Ramer's finite co-dimensional forms and stochastic analysis.

David Elworthy, Warwick University

"Stochastic Analysis"
Durham 10th- 20th July 2017.

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Roald Ramer

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Ramer's Thesis

"Integration on infinite-dimensional manifolds", University of Amsterdam (1974).
Supervisor: N. Kuiper

Our aim is to construct an integration theory "à la de Rham" on infinite dimensional manifolds. The two main ingredients...are exterior differential forms and the local integration.... The two are related by the fact that the transition functions for differential forms of top dimension are exactly Radon Nikodym derivatives of transformation of the measure (at least in the oriented case).

Main Content

- Definition of stochastic integrals with anticipating integrands, relating them to divergences of H-vector fields.
- Transformation of integral formula for Gaussian measures under H-differentiable mappings of the underlying Abstract Wiener Spaces, extending earlier work by H.H.Kuo, Len Gross, Cameron-Martin,...
- Theory of finite co-dimensional forms on abstract Wiener manifolds, with exterior derivatives given using his stochastic integrals. Trivial de Rham cohomology, in general; non-trivial L^2-deRham cohomology.

Impact

Main output: *On nonlinear transformations of Gaussian measures.* J. Functional Analysis 15 (1974), 166187.

Extended by Shigeo Kusuoka and Ustunel & Zakai, in particular.

Equations with random right hand side

$$F : P \to \mathcal{E}$$

Suppose $\mathcal{E}$ has a probability measure μ, as in a nice SPDE. If F has a Borel measurable inverse a.s. get $F^*(\mu) := F_*^{-1}(\mu)$ on P, "law of the solution" to $F(x) = y$.

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What if F is not injective?

Locally-injective case

Suppose there exists $Z \subset \mathcal{E}$ with $\mu(Z) = 0$ such that every $x \in P$ with $F(x) \notin Z$ has an open neighbourhood mapped homeomorphically onto an open set in $\mathcal{E}$.

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Then there exists $F^*(\mu)$ on P which somehow represents the law of the solution to the random problem, but will not be a probability measure or even finite in general.

$$F^*(\mu)(P) = \text{ expected number of solutions of F(x)=z.}$$

Sometimes can give a sign to solutions and get a signed measure...

Deterministic degree theory for proper Fredholm maps

Smooth $F : P \to \mathcal{E}$ between (open sets of) Banach spaces or Banach manifolds, separable metrisable. Proper.

F is **Fredholm index** k if

$$k = \dim KerDF(x) - \dim CokerDf(x) \in \mathbb{Z}.$$

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Smale-Sard Theorem: Set of critical values Z is the complement of an open dense set.
Same proof yields $\mu(Z) = 0$ if μ non-degenerate Gaussian.
"Sard Property"

k=0

If "orientable" get degree: $\mathrm{Deg}(F) \in \mathbb{Z}$ as algebraic number of solutions of $F(x) = y$ for generic y. $\{$Elworthy-Tromba following Smale$\}$

Reduces to Leray-Schauder theory when $F(x) = x + \alpha(x)$ for α a compact mapping.

$k > 0$

Generically $F^{-1}(y)$ is a k-dimensional submanifold of P. The degree is unoriented cobordism class $\{$Smale$\}$, framed cobordism class $\{$Elworthy-Tromba$\}$. The latter may relate to homotopy groups of maps $S^{n+k} \to S^n$ for n-large.

Applied by Nirenberg, 1971, to some semi-linear boundary value problems.

Measure theoretic versions for $k = 0$

$$F : P \to \mathcal{E}$$

as above with $k = 0$. If μ has Sard property then for measurable $f : P \to \mathbf{R}$

$$\int_P f(x)\, dF^*(u)(x) = \int_{\mathcal{E}} \sum_{F(x)=y} f(x)\, d\mu(y).$$

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Corollary: If orientable then for $g : \mathcal{E} \to \mathbf{R}$

$$\int_P g(F(x))\mathrm{sgn\ DF(x)\ dF^*}(\mu)(\mathrm{x}) = \mathrm{Deg(F)} \int_{\mathcal{E}} \mathrm{g(y)\ d}\mu(\mathrm{y}).$$

$k > 0$

?

