Large Deviations Principles for McKean-Vlasov SDEs, Skeletons, Supports and the law of iterated logarithm

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Overview

1. McKean Vlasov Equations
2. Large Deviations Principles
   - Skeleton ODE's of SDE's
   - LDPs
   - Results
3. Applications
   - Functional Strassen's law
4. Outlook and Further Extensions

Outline

McKean-Vlasov Stochastic Differential Equations

Definition

A **McKean-Vlasov SDEs** (**MV-SDE**) is an SDE where the coefficients are dependent on the law $\mathcal{L}(X(\cdot))$ of the solution process $(X(\cdot))$. We write

$$dX(t) = b\Big(t, X(t), \mathcal{L}(X(t))\Big)dt + \sigma\Big(t, X(t), \mathcal{L}(X(t))\Big)dW(t)$$
$$X(0) = x$$

Example (Mean Field Scalar Interaction)

Let us consider a simple example:

$$X(t) = x + \int_0^t \Big[\mathbb{E}[X(s)] - X(s)\Big]ds + W(t)$$

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▷ **Question**: Is this an standard SDE **after decoupling**?

Distributions and the Wasserstein Metric

Definition

Let (E, d) be a Polish space and σ-algebra $\mathcal{E}$. Let $\mathcal{P}_2(E)$ be the space of probability distributions on $(E, \mathcal{E})$ with finite second moments. Let $\mu, \nu \in \mathcal{P}_2(E)$. We define the **Wasserstein distance** to be

$$W^{(2)}(\mu, \nu) = \inf \left\{ \Big(\int_{E^2} d(x, y)^2 \pi(dx, dy) \Big)^{1/2} ; \pi \in \mathcal{P}(E \times E) \right\}$$

where $\mu(A) = \int_{E^2} \chi_A(x)\pi(dx, dy)$ and $\nu(B) = \int_{E^2} \chi_B(y)\pi(dx, dy)$.

See [Carmona, 2016] form more details.

Example

The Wasserstein distance between the law of a RV X and a constant y is

$$W^{(2)}(\mathcal{L}(X), \delta_y) = \mathbb{E}[|X - y|^2]^{1/2}$$

Existence and Uniqueness

Theorem (Existence and uniqueness)

Let $(X(t))_{t\geq 0}$ *satisfy the MV-SDE*

$$dX(t) = b(t, X(t), \mathcal{L}(X(t)))dt + \sigma(t, X(t), \mathcal{L}(X(t)))dW(t),$$
$$X(0) \sim \mu_0 \in \big(\mathcal{P}_2 \cap \mathcal{P}_4\big)(\mathbb{R}^d)$$

with: $\exists L > 0$, $\exists K \in \mathbb{R}\ \forall t \in [0, T]$, $\forall x, x' \in \mathbb{R}^d$, $\forall \mu, \mu' \in \mathcal{P}_2(\mathbb{R}^d)$ *s.th.*

$$|\sigma(t, x, \mu) - \sigma(t, x', \mu')| \leq L\Big(|x - x'| + W^{(2)}(\mu, \mu')\Big)$$
$$\langle x - y, b(t, x, \mu) - b(t, y, \mu)\rangle_{\mathbb{R}^d} \leq K|x - y|^2$$
$$|b(t, x, \mu) - b(t, x, \mu')| \leq LW^{(2)}(\mu, \mu')$$

Then there exists a unique solution X *and* $\exists C > 0$ *such that*

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |X(t)|^2\Big] \leq \Big(\mathbb{E}[|X(0)|^2] + C\Big)e^{CT}$$

Properties

Theorem (Properties)

