Hausdorff dimension of the boundary of Brownian bubbles

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Based on joint work with:

T. Mountford (EPF - Lausanne)

The Brownian sheet

A standard two-parameter Brownian sheet is a centered Gaussian random field $W = (W(t_1, t_2), (t_1, t_2) \in \mathbb{R}^2_+)$ defined on a probability space $(\Omega, \mathcal{F}, P)$, with continuous sample paths and covariance

$$E[W(s_1, s_2)W(t_1, t_2)] = \min(s_1, t_1) \min(s_2, t_2).$$

For fixed t_2, $t_1 \mapsto W(t_1, t_2)$ is a Brownian motion (with speed t_2).

References:
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Two books: R. Adler (1990), D. Khoshnevisan (2002)

Issues: Sample path properties, Markov properties, potential theory, level sets, small ball probabilities, hitting probabilities, multiple points.

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Level sets and bubbles

For $x \in \mathbb{R}$, the level set of W at level x is the random closed set

$$L(x) := \{(t_1, t_2) \in \mathbb{R}_+^2 : \; W(t_1, t_2) = x\}.$$

The complement of the level set is the union of two random open sets

$$L_+(x) := \{(t_1, t_2) \in \mathbb{R}_+^2 : \; W(t_1, t_2) > x\},$$

$$L_-(x) := \{(t_1, t_2) \in \mathbb{R}_+^2 : \; W(t_1, t_2) < x\}.$$

Definition. A Brownian bubble is one connected component of $L_+(x)$ or $L_-(x)$.

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(Recall that any open subset of $\mathbb{R}_+^2$ is a countable disjoint union of connected components.)

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Hausdorff dimension of level sets

Theorem 1 (R.J. Adler, 1978)

A.s., for all $x \in \mathbb{R}$, $dim_{\mathcal{H}} L(x) = 1.5$

Theorem 2 (T. Mountford, 1993)

Fix $x \in \mathbb{R}$. A.s., the Hausdorff dimension of the boundary of any Brownian bubble is: $\geqslant 1.25$ and < 1.5.

Interpretation: "Most of $L(x)$ is not part of the boundary of any bubble."

Comparison with standard Brownian motion:

bubbles $\longleftrightarrow$ excursions above/below level x;
boundaries of bubbles $\longleftrightarrow$ extremities of excursion intervals.

There are countably many extremities of excursion intervals (dimension 0), but the dimension of level sets of standard Brownian motion is $\frac{1}{2}$.

Question. Do all bubble boundaries have the same dimension? If so, what is it?

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Explanation for Adler's theorem

Upper bounds on Hausdorff dimension $\longleftarrow$ coverings.

Let

$$V_n := \{(1 + i2^{-2n}, 1 + j2^{-2n}) : i, j \in \{0, \dots, 2^{2n} - 1\}.$$

Then V_n = vertices of a grid in $[1, 2]^2$, $\quad \sharp V_n = 2^{4n}$.

For $t \in \mathbb{E}_n$, define $E_n(t) :=$ the square in the grid with lower left corner at t.

P2 One covering of $L(x) \cap [1, 2]^2$, with diameter $c2^{-2n}$, is:

$$\{E_n(t) : \ t \in V_n, \ E_n(t) \cap L(x) \neq \emptyset\}.$$

Calculation:

$$E\left[\sum_{t \in V_n} (2^{-2n})^\alpha 1_{\{E_n(t) \cap L(x) \neq \emptyset\}}\right] = (2^{-2n})^\alpha (2^{2n})^2 P\{E_n(t) \cap L(x) \neq \emptyset\}.$$

Now

$$P\{E_n(t) \cap L(x) \neq \emptyset\} \simeq P\{|W(t) - x| \leqslant 2^{-n}\} \simeq 2^{-n},$$

so the expectation above is

$$\leqslant 2^{(4-2\alpha)n} 2^{-n} = 2^{(3-2\alpha)n} \to 0$$

as $n \to \infty$ if and only if $\alpha > \frac{3}{2}$.

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Towards the dimension of bubble boundaries

Let $\mathcal{C}_1$ be a bubble of height $\geqslant 1$ (in $[1,2]^2$). Then:

$t \in \partial\mathcal{C}_1 \iff W(t) = x$ **and** for all $\varepsilon > 0$, there exists a path Γ with
$d(\Gamma(0), t) > \varepsilon$ and $W(\Gamma(\cdot)) - x$ hits 1 before 0.