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?

Transverse measures?
$(\infty - k)$-forms?

$(\infty - k)$-forms: basic idea in finite dimensions

If $\dim M = n$ and M is orientable there is duality between k-forms and $n-k$-forms; essentially via

$$(dx^1 \wedge ... \wedge dx^k) \times (dx^{k+1} \wedge .... \wedge dx^n) \mapsto dx^1 \wedge ... \wedge dx^n.$$

More precisely a choice of never zero top form on M gives an isomorphism

$$\bigwedge^{n-k} T^*M \cong \bigwedge^{k} TM.$$

Under this, exterior differentiation on sections of $\bigwedge^{n-k} T^*M$ corresponds to the "divergence" on sections of $\bigwedge^k TM$ and sections of $\bigwedge^n T^*M$ correspond to functions.

$(\infty - k)$-forms: basic idea

Assume P is an "abstract Wiener manifold" , i.e. locally modelled on an AWS $\tilde{\mathcal{E}}^* \to \tilde{\mathcal{H}} \to \tilde{\mathcal{E}}$, interchange of charts of the form $x \mapsto x + \alpha(x)$ with α having finite dim'l range in $\tilde{\mathcal{E}}^*$.

Oriented if each $\det(I_{\tilde{\mathcal{H}}} + D^H \alpha(z)) > 0$.

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Oriented if each $\det(I_{\tilde{\mathcal{H}}} + D^H\alpha(z)) > 0$.

Such is determined by $F : P \to \mathcal{E}$ index k if $\mathcal{E}$ has AWS structure, taking $\tilde{\mathcal{E}} = \mathcal{E} \times \mathbf{R}^k$

Change of variable formula: Gross, Kuo, Ramer,...

For our AWS if U, V are open in $\mathcal{E}$ and $\mathrm{Id} + \alpha : \mathrm{U} \to \mathrm{V}$ diffeo then

$$\begin{aligned}(\mathrm{Id}+\alpha)^*(\gamma)_{\mathrm{x}} &= |\det(\mathrm{Id}+\mathrm{D}^{\mathrm{H}}\alpha(\mathrm{x}))| \exp\{\langle \mathrm{x}, \alpha(\mathrm{x})\rangle - \frac{1}{2}|\alpha(\mathrm{x})|^2\}\gamma \\ &= \det{}_2(\mathrm{Id}+\mathrm{D}^{\mathrm{H}}\alpha(\mathrm{x})) \\ &\quad \times \exp\{-(\text{“}\langle \alpha(x), x\rangle - \mathrm{trace}\ \mathrm{D}^{\mathrm{H}}\alpha(\mathrm{x})\text{”}) - \frac{1}{2}|\alpha(\mathrm{x})|^2\}\gamma\end{aligned}$$

$\therefore$ need more than just forms. The “$(\infty - k)$-volumes” $V^{\infty-k}$ are sections of $\mathcal{P} \bigotimes \wedge^{\infty-k}\mathcal{H}^*$. They can be integrated over k-codimensional submanifolds or wedged with an H k-form to give a volume form i.e. in V^{∞}.

Pull backs

For $F : P \to \mathcal{E}$ with index $k \geq 0$ and $\mathcal{E}$ with Gaussian γ.

Assume orientable.

Let $\omega \in V^\infty$ correspond to the Gaussian measure γ. Obtain $F^*(\omega) \in V^{\infty - k}(P)$.