- *Integrability*
  1. $\forall p > 1$ *we have* $\mathbb{E}[\sup_t |X(t)|^p] < \infty$
     (with agreeing integrability of $X(0)$, $b(\cdot, 0, \delta_0)$, $\sigma(\cdot, 0, \delta_0)$*)*
- *Continuity*
  1. *paths of* $t \mapsto X(t)(\omega)$ *are a.s. continuous in* C^α, $\alpha < 1/2$.
  2. $t \mapsto \mathcal{L}(X(t))$ *is* $C^{1/2}$ *in the* $W^{(p)}$*-metric*
- *Differentiability*
  1. *for any* $p \geq 1$ *the map* $t \mapsto \mathbb{E}[|X^p(t)|^p] \in C^1$
  2. *Malliavin differentiability:* $X \in \mathbb{D}^{1,2}$
     (with deterministic coefficients)

Outline

Deterministic Approximation of SDE

Definition

Let H be the Cameron Martin space, the space of all absolutely continuous paths $h(t) = \int_0^t \dot{h}(s)ds$ such that $\dot{h} \in L^2([0,1])$.

Definition

We approximate the McKean Vlasov SDE

$$dX(t) = b_\varepsilon(t, X(t), \mathcal{L}(X(t)))dt + \varepsilon\sigma(t, X(t), \mathcal{L}(X(t)))dW(t)$$
$$X(0) = x$$

by the ODE

$$d\Phi(h)(t) = b(t, \Phi(h)(t), \delta_{\Phi(t)})dt + \sigma(t, \Phi(h)(t), \delta_{\Phi(t)})\dot{h}(t)dt$$
$$\Phi(0) = x$$

and we call this the **Skeleton**.

Deterministic Appoximation of SDE

Example

The SDE

$$X(t) = x + \int_0^t \Big[X(s) - \mathbb{E}\Big[|X(s)|^3\Big]\Big] ds + \varepsilon W(t)$$

has a *Skeleton* $\forall h \in H$

$$\Phi(h)(t) = x + \int_0^t \Big[\Phi(h)(s) - |\Phi(h)(s)|^3\Big] ds + h(t)$$

since $\int_\Omega |x|^3 d\delta_x(y) = |y|^3$.

Large Deviations Principles

Definition (Large Deviations Principle)

Let (E, d) be a Polish space and let $\{\mathbb{P}_N\}_{N\in\mathbb{N}}$ be a sequence of Borel probability measures on E. Let $I : E \to [0, \infty]$ be a lower semicontinuous functional on E. The sequence $\{\mathbb{P}_N\}_{N\in\mathbb{N}}$ is said to satisfy a **Large Deviations Principle** with rate function $I \iff$

$$-\inf_{x\in\mathring{A}} I(x) \leq \liminf_{N\to\infty} \frac{\log(\mathbb{P}_N[A])}{N^2} \leq \limsup_{N\to\infty} \frac{\log(\mathbb{P}_N[A])}{N^2} \leq -\inf_{x\in\overline{A}} I(x)$$

for any Borel measurable set $A \subset E$.

LDP for Brownian Motion

Consider the following simple example for Brownian motion with a supremum norm.

Example (LDP in uniform Norm for BM)

Consider the simple example of $(\varepsilon B_t)_t$.

We know for $R >> 1$ fixed that $\mathbb{P}\Big[\big\|\varepsilon B_\cdot\big\|_\infty > R\Big] \lesssim e^{-R^2/(2\varepsilon^2)}$.

Therefore (apply log + scalling & limits)

$$\limsup_{\varepsilon\to 0} \varepsilon^2 \log\left(\mathbb{P}\Big[\big\|\varepsilon B_\cdot\big\|_\infty > R\Big]\right) \leq \lim_{\varepsilon\to 0} \varepsilon^2 c - \frac{R^2}{2} = -\frac{R^2}{2}$$

▷ The rate function for Brownian motion would output $\frac{R^2}{2}$ for the set $\{x(t) \in C([0,1]) : \|x\|_\infty > R\}$ the set of continuous paths starting at 0 such that the supremum of the path is greater than R.

