Ergodicity on Sublinear Expectation Spaces

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Talk at
LMS Durham Symposium on Stochastic Analysis
Durham, 10-20 July 2017
joint work with Huaizhong Zhao

Ergodic theory was one of the most important observations in mathematics made in the last century.

Consider a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$ and measure preserving map $\hat{\theta} : (\hat{\Omega}, \hat{\mathcal{F}}) \to (\hat{\Omega}, \hat{\mathcal{F}})$ i.e.

$$\hat{\theta}\hat{P} = \hat{P}.$$

The measurable dynamical system $\{\hat{\theta}^n\}_{n\in\mathbb{N}}$ is called *ergodic* if for any invariant set $A \in \hat{\mathcal{F}}$, i.e. $\hat{\theta}^{-1}A = A$,

$$\text{either } \hat{P}(A) = 0 \text{ or } \hat{P}(A) = 1.$$

Define a transformation operator U_1 from $L^2(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$ into itself,

$$U_1\xi(\hat{\omega}) = \xi(\hat{\theta}\hat{\omega}), \quad \xi \in L^2.$$

Then

$$\text{ergodic}$$

$$<=>$$

$$\text{the eigenvalue } 1 \text{ of } U_1 \text{ is simple}$$

$$<=>$$

$$\frac{1}{N}\sum_{n=0}^{N-1}\xi(\hat{\theta}^n\hat{\omega}) \to \int \xi d\hat{P} \ \ (L^2 \text{ or } a.s.), \ \forall \xi \in L^2$$

$$\text{(Birkhoff's SLLN)}.$$

Due to the spreading nature of random forcing, ergodicity is an important feature of stochastic systems.

Consider a Markovian cocycle stochastic dynamical system Φ on a separable Banach space $\mathbb{X}$. Define the transition probability function of $\Phi(t)x$ as for any $\Gamma \in \mathcal{B}$

$$P(t, x, \Gamma) = P\{\omega : \Phi(t, \omega)x \in \Gamma\}, \ t \in \mathbb{R}^+.$$

Let ρ be an invariant measure on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ i.e.

$$\int_{\mathbb{X}} P(t, x, \Gamma)\rho(dx) = \rho(\Gamma), \ t \in \mathbb{R}^+.$$

Markovian transition semigroup $P_t, t \geq 0$ is defined as

$$P_t\varphi(x) = \int_{\mathbb{X}} P(t, x, dy)\varphi(y), \ t \geq 0, \ \varphi \in B_b(\mathbb{X}),$$

where $B_b(\mathbb{X})$ is the set of all real bounded Borel functions on $\mathbb{X}$.

Denote $\mathbb{X}^{\mathbb{R}}$ the space of all $\mathbb{X}$-valued functions defined on $\mathbb{R}$, $\mathcal{B}(\mathbb{X}^{\mathbb{R}})$ is the smallest σ-field containing all cylindrical sets in $\mathbb{X}^{\mathbb{R}}$.

For any finite set $J = \{t_1, \cdots t_n\}$, define a probability measure P_J^ρ on $(\mathbb{X}^J, \mathcal{B}(\mathbb{X}^J))$, (c.f. Da Prato-Zabczyk 1996)

$$P_J^\rho(\Gamma) = \int_{\mathbb{X}} \rho(dx_1) \int_{\mathbb{X}} P_{t_2-t_1}(x_1, dx_2) \cdots \int_{\mathbb{X}} P_{t_n-t_{n-1}}(x_{n-1}, dx_n) I_\Gamma(x_1, \cdots, x_n)$$

By the Kolmogorov extension theorem, there exists a unique probability measure P^ρ on $(\Omega^*, \mathcal{F}^*) = (\mathbb{X}^{\mathbb{R}}, \mathcal{B}(\mathbb{X}^{\mathbb{R}}))$ generated from the invariant measure ρ on $\mathbb{X}$ such that

$$P^\rho(\{\omega^* : (\omega^*_{t_1}, \cdots, \omega^*_{t_n}) \in \Gamma\}) = P_J^\rho(\Gamma),\ \Gamma \in \mathcal{B}(\mathbb{X}^J).$$