P3

Covering of $\partial\mathcal{C}_1 \cap [1,2]^2$:

$$\{E_n(t) : \; E_n(t) \cap L(x) \neq \emptyset \text{ and } F(t) \text{ occurs}\},$$

where

$$F(t) = \{\exists\Gamma : \Gamma(0) = t \text{ and } W(\Gamma(\cdot)) - x \text{ hits 1 before 0}\}.$$

Should examine the behavior as $n \to \infty$ of

$$\sum_{t \in V_n} (2^{-2n})^{\alpha} \, P\{|W(t) - x| \leqslant 2^{-n}\} \, P\{F(t) \,\big|\, |W(t) - x| \leqslant 2^{-n}\}$$
$$\simeq 2^{4n} 2^{-2\alpha n} 2^{-n} \; P\{F(t) \,\big|\, |W(t) - x| \leqslant 2^{-n}\}.$$

Main difficulty in estimating $P\{F(t) \,\big|\, |W(t) - x| \leqslant 2^{-n}\}$: there are infinitely many possible paths, and these can be arbitrarily "twisty" [D. & Walsh, 1993].

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Local decomposition of the Brownian sheet

The event $F(t)$ is "local": either 0 is hit rather quickly, or not, and in this case, $W - x$ will typically escape to a height of order 1 (the same occurs for Brownian motion).

Local decomposition of W [W. Kendall, 1980]: Fix $t = (t_1, t_2)$. For $u_1, u_2 \in \mathbb{R}$,

$$W(t_1 + u_1, t_2 + u_2) = W(t_1, t_2) + B_1^t(u_1) + B_2^t(u_2) + \mathcal{E}^t(u_1, u_2),$$

where:

B_1^t, B_2^t are independent (two-sided) BM's, and
$\mathcal{E}^t$ is "small" (of order $\sqrt{|u_1 u_2|}$).

This suggest to study additive Brownian motion:

$$X(u_1, u_2) := X(0,0) + B_1(u_1) + B_2(u_2), \qquad u_1, u_2 \in \mathbb{R}.$$

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Gambler's ruin problem for additive BM

Let $X = (X(u_1, u_2),\ (u_1, u_2) \in \mathbb{R}^2)$ be an additive Brownian motion.

For $x \in [0, 1]$, define

$$\mathbb{E}(x) := P\{\exists \text{ path } \Gamma : \Gamma(0) = (0,0),\ X(\Gamma(\cdot)) \text{ hits } 1 \text{ before } 0 \mid X(0,0) = x\}.$$

Problem. Estimate $\mathbb{E}(x)$.

P4

Main difficulty: there is no constraint on the path Γ: one has to consider all paths, with no restrictions.

Related problem. For $X(0,0) \neq 0$, let $\mathcal{C}_{(0,0)}$ be the bubble "stradling" $(0,0)$.

Question. For $a > 0$, what is the probability that the bubble $\mathcal{C}_{(0,0)}$ extends at least a units away from the origin?

P5

That is, estimate

$$\mathbb{D}(x, a) = P\{\mathcal{C}_{(0,0)} \not\subset [-a, a]^2 \mid X(0,0) = x\}.$$

By scaling, $\mathbb{D}(x, a) = \mathbb{D}(x/\sqrt{a}, 1)$, and we expect $\mathbb{D}(x, 1) \simeq \mathbb{E}(x)$ for $x \downarrow 0$.

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Gambler's ruin

Theorem 3 (D. & Mountford)

For $x \in [0,1]$,

$$\mathbb{E}(x) = \alpha_1 x^{\lambda_1} + \alpha_2 x^{\lambda_2} + \alpha_3 x^{\lambda_3} + \alpha_4 x^{\lambda_4},$$

where

$$\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} = \left\{ \tfrac{1}{2}\left(5 \pm \sqrt{13 \pm 4\sqrt{5}}\right) \right\},$$

$\lambda_1 = \frac{1}{2}\left(5 - \sqrt{13 + 4\sqrt{5}}\right) \simeq 0.158 \quad < \quad \lambda_2 \simeq 1.49 \quad < \cdots$

$\alpha_1 \simeq 0.939$, $\alpha_2 = \ldots$ *(exact, explicit formulas are given).*

In particular, $\mathbb{E}(x) \simeq x^{\lambda_1}$ *as* $x \downarrow 0$.