LDP for MV-SDEs

Our goals:

1. an $\|\cdot\|_\infty$-topology LDP for X
   (see [Gärtner, 1988], [Budhiraja et al, 2012])

2. a conditional $\|\cdot\|_\alpha$-topology type LDP for X,

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Theorem

$\forall R, \rho > 0$, $\exists \delta, \nu > 0$ *such that* $\forall 0 < \varepsilon < \nu$,

$$\mathbb{P}\Big[\|X_\varepsilon^x - \Phi^x(0)\|_\alpha \geq \rho, \|\varepsilon W\|_\infty \leq \delta\Big] \lesssim \exp(-R/\varepsilon^2)$$

Hölder Norms using Ciesielski Isomorphism

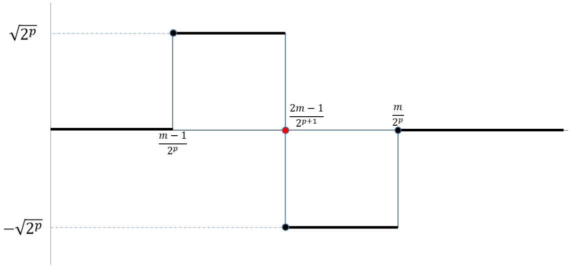
Definition

Let $H_{00}(t) = 1$ and

$$H_{pm}(t) = \begin{cases} \sqrt{2^p}, & \text{if } t \in [\frac{m-1}{2^p}, \frac{2m-1}{2^{p+1}}), \\ -\sqrt{2^p}, & \text{if } t \in [\frac{2m-1}{2^{p+1}}, \frac{m}{2^p}), \\ 0, & \text{otherwise.} \end{cases}$$

where $m \in \{1, ..., 2^p\}$ and $p \in \mathbb{N} \cup \{0\}$. These are called the **Haar functions**.

Figure: Haar Function $H_{pm}(t)$



Hölder Norms using Ciesielski Isomorphism

Ciesielski's Isomorphism

Define the Fourier coefficients $\psi_{pm} = \int_0^1 H_{pm}(s)\psi(s)ds$,

$$\psi_{pm} := \langle H_{pm}, d\psi \rangle := \sqrt{2^p}\Big[2\psi\Big(\frac{2m-1}{2^{p+1}}\Big) - \psi\Big(\frac{m-1}{2^p}\Big) - \psi\Big(\frac{m}{2^p}\Big)\Big],$$

additionally $\psi_{00} := \langle H_{00}, d\psi \rangle = \psi(1) - \psi(0)$.
Let $G_{pm}(t) = \int_0^t H_{pm}(s)ds$. Then

$$\psi(t) = \psi_{00} G_{00}(t) + \sum_{p=0}^{\infty} \sum_{m=1}^{2^p} \psi_{pm} G_{pm}(t)$$

Hölder Norms using Ciesielski Isomorphism

The Hölder Norm

The **Hölder Norm** is defined to be

$$||f||_\alpha = |f(0)| + \sup_{t,s\in[0,1]} \frac{|f(t) - f(s)|}{|t-s|^\alpha}$$

We have that $\|\cdot\|_\alpha$ is equivalent to (see [Ciesielsky, 1960])

$$\|\psi\|'_\alpha = \sup_{p,m} 2^{(\alpha-1/2)p} |\psi_{pm}|.$$

Throughout this talk, we will assume that $\alpha < 0.5$.

Auxilliary Lemmas

Lemma 1

$\exists C > 0$ such that $\forall u > 0$ and for all processes K on $[0, 1]$ we have

$$\mathbb{P}\Big[\Big\| \int_0^{\cdot} K(s)dW(s)\Big\|_\alpha \geq u, ||K||_\infty \leq 1\Big] \leq C \exp(-u^2/C)$$

Lemma 2

$\exists C' > 0$ such that $\forall u, v > 0$ we have

$$\mathbb{P}\Big[||W||_\alpha \geq u, ||W||_\infty \leq v\Big] \leq C' \max\Big(1, \Big(\frac{u}{v}\Big)^{1/\alpha}\Big) \exp\Big(\frac{-1}{C'} \cdot \frac{u^{1/\alpha}}{v^{1/\alpha - 2}}\Big).$$

$\triangleright$ These are proved via the equivalence of norms from Ciesielski's isomorphism + Chernoff's inequality.