Then for any H k-form ϕ there is the co-area formula:

$$\int_P F^*(\omega) \wedge \phi = \int_{\mathcal{E}} \left(\int_{F^{-1}(y)} \phi \right) \, d\gamma(y).$$

Example: periodic orbits, Kokarev & Kuksin 2006

$\mathcal{E}$ is a space of non-autonomous periodic vector fields $g : S^1 \times M \to TM$ on compact M.

Seek

$$u : S^1 \to M \ \text{ with } \frac{\mathrm{du}}{\mathrm{dt}} = \mathrm{g}(\mathrm{t}, \mathrm{u}(\mathrm{t})).$$

For this take

$$P = \{(u, g) \ \text{ with } \ \frac{\mathrm{du}}{\mathrm{dt}} = \mathrm{g}(\mathrm{t}, \mathrm{u}(\mathrm{t})), \mathrm{g} \in \mathcal{E}\}$$

Define $F : P \to \mathcal{E}$ as the projection. Then $k = 0$.

example: harmonic maps with force, Kokarev & Kuksin 2006

M and N finite dimensional, Riemannian,
$\mathcal{F} = \mathcal{F}(M, N)$ a space of maps from M to N;
E a suitable Banach space of ”non-autonomous” vector fields $v : M \to TN$ on N.

$$P := \{(f, v) \in \mathcal{F} \times E : \triangle f = v(x, f(x))\}$$

Take the projection $F : P \to E$. In certain cases it is a proper Φ_0-map, giving an orientable structure.

Integration along fibres

Consider $\pi : P \to M$ a submersion, so fibres $P_z := \pi^{-1}(z)$ have codimension n.

Given $\Psi \in V^{\infty - k}$ with $k \leq n = \dim M$, get an $(n-k)$-form $\pi_*(\Psi)$ on M by

$$\pi_*(\Psi)(v^1 \wedge \ldots \wedge v^{n-k}) = \int_{P_z} \iota_{\tilde{v}^1 \wedge \ldots \wedge \tilde{v}^{n-k}}(\Psi) \qquad v^j \in T_z M, \tilde{v}^j \text{ lift to P.}$$

Then

$$\int_P \Psi \wedge \pi^*(\phi) = \int_M \pi_*(\Psi) \wedge \phi.$$

Gauss-Bonnet-Chern-Poincaré-Hopf etc

- Euler characteristic of M is

$$\chi(M) = \sum_0^n (-1)^j \dim H^j(M; \mathbf{R}).$$

- Poincaré-Hopf: V a generic vector field, $Z_V =$ zero set, discrete, then

$$\text{algebraic number of zeros} = \chi(M).$$

- Euler class: If $p : E \to M$ is vector bundle rank $2q \leq n$ oriented. We have $e(E) \in H^{2q}(M : \mathbf{R})$.
  If $U : M \to E$ is a generic section then

$$\int_{Z_U} \phi = \int_M e(E) \wedge \phi \quad \text{for any closed} \quad (n-2q) - \text{form } \phi.$$

- $\int_M e(E) = \chi(M)$
- Generalized GBC: $e(E) =$ 'geometric Euler class' $e_g(E)$.

Example for “Gauss-Bonnet-Chern” Nicolaescu & Savale 2014, Nicolaescu PTRF 2016

$p : E \to M$ vector bundle fibre dimension $n - k$ with $k \leq n$.
γ non-degenerate Gaussian on ample Banach space of smooth sections $\mathcal{E}$ of E.
Define:

$$P = \{(U, z) \in \mathcal{E} \times M : U(z) = 0\} \quad \text{with } \mathrm{F} : \mathrm{P} \to \mathcal{E}.$$

Proper Fredholm, index k.
Get $F^*(\omega) \in V^{\infty - k}(P)$. Then for a k-form ϕ on M:

$$\begin{aligned} \int_{\mathcal{E}} \left(\int_{F^{-1}(U)} \pi^* \phi \right) d\gamma(U) &= \int_P F^*(\omega) \wedge \pi^* \phi \\ &= \int_M \pi_*(F^*(\omega)) \wedge \phi. \end{aligned}$$

$\pi : P \to M$ the projection.