Comparison. For standard BM, we would have $\mathbb{E}(x) \simeq x \ll x^{\lambda_1}$.

Theorem 3 is somewhat surprising!

Escape probabilities

Corollary 1

There exist $0 < c < C < \infty$ *such that, for all* $a \geqslant x^2$,

$$c\left(\frac{x}{\sqrt{a}}\right)^{\lambda_1} \leqslant \mathbb{D}(x,a) \leqslant C\left(\frac{x}{\sqrt{a}}\right)^{\lambda_1}.$$

Proving Corollary 1 from Theorem 3 requires some effort.

Main result

Theorem 4 (D. & Mountford)

Fix $x \in \mathbb{R}$. For the Brownian sheet, the Hausdorff dimension of the boundary of every x-bubble is

$$\frac{3}{2} - \frac{\lambda_1}{2} = \tfrac{1}{4}\left(1 + \sqrt{13 + 4\sqrt{5}}\right) \simeq 1.421.$$

Once Theorem 3 and Corollary 1 are proved, the road map to prove Theorem 4 is fairly clear. Carrying out these steps requires some effort.

Will explain why Theorem 3 is true, then give some ideas on how to deduce Theorem 4.

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Proving Theorem 3 (gambler's ruin probabilities for ABM)

Theorem 5 (D. & Walsh, 1993)

There is a specific path Γ^o such that

$$\mathbb{E}(x) = P\{X(\Gamma^o(\cdot)) \text{ hits } 1 \text{ before } 0 \mid X(0,0) = x\}.$$

P6 Explain construction of Γ^0: the DW-algorithm.

Lemma

The sequence $M_0 = x, M_1, M_2, \ldots$ of successive maxima encountered along the horizontal/vertical segments of the path Γ^o is Markov of order 2, with transition probabilities

$$P\{M_{n+1} \in dz \mid M_n = y,\ M_{n-1} = x\} = f(x,y,z)\,dz, \qquad z > y > x,$$

where

$$f(x,y,z) = \frac{2(y-x)}{z^2} - \frac{2(y-x)^2}{z^3},$$

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$$P\{M_{n+1} = y \mid M_n = y,\ M_{n-1} = x\} = \left(\frac{x}{y}\right)^2.$$

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Study of the Markov chain $\Theta_n = (M_{n-1}, M_n)$

State space: $\mathcal{S} = \{(y_1, y_2) \in \mathbb{R}_+^2 : 0 < y_1 \leqslant y_2\}$

P7 Consider the paths of (Θ_n)

Define the subsets:
WIN $:= \{(y_1, y_2) \in \mathcal{S} : y_2 \geqslant 1\}$,
LOSE $:= \{(y_1, y_2) \in \mathcal{S} : y_2 = y_1\}$.

P8 and set

$$\alpha(x, y) = P\{(\Theta_n) \text{ visits LOSE before WIN } \big| \Theta_1 = (x, y)\}.$$

Then

$$\alpha(x, y) = \left(\frac{x}{y}\right)^2 + \int_y^1 dz\, f(x, y, z)\, \alpha(y, z). \tag{1}$$

This is an unusual sort of linear integral equation (but similar to the system of equations for absorption probabilities for Markov chains). After several manipulations, one checks that:

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Solving the integral equation

Solving (1) is equivalent to soving the linear system of o.d.e.'s

$$\dot{\underline{x}}(y) = A \cdot \underline{x}(y) + \underline{b}, \qquad y > 0,$$

where A is the 6×6 matrix and $\underline{b}$ and $\underline{x}(0)$ are the column vectors

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -9 & 6 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -8 & 2 & 0 & 28 & -26 & 9 \end{pmatrix}, \quad \underline{b} = \begin{pmatrix} 0 \\ 0 \\ -2 \\ 0 \\ 0 \\ -6 \end{pmatrix}, \quad \underline{x}(0) = \begin{pmatrix} 0 \\ -1 \\ -3 \\ 0 \\ 1 \\ -4 \end{pmatrix}.$$

This yields an explicit formula for $\alpha(x, y)$, via the 4 real eigenvalues $\lambda_1, \ldots, \lambda_4$ and eigenvectors of A. Finally,

$$\mathbb{E}(x) = 1 - E[\alpha(x, H_1)],$$

where $P\{H_1 \leqslant y\} = (P_x\{B(\cdot) \text{ hits } 0 \text{ before } y\})^2 = \left(\frac{y-x}{y}\right)^2$.