Main Results

Definition

Let $h \in H$ be an element of the Cameron Martin Space, σ bdd. We consider the SDE

$$X_\varepsilon^x = x + \int_0^t b_\varepsilon(s, X_\varepsilon^x(s), \mathcal{L}(X_\varepsilon^x(s)))ds \\ + \varepsilon \int_0^t \sigma_\varepsilon(s, X_\varepsilon^x(s), \mathcal{L}(X_\varepsilon^x(s)))dW(s)$$

with Skeleton ($b_\varepsilon \to b$, $\sigma_\varepsilon \to \sigma$ uniformly as $\varepsilon \searrow 0$)

$$\Phi^x(h)(t) = x + \int_0^t b(s, \Phi^x(h)(s), \delta_{\Phi^x(h)(s)})ds \\ + \int_0^t \sigma(s, \Phi^x(h)(s), \delta_{\Phi^x(h)(s)})\dot{h}(s)ds$$

Main Results

Theorem

$\forall R, \rho > 0$, $\exists \delta, \nu > 0$ *such that* $\forall 0 < \varepsilon < \nu$,

$$\mathbb{P}\Big[\|X_\varepsilon^x - \Phi^x(h)\|_\alpha \geq \rho, \|\varepsilon W - h\|_\infty \leq \delta\Big] \lesssim \exp(-R/\varepsilon^2)$$

Main Results

Theorem

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$$\mathbb{P}\Big[\|X^x_\varepsilon - \Phi^x(h)\|_\alpha \geq \rho, \|\varepsilon W - h\|_\infty \leq \delta\Big] \lesssim \exp(-R/\varepsilon^2)$$

From the above inequality follows

Theorem

Let A be a Borel set of the space of $\mathbb{R}$-valued continuous paths over $[0,1]$ in the Hölder topology. Let $\Delta(A) := \inf\Big\{\|\dot{h}\|_2^2/4; h \in H, \Phi^x(h)(\cdot) \in A\Big\}$. *Then*

$$-\Delta(\mathring{A}) \leq \liminf_{\varepsilon \to 0} \frac{\varepsilon^2}{2} \log \mathbb{P}[X^x_\varepsilon \in A] \leq \limsup_{\varepsilon \to 0} \frac{\varepsilon^2}{2} \log \mathbb{P}[X^x_\varepsilon \in A] \leq -\Delta(\bar{A})$$

where $\mathring{A}$ and $\bar{A}$ are the interior and closure of the set A with respect to the topology generated by the Hölder norm.

Our proof follows loosely the methods of [Arous, 1994].

Proof of Main Result

Proof.

We condition on the event that the process $X^x_\varepsilon(t)$ remains in the ball of radius N and we see

$$\mathbb{P}\Big[\|X^x_\varepsilon - \Phi^x(h)\|_\alpha \geq \rho, \|\varepsilon W - h\|_\infty \leq \delta\Big] \\ \leq \mathbb{P}\Big[\|X^x_\varepsilon - \Phi^x(h)\|_\alpha \geq \rho, \|\varepsilon W - h\|_\infty \leq \delta, \|X^x_\varepsilon\|_\infty < N\Big] + \mathbb{P}\Big[\|X^x_\varepsilon\|_\infty \geq N\Big]$$

We use that we have the LDP result for X^x_ε in a supremum norm and choose N large enough so that

$$\mathbb{P}\Big[\|X^x_\varepsilon\|_\infty \geq N\Big] < \exp\Big(-\frac{N}{\varepsilon^2}\Big).$$

□

▷ (We give & prove LDP in $\|\cdot\|_\infty$-topology, we do not state it here.)

Proof of Main Result

Proof.