For any $\omega^* \in \Omega^*$, denote its canonical process by $X_t^*(\omega^*) = \omega^*(t)$, which is a Markovian process. Define

$$\theta_t^* \omega^*(s) = \omega^*(t+s). \tag{1}$$

Then for any $t \in \mathbb{R}$,

$$\theta_t^* P^\rho = P^\rho.$$

The invariant measure ρ is called *ergodic* if the dynamical system $S^\rho := (\Omega^*, \mathcal{F}^*, \theta_t^*, P^\rho)$ is ergodic.

Well-known results

ρ is ergodic.

<=>

if $U_t\xi = \xi$, then ξ is constant.

<=>

if $P(t)\phi = \phi$, then ϕ is a constant.

<=>

a set $\Gamma \in \mathcal{B}(\mathbb{X})$ satisfies for all $t > 0$,

$$P_t I_\Gamma = I_\Gamma,\ \rho - a.e.$$

then either $\rho(\Gamma) = 0$ or $\rho(\Gamma) = 1$.

<=>

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T P(s,x,\Gamma)ds \to \rho(\Gamma),\ \text{in } L^2(\mathbb{X},\rho(dx)).$$

[stationary processes and random periodic processes]

Uncertainty in probability spaces/nonlinearity

Economists already observed “nonlinearities” in the behaviour of real world trading in financial market due to heterogeneity of expectation-formation processes.

Potentially biased beliefs of future price movements drive the decision of stock-market participants and create ambiguous volatility.

Coherent risk measure <=> sublinear expectation.

...

Stochastic analysis of sublinear expectation spaces

Stochastic analysis of nonlinear/sublinear expectations have been studied by Artzner-Delbaen-Eber-Heath (1999), Delbaen (2002), Follmer-Schied (2002), Peng (2006,2010) and lot of important stochastic analysis tools have been built especially by Peng (2010) on the observation of G-Brownian motion and associated stochastic analysis. Peng observed

G-diffusion => Fully nonlinear PDEs (Krylov et al)

Dynamics/ergodic theory on sublinear expectation spaces

Question:

How about dynamics? Does there exist an ergodic theory?

Sublinear expectation space

Let $(\hat{\Omega}, \hat{\mathcal{F}})$ be a measurable space, $L_b(\hat{\mathcal{F}})$ be the space of all $\hat{\mathcal{F}}$-measurable real-valued functions with $\sup_{\hat{\omega}\in\hat{\Omega}} |X(\hat{\omega})| < \infty$, $\hat{\mathcal{D}}$ be vector lattice of real valued functions defined on $\hat{\Omega}$ such that $1 \in \hat{\mathcal{D}}$ and $|X| \in \hat{\mathcal{D}}$ if $X \in \hat{\mathcal{D}}$.

Definition 1

(c.f. (Peng2010)) A sublinear expectation $\hat{\mathbb{E}}$ is a functional $\hat{\mathbb{E}} : \hat{\mathcal{D}} \to \mathbb{R}$ satisfying
(i) Monotonicity: $\hat{\mathbb{E}}[X] \geq \hat{\mathbb{E}}[Y], \text{ if } X \geq Y.$
(ii) Constant preserving: $\hat{\mathbb{E}}[c] = c, \text{ for } c \in \mathbb{R}.$
(iii) Sub-additivity: for each $X, Y \in \hat{\mathcal{D}}$, $\hat{\mathbb{E}}[X+Y] \leq \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y].$
(iv) Positive homogeneity: $\hat{\mathbb{E}}[\lambda X] = \lambda\hat{\mathbb{E}}[X], \text{ for } \lambda \geq 0.$
The triple $(\hat{\Omega}, \hat{\mathcal{D}}, \hat{\mathbb{E}})$ is called a sublinear expectation space.