G-B-C ctd

$$\begin{aligned}\int_{\mathcal{E}}\left(\int_{F^{-1}(U)}\pi^*\phi\right)\,d\gamma(U) &= \int_P F^*(\omega)\wedge\pi^*\phi \\ &= \int_M \pi_*(F^*(\omega))\wedge\phi.\end{aligned}$$

Thus if Z_U denotes the zero set of U:

$$\mathbf{E}\int_{Z_U}\phi = \int_M \pi_*(F^*(\omega))\wedge\phi.$$

A calculation yields $\pi_*(F^*(\omega))$ represents the geometric Euler class of E, when $n-k=2q$:

$$[\pi_*(F^*(\omega))] = e_g := [(-1/2\pi)^q\mathrm{Pf}(\Omega)] \in H^{n-k}(M).$$

Final Result of Nicolaescu PTRF 2016

$$\mathbf{E}\int_{Z_U}\phi = \int_M (-1/2\pi)^q \mathrm{Pf}(\Omega)\wedge\phi \quad \text{all k-forms } \phi$$

Consequently, for generic sections U of E:

$$\int_{Z_U}\phi = \int_M (-1/2\pi)^q \mathrm{Pf}(\Omega)\wedge\phi \quad \text{all closed k-forms } \phi.$$

To Calculate $\pi_*(F^*(\omega))$. Step 1

The Gaussian γ on $\mathcal{E}$ determines a Riemannian metric on E and metric connection $\breve{\nabla}$ so for $U \in \mathcal{E}$

$$\breve{\nabla}_- U : TM \to E.$$

Properties:

- $$\breve{\nabla}_v U = 0 \quad \text{if} \;\; v \in T_z M \;\; \& \;\; U \perp P_z.$$
- $$\mathbf{E}\breve{\nabla}_- U \wedge \breve{\nabla}_- U = \breve{\mathcal{R}} : \wedge^2 TM \to \wedge^2 E$$

  for $\breve{\mathbf{R}}$ the curvature operator of E.

El-LeJan-Li , Taniguchi Symposium Proc 1997 & LNM 1720 "redundant noise theory"; Nicolaescu & Savale 2014,

Step 2

Take $v^1, ..., v^{n-k} \in T_z M$.

A lift of $\tilde{v}^1 \wedge ... \wedge \tilde{v}^{n-k}$ at $U \in P_z$ is given by

$$(-\mathrm{ev}_z^*(\breve{\nabla}_{v^1} U), v^1) \wedge ... \wedge (-\mathrm{ev}_z^*(\breve{\nabla}_{v^{n-k}} U), v^{n-k}) \in \wedge^{n-k}(\mathcal{H} \times T_z M),$$

for $\mathrm{ev}_z : \mathcal{E} \to E_z$ the evaluation at $z \in M$.

Step 3

$F^*(\omega)$ is the restriction to P of the $(\infty - n) - \mathrm{volume}\ p_1^*(\omega)$ induced on $\mathcal{E} \times M$ by the projection $p_1 : \mathcal{E} \times M \to \mathcal{E}$.

Write $\mathcal{E} = P_z \oplus P_z^\perp \simeq P_z \oplus E_z$ and $\omega = \omega^0 \otimes \omega^\perp$. Then:

$$\begin{aligned}
\pi_*(F^*(\omega))(v^1 \wedge ... \wedge v^{n-k}) &= \int_{P_z} \iota_{(\mathrm{ev_z}^* \check{\nabla}_{\mathrm{v}^1} \mathrm{U} \wedge ...)}(\omega^\perp).\omega_U^0 \\
&= (2\pi)^{-q} \int_{P_z} \omega_{E_z}(\check{\nabla}_{v^1} U \wedge ...).\omega_U^0 \\
&= (2\pi)^{-q} \mathbf{E} \omega_{E_z}(\check{\nabla}_{v^1} U \wedge ...).
\end{aligned}$$

for ω_{E_z} the top form of E_z, since writing $U = U^0 + U^\perp$ gives $\check{\nabla}_{v^1} U = \check{\nabla}_{v^1} U^0$.