Let $X^{x,l}_\varepsilon$ be a step function approximation of X^x_ε.

$$\mathbb{P}\Big[\|\varepsilon\int_0^\cdot \sigma_\varepsilon(s,X^x_\varepsilon(s),\mathcal{L}(X^x_\varepsilon(s)))dW(s)\|_\alpha \geq \rho, \|\varepsilon W\|_\infty \leq \delta, \|X^x_\varepsilon\|_\infty < N\Big]$$

$$\leq \mathbb{P}\Big[\|\varepsilon\int_0^\cdot [\sigma_\varepsilon(s,X^x_\varepsilon(s),\mathcal{L}(X^x_\varepsilon(s))) - \sigma_\varepsilon(\tfrac{\lfloor sl\rfloor}{l},X^{x,l}_\varepsilon,\mathcal{L}(X^x_\varepsilon(\tfrac{\lfloor sl\rfloor}{l})))]dW(s)\|_\alpha \geq \frac{\rho}{2},$$

$$\tfrac{1}{l^\beta} + \|X^x_\varepsilon - X^{x,l}_\varepsilon\|_\infty + \mathbb{E}[\|X^x_\varepsilon - X^{x,l}_\varepsilon\|^2_\infty]^{1/2} \leq \gamma\Big]$$

$$+\,\mathbb{P}\Big[\tfrac{1}{l^\beta} + \|X^x_\varepsilon - X^{x,l}_\varepsilon\|_\infty + \mathbb{E}[\|X^x_\varepsilon - X^{x,l}_\varepsilon\|^2_\infty]^{1/2} > \gamma, \|X^x_\varepsilon\|_\infty < N\Big]$$

$$+\,\mathbb{P}\Big[\|\varepsilon\int_0^\cdot \sigma_\varepsilon(\tfrac{\lfloor sl\rfloor}{l},X^{x,l}_\varepsilon(s),\mathcal{L}(X^x_\varepsilon(\tfrac{\lfloor sl\rfloor}{l})))dW(s)\|_\alpha \geq \frac{\rho}{2}, \|\varepsilon W\|_\infty \leq \delta\Big]$$

$$\lesssim \exp\left(\frac{-R}{\varepsilon^2}\right)$$

□

Outline

Strassens Law for Brownian Motion

Theorem

Let $W(t)$ be a Brownian Motion. Then

$$X_n(t) = \frac{W(nt)}{\sqrt{n}} \qquad Y_n(t) = \frac{W(nt)}{n}$$

X_n is a Brownian motion but Y_n converges almost surely to 0 as $n \to \infty$. **Strassens Law** states that

$$Z_n(t) = \frac{W(nt)}{\sqrt{n \log(\log(n))}}$$

converges to 0 in probability but does not converge almost surely. Therefore we get the well known result

$$\limsup_{n\to\infty} \frac{W(n)}{\sqrt{n \log(\log(n))}} = \sqrt{2}$$

Contraction Operators

Definition

Let $\alpha \in \mathbb{R}^+$. A family of continuous bijections $\Gamma_\alpha : \mathbb{R}^d \to \mathbb{R}^d$ is said to be a **System of Contractions** centered at x if

1. $\Gamma_\alpha(x) = x$ for every $\alpha \in \mathbb{R}^+$.
2. If $\alpha \geq \beta$ then $|\Gamma_\alpha(y_1) - \Gamma_\alpha(y_2) - \Gamma_\alpha(z_1) + \Gamma_\alpha(z_2)| \leq |\Gamma_\beta(y_1) - \Gamma_\beta(y_2) - \Gamma_\beta(z_1) + \Gamma_\beta(z_2)|$ for every $y_1, y_2, z_1, z_2 \in \mathbb{R}^d$.
3. Γ_1 is the identity and $(\Gamma_\alpha)^{-1} = \Gamma_{\alpha^{-1}}$.
4. For every compact set $\mathcal{K} \subset C^\alpha([0,1]; \mathbb{R}^d)$, $\forall\, f \in \mathcal{K}$ and $\varepsilon > 0$, $\exists \delta > 0$ such that $|pq - 1| < \delta$ implies

$$\|\Gamma_p \circ \Gamma_q(f) - f\|_\alpha < \varepsilon, \qquad p, q \in \mathbb{R}^+.$$