The representation result (Artzner-Delbaen-Eber-Heath (1999), Delbaen (2002), Follmer-Schied (2002)) says that there exists a family of linear expectations $\{E_\theta : \theta \in \Theta\}$ defined on $\hat{\mathcal{D}}$ such that

$$\hat{\mathbb{E}}[X] = \sup_{\theta \in \Theta} E_\theta[X] \text{ for } X \in \hat{\mathcal{D}}.$$

Ergodicity of expectation preserving maps

Now we introduce a measurable transformation $\hat{\theta} : \hat{\Omega} \to \hat{\Omega}$ that preserves the sublinear expectation $\hat{\mathbb{E}}$, i.e.

$$\hat{\theta}\hat{\mathbb{E}} = \hat{\mathbb{E}}. \tag{2}$$

Here $\hat{\theta}\hat{\mathbb{E}}$ is defined as

$$\hat{\theta}\hat{\mathbb{E}}[X(\cdot)] = \hat{\mathbb{E}}[X(\hat{\theta}\cdot)] \ \text{ for any } X \in \hat{\mathcal{D}}.$$

Definition 2

Let $(\hat{\Omega}, \hat{\mathcal{D}}, \hat{\mathbb{E}})$ be a sublinear expectation space. An expectation preserving transformation $\hat{\theta}$ of $(\hat{\Omega}, \hat{\mathcal{D}}, \hat{\mathbb{E}})$ is called ergodic if any invariant measurable set $B \in \hat{\mathcal{F}}$ satisfies

$$\textit{either } \hat{\mathbb{E}}\mathrm{I}_B = 0 \textit{ or } \hat{\mathbb{E}}\mathrm{I}_{B^c} = 0.$$

Remark 3

The difference with the classical measure theoretical ergodic theory is that $\hat{\mathbb{E}}\mathrm{I}_B = 1$ *does not imply* $\hat{\mathbb{E}}\mathrm{I}_{B^c} = 0$ *as the sublinear expectation* $\hat{\mathbb{E}}$ *only satisfies*

$$\hat{\mathbb{E}}\mathrm{I}_B + \hat{\mathbb{E}}\mathrm{I}_{B^c} \geq 1. \tag{3}$$

In fact it is quite possible that $\hat{\mathbb{E}}\mathrm{I}_B = 1$ *and* $\hat{\mathbb{E}}\mathrm{I}_{B^c} = 1$. *However it is noted that* $\hat{\mathbb{E}}\mathrm{I}_B = 0$ *implies* $\hat{\mathbb{E}}\mathrm{I}_{B^c} = 1$ *and* $\hat{\mathbb{E}}\mathrm{I}_{B^c} = 0$ *implies* $\hat{\mathbb{E}}\mathrm{I}_B = 1$.

Definition 4

The functional $\hat{\mathbb{E}}[\cdot]$ is said to be strongly regular if for any $A_n \in \hat{\mathcal{F}}$, $A_n \downarrow \emptyset$, we have $\hat{\mathbb{E}}[I_{A_n}] \downarrow 0$.

A similar condition as strong regularity was introduced in Peng (2005).

Theorem 5

If $\hat{\theta} : \hat{\Omega} \to \hat{\Omega}$ is a measurable expectation preserving transformation of the sublinear expectation space $(\hat{\Omega}, \hat{\mathcal{D}}, \hat{\mathbb{E}})$, then the following four statements:

(i) The map $\hat{\theta}$ is ergodic;

(ii) If $B \in \hat{\mathcal{F}}$ and $\hat{\mathbb{E}}\mathrm{I}_{\hat{\theta}^{-1}B\Delta B} = 0$, then either $\hat{\mathbb{E}}\mathrm{I}_B = 0$ or $\hat{\mathbb{E}}\mathrm{I}_{B^c} = 0$;

(iii) For every $A \in \hat{\mathcal{F}}$ with $\hat{\mathbb{E}}\mathrm{I}_A > 0$, we have $\hat{\mathbb{E}}\mathrm{I}_{(\bigcup_{n=1}^{\infty} \hat{\theta}^{-n}A)^c} = 0$;

(iv) For every $A, B \in \hat{\mathcal{F}}$ with $\hat{\mathbb{E}}\mathrm{I}_A > 0$ and $\hat{\mathbb{E}}\mathrm{I}_B > 0$, there exists $n \in \mathbb{N}^+$ such that $\hat{\mathbb{E}}\mathrm{I}_{(\hat{\theta}^{-n}A \cap B)} > 0$;

have the following relations: (i)<=>(ii); (iii)=>(iv)=>(i).