Step 4:Pfaffian of curvature

Liviu Nicolaescu "A stochastic Gauss-Bonnet- Chern formula" PTRF 2016, Also Adler & Jonathan Taylor.

$$\mathbf{E}\left\{\wedge^{n-k}\breve{\nabla}_{.}U\right\} = \mathrm{Pf}(\breve{\mathcal{R}})$$

where the Pfaffian of the curvature has local coordinate expression

$$\mathrm{Pf}(\breve{\mathcal{R}})_{1,\ldots,n-k} = c\sum_{\sigma}\sum_{\rho}\mathrm{sgn}(\sigma)\mathrm{sgn}(\rho)\breve{\mathcal{R}}_{\sigma(1)\sigma(2)}^{\rho(1)\rho(2)}\ldots\breve{\mathcal{R}}_{\sigma(n-1)\sigma(n)}^{\rho(n-1)\rho(n)}.$$

To believe this: Use Wick formula:
If $A_1, \ldots, A_{2p}$ are a Gaussian family, real valued, mean-zero. Then

$$\mathbf{E}\left\{A_1 A_2 \ldots A_{2p}\right\} = \sum_{\pi}\mathbf{E}\left\{A_{\pi(1)}A_{\pi(2)}\right\}\ldots\mathbf{E}\left\{A_{\pi(2p-1)}A_{\pi(2p)}\right\}$$

π such that $\pi(2r-1) < \pi(2r)$.

McKean-Singer formula

Let $\{P_t^*\}_{t\geq 0}$ be the heat semi-group on forms. Then

$$\chi_M = STr(P_t^*) \quad any \quad t > 0.$$

Supertraces

$$
\begin{aligned}
STr(P_t^*) : &= \sum_0^n (-1)^q \, Tr \; P_t^q \\
&= \sum_0^n (-1)^q \int_M trace \; k_t^q(x,x) dx \\
&= \int_M Str \; k_t^*(x,x) dx
\end{aligned}
$$

for fundamental solution $k_t^q(x,y) : \wedge^q T_x^* M \to \wedge^q T_y^* M$.

Kusuoka's approach, sort of

$$C_{id}Diff(M) = \{\xi : [0, T] \to Diff(M) \text{ with } \xi_0 = \text{identity}\}.$$

As before F is the projection:

$$P := \{(x, \xi) \in M \times C_{id}Diff(M) : \xi_T(x) = x\} \to C_{id}Diff(M).$$

It is proper Φ_0, and $\mathrm{Deg}(F) = \chi(M)$
Give $C_{id}Diff(M)$ the measure which is the law of a suitable flow of BM's on M

$$
\begin{aligned}
Deg\, F &= \int_M \int_{\{\xi_t^\beta(x)=x\}} \det(I - T_x\xi_t^\beta)\, d\nu_t^x(\xi_\cdot)\, p_t^\beta(x,x)\, dx \\
&= \int_M \int_{\{\xi_t^\beta(x)=x\}} \sum_{q=1}^{n} (-1)^q \mathrm{tr}(\wedge^q(T_x\xi_t))\ d\nu_t^x(\xi_\cdot) p_t^\beta(x,x)\, dx \\
&= StrP_t^{\beta,*} \qquad\qquad \text{for all } t>0 \text{ and } \beta>0,
\end{aligned}
$$

agreeing with McKean & Singer.

McKean -Singer for foliations

Consider only diffeos preserving a foliation, and stochastic flow of BM along the leaves. Get an F with index $n - k > 0$. Get GBC as before; should get McKean -Singer using Kusuoka approach +Ramer.

Some additional references

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That’s it!

THANK YOU