Definition

Let Y be the solution to the SDE

$$dY(t) = b(Y(t), \mathcal{L}(Y(t)))dt + \sigma(Y(t), \mathcal{L}(Y(t)))dW(t), \quad Y(0) = x \in \mathbb{R}^d$$

Denote $\phi(u) = \sqrt{u \log(\log(u))}$. Consider the coefficients

$$\hat{\sigma}_u(y, \mu) = \phi(u)\nabla\Big[\Gamma_{\phi(u)}\Big]\Big(\Gamma_{\phi(u)^{-1}}(y)\Big)^T \sigma\Big(\Gamma_{\phi(u)^{-1}}(y), \mu \circ \Gamma_{\phi(u)}\Big)$$

$$\hat{b}_u(y, \mu) = u\mathbf{L}(y, \mu)\Big[\Gamma_{\phi(u)}\Big]\Big(\Gamma_{\phi(u)^{-1}}(y)\Big)$$

where the operator (with $\tilde{a} = \sigma^T\sigma$)

$$\mathbf{L}(y, \mu)\Big[f\Big]\Big(z\Big) = \sum_{i=1}^{d} \frac{\partial f}{\partial y_i}\Big(\Gamma_{\phi(u)^{-1}}(z)\Big) b_i\Big(\Gamma_{\phi(u)^{-1}}(y), \mu \circ \Gamma_{\phi(u)}\Big) + \frac{1}{2}\sum_{i,j=1}^{d} \tilde{a}_{i,j}\Big(\Gamma_{\phi(u)^{-1}}(y), \mu \circ \Gamma_{\phi(u)}\Big)\frac{\partial^2 f}{\partial y_i \partial y_j}\Big(\Gamma_{\phi(u)^{-1}}(z)\Big).$$

Law of Iterated Logarithms for McKean Vlasov SDEs

Assumption

Assume that $(\hat{\sigma}_u, \hat{b}_u) \to (\hat{\sigma}, \hat{b})$ uniformly on $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$ as $u \to \infty$. Further, assume that $\hat{\sigma}(y, \mu)$ is bounded and Lipschitz and that $\hat{b}(y, \mu)$ has monotone growth in y and is Lipschitz in μ.

Definition

Let $Z_u(t) = \Gamma_{\phi(u)}\big[Y(ut)\big]$ and note that $\mathcal{W}_u(t) = \frac{W(ut)}{\sqrt{u}}$ is a Brownian motion

$$dZ_u(t) = \frac{\hat{\sigma}_u\Big(Z_u(t), \mathcal{L}(Z_u(t))\Big)}{\sqrt{\log\log(u)}} d\mathcal{W}_u(t) + \hat{b}_u\Big(Z_u(t), \mathcal{L}(Z_u(t))\Big)dt$$

with Skeleton

$$d\Phi(h)(t) = \hat{\sigma}\Big(\Phi(h)(t), \delta_{\Phi(h)(t)}\Big)\dot{h}(t)dt + \hat{b}\Big(\Phi(h)(t), \delta_{\Phi(h)(t)}\Big)dt$$

Law of Iterated Logarithms for McKean Vlasov SDEs

$\alpha < 1/2$

Theorem

With probability 1, the set of paths $\{Z_u; u > 3\}$ is relatively compact in the Hölder topology C^α
and
its set of limit points coincides with $K = \{\Phi(h) : \frac{||\dot{h}||_2^2}{2} \leq 1\}$.