This leads to the explicit formula for $\mathbb{E}(x)$.

Solving the integral equation

Solving (1) is equivalent to soving the linear system of o.d.e.'s

$$\dot{\underline{x}}(y) = A \cdot \underline{x}(y) + \underline{b}, \qquad y > 0,$$

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Proving Theorem 4

Theorem. $\dim_H$ "bubble" $= \frac{3}{2} - \frac{\lambda_1}{2}$.

Part 1. Upper bound: $\dim_H$ "bubble" $\leqslant \frac{3}{2} - \frac{\lambda_1}{2}$.

Use the covering argument discussed previously:

$$\begin{aligned}
&\sum_{t \in V_n} (2^{-2n})^{\alpha} \, P\{|W(t) - x| \leqslant 2^{-n}\} \, P\{F(t) \,\big|\, |W(t) - x| \leqslant 2^{-n}\} \\
&\quad \simeq 2^{4n} \, 2^{-2\alpha n} \, 2^{-n} \; P\{F(t) \,\big|\, |W(t) - x| \leqslant 2^{-n}\} \\
&\quad \simeq 2^{(3-2\alpha)n} \, (2^{-n})^{\lambda_1} \\
&\quad = 2^{(3-2\alpha-\lambda_1)n} \\
&\quad \longrightarrow 0 \qquad \text{if } \alpha > \frac{3-\lambda_1}{2}.
\end{aligned} \tag{2}$$

Note. (2) concerns the Brownian sheet, not ABM: some effort is needed to go from one to the other ("robustness" of the DW-algorithm).

Proving Theorem 4

Part 2. Lower bound: $\dim_H$ "bubble" $\geqslant \frac{3}{2} - \frac{\lambda_1}{2}$.

Energy method: For $\alpha < \frac{3}{2} - \frac{\lambda_1}{2}$, seek a measure μ supported on the boundary of a bubble, such that

$$\int\int \frac{\mu(ds)\mu(dt)}{|t-s|^\alpha} < \infty.$$

Via a "second moment argument", the key estimate is:

Part 2 (continued)

Lemma

For $s, t \in [1,2]^2$, with $|s_1 - t_1| \simeq 2^{2(k-n)}$, $|s_2 - t_2| \simeq 2^{2(\ell-n)}$ $(1 \leqslant k < \ell \leqslant n)$,

$$P\{|W(t)| \leqslant 2^{-n}, F(t), |W(s)| \leqslant 2^{-n}, F(s)\} \leqslant 2^{-n} 2^{-\ell} (2^{-k\lambda_1})^2 (2^{\ell-n})^{\lambda_1}.$$

(recall that $F(t) = \{\exists \Gamma : \Gamma(0) = t$ and $W(\Gamma(\cdot))$ hits 1 before 0$\}$; here $x = 0$.)

Explanation of each factor:

P9 $W(t) \simeq 2^{-n}$: prob. $\simeq 2^{-n}$

$W(s) \simeq 2^{-n}$ (given $W(t) \simeq 2^{-n}$): prob. $\simeq \frac{2^{-n}}{2^{\ell-n}} = 2^{-\ell}$.

$F(t) \cap F(s)$: first both paths reach level 2^{k-n} units: prob. $\left[(2^{-k})^{\lambda_1}\right]^2$.

In the big rectangle, the maximum of W is $\simeq 2^{\ell-n}$. Starting from this level, one path (at least) must reach level 1 before 0: prob. $\simeq (2^{\ell-n})^{\lambda_1}$.

A good bound is obtained by multiplying these factors (even though the events are not independent!).

Part 2 (continued)

Lemma

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References

Mountford, T.S. (1993). Estimates of the Hausdorff dimension of the boundary of positive Brownian sheet components. *Sém. de Probabilités* XXVII *Lect. Notes in Math.* vol. 1557, pp.233-255. Springer.

Dalang, R.C. & Walsh, J.B. (1993) The structure of a Brownian bubble. *Probab. Th. Relat. Fields* 96, 475-501.

Mörters, P. & Peres, Y. (2010). *Brownian motion.* Cambridge University Press.

Dalang, R.C. & Mountford, T. (2017). Hausdorff dimension of the boundary of bubbles of additive Brownian motion and of the Brownian sheet (145pp). http://arxiv.org/abs/1702.08183