Moreover, if $\hat{\mathbb{E}}$ is strongly regular, then (ii)=>(iii); so (i)<=>(ii)<=>(iii)<=>(iv).

Set the transformation operator $U_1 : \hat{\mathcal{D}} \to \hat{\mathcal{D}}$ by

$$U_1\xi(\hat{\omega}) = \xi(\hat{\theta}\hat{\omega}),\ \xi \in \hat{\mathcal{D}}.$$

Then expectation preserving of $\hat{\theta}$ is equivalent to

$$\hat{\mathbb{E}}[U_1\xi] = \hat{\mathbb{E}}[\xi], \quad \text{for any } \xi \in \hat{\mathcal{D}}.$$

Theorem 6

If $(\hat{\Omega}, \hat{\mathcal{D}}, \hat{\mathbb{E}})$ is a sublinear expectation space and the measurable map $\hat{\theta} : \hat{\Omega} \to \hat{\Omega}$ is expectation preserving, then the following statements are equivalent:

(i). The map $\hat{\theta}$ is ergodic;

(ii). Whenever $\xi \in L^1_{\mathbb{R}}(\hat{\mathbb{E}})$ $(or\ L^1_{\mathbb{C}}(\hat{\mathbb{E}}))$ is measurable and $U_1\xi = \xi$ quasi-surely, then ξ is constant quasi-surely.

Definition 7

A dynamical system $\hat{S} = \{\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{E}}, (\hat{\theta}^n)_{n\in\mathbb{N}}\}$ is said to satisfy the strong law of large numbers (SLLN) if

$$
\begin{aligned}
-\hat{\mathbb{E}}[-\xi] \quad &\le \quad \underline{\xi}(\hat{\omega}) := \liminf_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \xi(\hat{\theta}^n\hat{\omega}) \\
&\le \quad \limsup_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \xi(\hat{\theta}^n\hat{\omega}) =: \bar{\xi}(\hat{\omega}) \le \hat{\mathbb{E}}\xi \quad quasi-surely,
\end{aligned} \tag{4}
$$

for any $\xi \in L^1$. Here $\underline{\xi}(\hat{\omega})$ and $\bar{\xi}(\hat{\omega})$ satisfy $\underline{\xi}(\hat{\theta}\hat{\omega}) = \underline{\xi}(\hat{\omega})$ and $\bar{\xi}(\hat{\theta}\hat{\omega}) = \bar{\xi}(\hat{\omega})$ quasi-surely. Moreover equalities in all the three inequalities in (4) hold for ξ satisfying $\xi(\hat{\theta}\hat{\omega}) = \xi(\hat{\omega})$ quasi-surely.

Theorem 8

Assume $\hat{\mathbb{E}}$ is strongly regular, then $\hat{\theta}$ is ergodic implies the SLLN holds.
Conversely, if $\hat{\theta}$ satisfies SLLN, then the eigenvalue 1 of U_1 on L^1 is simple and $\hat{\theta}$ is ergodic. (strong regularity is not needed)

Sublinear Markov semigroups

Consider a measurable space $(\Omega, \mathcal{F})$. Let $(\Omega, \mathcal{D}, \mathbb{E})$ be a sublinear expectation space where $\mathbb{E}[\cdot]$ is a sublinear expectation on $\mathcal{D}$. Let $\xi \in (L_b(\mathcal{F}))^{\otimes d}$ be given. The nonlinear distribution of ξ under $\mathbb{E}[\cdot]$ is defined by

$$T[\varphi] := \mathbb{E}[\varphi(\xi)], \ \varphi \in L_b(\mathcal{B}(\mathbb{R}^d)).$$

This distribution $T[\cdot]$ is again a sublinear expectation defined on $L_b(\mathcal{B}(\mathbb{R}^d))$.

