degree i,

$$\limsup_n \|(\frac{n}{m})! \int_0^1 \int_0^{t_n} \ldots \int_0^{t_1} \mathrm{d}X_{t_1} \otimes \ldots \otimes \mathrm{d}X_{t_n}\|^{\frac{m}{n}} \leq \|P_m\|.$$

Open problem: Lower bound