Proof of the Law of Iterated Logarithms

Proof

We prove 2 Propositions:

1. **Relatively compact.** For every $\varepsilon > 0$ there exists a.s. a positive real number $u_0(\omega)$ such that for every $u > u_0$

$$d_\alpha(Z_u(\omega), K) < \varepsilon$$

   where for $x \in C^\alpha([0,1])$ and $M \subset C^\alpha([0,1])$

$$d_\alpha(x, M) = \inf_{y \in M} ||x - y||_\alpha$$

2. **Limit point.** Let $g \in K$. Then $\forall \varepsilon > 0$, $\exists c_\varepsilon > 1$ such that $\forall c > c_\varepsilon$

$$\mathbb{P}\Big[||Z_{c^j} - g||_\alpha < \varepsilon \text{ for } j \text{ i.o.} \Big] = 1$$

Proof of the Law of Iterated Logarithms

Proof of Proposition (1) - Relative compactness

To prove the first Proposition, we argue ($c > 1$, $j \in \mathbb{N}$ and $j >> 1$)

$$\begin{aligned} d_\alpha(Z_u, K) \leq & d_\alpha(Z_{c^j}, K) \\ & + ||\Gamma_{\phi(u)} \circ \Gamma^{-1}_{\phi(c^j)}(Z_{c^j}) - Z_{c^j}||_\alpha \\ & + ||Z_u - \Gamma_{\phi(u)} \circ \Gamma_{\phi(c^j)^{-1}}(Z_{c^j})||_\alpha \end{aligned}$$

Then we use the following Lemma

Lemma

$\forall c > 1$, $\forall \varepsilon > 0$ then there exists a.s. $j_0(\omega) \in \mathbb{N}$ such that $\forall j > j_0$

$$d_\alpha(Z_{c^j}, K) < \varepsilon$$

Proof of Auxilliary Lemmas

Proof of Lemma

Let $K'_\varepsilon = \{g; d_\alpha(g, K) > \varepsilon\}$.

Then $\exists \delta > 0$ such that $\Delta(K'_\varepsilon) > 1 + 2\delta$.

$$\mathbb{P}\Big[Z_{c^j} \in K'_\varepsilon\Big] \leq \exp\Big(-(1+\delta)\log\log(c^j)\Big) \lesssim \frac{1}{j^{1+\delta}}$$

Hence by Borel Cantelli $\mathbb{P}\Big[Z_{c^j} \in K'_\varepsilon \text{ for } j \text{ i.o.}\Big] = 0$.

Proof of Law of Iterated Logarithms

Proof of Proposition (2) - The limit Points

Let $h \in H$ s.th. $\frac{\|h\|_2^2}{2} \leq 1 \Rightarrow \Phi(h) \in K$. Define (recall $\mathcal{W}_u(t) = W(ut)/\sqrt{u}$)

$$E_j = \left\{ \left\| \frac{\mathcal{W}_{c^j}(t)}{\sqrt{\log\log(c^j)}} - h \right\|_\infty \leq \beta \right\} \quad \text{and} \quad F_j = \left\{ \left\| Z_{c^j} - \Phi(h) \right\|_\alpha \leq \varepsilon \right\}$$

By the Hölder topology LDP:

$$\mathbb{P}[E_j] - \mathbb{P}[F_j] = \mathbb{P}\Big[E_j \cap F_j^c\Big] \leq \exp\Big(-2\log\log(c^j)\Big) \lesssim \frac{1}{j^2}$$

However, we also have

$$\sum_j \mathbb{P}\Big[E_j\Big] = \infty, \quad \sum_j \Big(\mathbb{P}[E_j] - \mathbb{P}[F_j]\Big) < \infty \quad \Rightarrow \quad \sum_j \mathbb{P}\Big[F_j\Big] = \infty.$$

Hence

$$\mathbb{P}\Big[||Z_{c^j} - \Phi(h)||_\alpha < \varepsilon \text{ i.o.}\Big] = 1. \qquad \square$$

Outline

Outlook and Further Extensions

- Takeway...
  - Existence & uniqueness results, regularity, LDPs in path space, Iterated logarithm law
  - Techniques able to *directly* deal with the MV-SDE law
- Outlook
  - (Topological characterization) Support Theorem for MV-SDEs
  - Existence and uniqueness for the fully coupled case

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Thank you