Consider a family of sublinear expectation parameterized by $t \in \mathbb{R}^+$:

$$T_t : L_b(\mathcal{B}(\mathbb{R}^d)) \to L_b(\mathcal{B}(\mathbb{R}^d)), \ \mathrm{t} \geq 0.$$

Definition 9

(Peng (2005)) The operator T_t is called a sublinear Markov semigroup if it satisfies
(m1) For each fixed $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^d$, $T_t[\varphi](x)$ is a sublinear expectation defined on $L_b(\mathcal{B}(\mathbb{R}^d))$.
(m2) $T_0[\varphi](x) = \varphi(x)$.
(m3) $T_t[\varphi](x)$ satisfies the following Chapman semigroup formula

$$(T_t \circ T_s)[\varphi] = T_{t+s}[\varphi], \ t, s \geq 0.$$

Invariant sublinear expectation

Definition 10

An invariant sublinear expectation $\tilde{T} : L_b(\mathcal{B}(\mathbb{R}^d)) \to \mathbb{R}$ is a sublinear expectation satisfying

$$(\tilde{T}T_s)(\varphi) = \tilde{T}(\varphi), \text{ for any } \varphi \in L_b(\mathcal{B}(\mathbb{R}^d)),$$

where $T_s, s \geq 0$ is a sublinear Markov semigroup.

The definition in the case of G-diffusion processes was given in Hu-Li-Wang-Zheng (2015).

G-Brownian motion on S^1

Consider a G-Brownian motion on the unit circle $S^1 = [0, 2\pi]$ defined by $X(t) = x + B(t) \bmod 2\pi$, where B is a one-dimensional G-Brownian motion such that $B(1)$ has normal distribution $N(0, [\underline{\sigma}^2, \overline{\sigma}^2])$. Here $\overline{\sigma}^2 \geq \underline{\sigma}^2 > 0$ are constants. For $\varphi \in C_{b,lip}(S^1)$, set

$$T_t\varphi(x) = u(t, x) = \mathbb{E}\varphi(X(t)). \tag{5}$$

Then u is a viscosity solution of the following fully nonlinear PDE (Peng (2006))

$$\frac{\partial}{\partial t}u = \frac{1}{2}\overline{\sigma}^2 u_{xx}^+ - \frac{1}{2}\underline{\sigma}^2 u_{xx}^-. \tag{6}$$

Then according to Krylov (1986,1987), when $t > 0$, $u(t,x)$ is $C^{1,2}$ in (t,x), thus a classical solution for any $t > 0$. In fact, we can extend the solution to the case when φ is bounded and measurable and obtain a classical solution for any $t > 0$.

Lemma 11

Assume $\underline{\sigma}^2 > 0$, for T_t defined in (5), we have for any $t > 0$, $A_n \in \mathcal{B}(S^1)$ such that $A_n \downarrow \emptyset$, we have $(T_t I_{A_n})(x) \downarrow 0$.

Idea of the proof: We will get

$$\begin{aligned} & (T_t I_{A_n})(x) \\ \leq\ & \mathrm{Leb}(A_n) \frac{1}{\sqrt{2\pi\underline{\sigma}^2 t}} \mathrm{e}^{\frac{(2\pi)^2}{2\underline{\sigma}^2 t}} \frac{1}{1 - e^{-\frac{\pi^2}{\underline{\sigma}^2 t}}} \\ \to\ & 0, \end{aligned}$$

since $\mathrm{Leb}(A_n) \to 0$ as $n \to \infty$.

Lemma 12

Assume $\underline{\sigma}^2 > 0$ and $\varphi \in L_b(\mathcal{B}(S^1))$, then for any $t > 0$, $u(t,x) = T_t\varphi(x)$ given by (5) is $C^{1,2}$ and a classical solution of (6).

Invariant expectation of G-Brownian motion on S^1

Theorem 13

Let $\overline{\sigma}^2 \geq \underline{\sigma}^2 > 0$. *Then*

$$\tilde{T}\varphi = \frac{1}{2\pi}\int_0^{2\pi}(T_\delta\varphi)(x)dx, \ \varphi \in L_b(\mathcal{B}(S^1)), \ \delta > 0. \tag{7}$$

is independent of $\delta > 0$ *and is the unique invariant expectation of* $T_t, \ t \geq 0$. *Moreover,* $T_t\varphi \to \tilde{T}\varphi$ *as* $t \to \infty$.

Remark 14

When $\varphi \in C([0, 2\pi])$, $\tilde{T}\varphi = \frac{1}{2\pi}\int_0^{2\pi}\varphi(x)dx$.

The canonical dynamical system on a sublinear expectation space from a Markov semigroup and invariant expectation

Define $\Omega^* = C(\mathbb{R}, \mathbb{R}^d)$, the space of all $\mathbb{R}^d$-valued continuous functions $(\omega_t^*)_{t\in\mathbb{R}}$ equipped with the distance

$$\rho(\omega^{*1}, \omega^{*2}) := \sum_{i=1}^{\infty} 2^{-i}[\max_{t\in[-i,i]} |\omega_t^{*1} - \omega_t^{*2}| \wedge 1]$$

with $\mathcal{F}^* = \mathcal{B}(C(\mathbb{R}, \mathbb{R}^d))$. Moreover, set $\hat{\Omega} = (\mathbb{R}^d)^{(-\infty,+\infty)}$ as the space of all $\mathbb{R}^d$-valued functions on $(-\infty, +\infty)$, $\hat{\mathcal{F}}$ is the smallest σ-field containing all cylindrical sets of $\hat{\Omega}$.
Given a nonlinear Markov semigroup $T_t, t \geq 0$ and the invariant sublinear expectation $\tilde{T}[\cdot]$, we can define the family of finite-dimensional nonlinear distributions of the canonical process $(\hat{\omega}_t)_{t\in\mathbb{R}^+} \in \hat{\Omega}$ under a sublinear expectation $\mathbb{E}^{\tilde{T}}[\cdot]$ on $((\mathbb{R}^d)^m, \mathcal{B}[(\mathbb{R}^d)^m])$ as follows.

For each integer $m \geq 1$, $\varphi \in L_b(\mathcal{B}[(\mathbb{R}^d)^m])$ and $t_1 < t_2 < \cdots < t_m$, we successively define functions $\varphi_i \in L_b(\mathcal{B}[(\mathbb{R}^d)^{(m-i)}])$, $i = 1, \cdots, m$, by

$$
\begin{aligned}
\varphi_1(x_1, \cdots, x_{m-1}) &:= T_{t_m - t_{m-1}}[\varphi(x_1, \cdots, x_{m-1}, \cdot)](x_{m-1}), \\
\varphi_2(x_1, \cdots, x_{m-2}) &:= T_{t_{m-1} - t_{m-2}}[\varphi_1(x_1, \cdots, x_{m-2}, \cdot)](x_{m-2}), \\
&\vdots \\
\varphi_{m-1}(x_1) &:= T_{t_2 - t_1}[\varphi_{m-2}(x_1, \cdot)](x_1).
\end{aligned}
$$

We now consider two different set-ups. The first one is to consider $\varphi_m := \tilde{T}[\varphi_{m-1}(\cdot)]$ and

$$\mathbb{E}^{\tilde{T}}[\varphi(\hat{\omega}_{t_1}, \hat{\omega}_{t_2}, \cdots, \hat{\omega}_{t_m})] := T^{\tilde{T}}_{t_1,t_2,\cdots,t_m}[\varphi(\cdot)] := \varphi_m.$$

In fact, $T^{\tilde{T}}_t = \tilde{T}$, for $t \geq 0$ and $T^{\tilde{T}}_{t_1,t_2,\cdots,t_m}[\varphi(\cdot)]$ is a sublinear expectation defined on $L_b(\mathcal{B}[(\mathbb{R}^d)^m])$. Denote

$$\tilde{\mathcal{E}}(\varphi(\hat{\omega}_0)) = \tilde{T}[\varphi], \text{ for any } \varphi \in L_b(\mathcal{B}(\mathbb{R}^d)),$$

then

$$\tilde{\mathcal{E}}(\varphi(\hat{\omega}_t)) = \tilde{\mathcal{E}}(\varphi(\hat{\omega}_0)) = \tilde{T}[\varphi], \text{ for any } \varphi \in L_b(\mathcal{B}(\mathbb{R}^d)),$$

The second set-up is to set $\varphi_m(x) := T_{t_1}[\varphi_{m-1}(\cdot)](x)$ for $t_1 \geq 0$ following Peng (2005). Then

$$\mathbb{E}^x[\varphi(\hat{\omega}_{t_1}, \hat{\omega}_{t_2}, \cdots, \hat{\omega}_{t_m})] := T^x_{t_1,t_2,\cdots,t_m}[\varphi(\cdot)] := \varphi_m(x),$$

and $T^x_{t_1,t_2,\cdots,t_m}[\cdot]$ defines a sublinear expectation.

Set

$$
\begin{aligned}
L_0(\hat{\mathcal{F}}): \quad = \quad & \{\varphi(\hat{\omega}_{t_1}, \hat{\omega}_{t_2}, \cdots, \hat{\omega}_{t_m}), \text{ for any } m \geq 1, \\
& t_1, t_2, \cdots, t_m \in \mathbb{R}, \varphi \in L_b(\mathcal{B}[(\mathbb{R}^d)^m])\}.
\end{aligned}
$$

It is clear that $L_0(\hat{\mathcal{F}})$ is a linear subspace of $L_b(\hat{\mathcal{F}})$. Denote $L_0^p(\hat{\Omega})$ that is the completion of $L_0(\hat{\mathcal{F}})$ under the norm $(\mathbb{E}^{\tilde{T}}[|\cdot|^p])^{\frac{1}{p}}$, $p \geq 1$. Define the space

$$
\begin{aligned}
Lip_{b,cyl}(\hat{\Omega}) := & \{\varphi(\hat{\omega}_{t_1}, \hat{\omega}_{t_2}, \cdots, \hat{\omega}_{t_m}), \text{ for any } m \geq 1, \\
& t_1, t_2, \cdots, t_m \in \mathbb{R}, \varphi \in C_{b,Lip}((\mathbb{R}^d)^m)\},
\end{aligned}
$$

and $L_G^p(\hat{\Omega})$ the completion of $Lip_{b,cyl}(\hat{\Omega})$ under the norm $||\cdot||_{L_G^p} = (\mathbb{E}^{\tilde{T}}[|\cdot|^p])^{\frac{1}{p}}$. From Denis-Hu-Peng (2011), we know that the completion of $C_b(\hat{\Omega})$ and $Lip_{b,cyl}(\hat{\Omega})$ under the norm $||\cdot||_{L_G^p}$ are the same, and $L_G^2(\hat{\Omega}) \subset L_0^2(\hat{\Omega})$.

Applying nonlinear Kolmogorov extension theorem (Peng (2005)), there exists a unique sub-linear expectation $\mathbb{E}^{\tilde{T}}$ on $L_0^1(\hat{\Omega})$ such that

$$\mathbb{E}^{\tilde{T}}[Y] = T^{\tilde{T}}_{t_1,t_2,\cdots,t_m}[\varphi(\cdot)],$$

for any $m \geq 1$, $t_1, t_2, \cdots, t_m \in \mathbb{R}$, $Y \in L_0(\hat{\mathcal{F}})$ with $Y(\hat{\omega}) = \varphi(\hat{\omega}_{t_1}, \hat{\omega}_{t_2}, \cdots, \hat{\omega}_{t_m})$, $\varphi \in L_b(\mathcal{B}[(\mathbb{R}^d)^m])$. Now we write the canonical process as

$$\hat{X}_t(\hat{\omega}) = \hat{\omega}(t),\ \hat{\omega} \in \hat{\Omega},\ t \in \mathbb{R}. \tag{8}$$

Now we introduce a group of invertible measurable transformation

$$\hat{\theta}_t\hat{\omega}(s) = \hat{\omega}(t+s),\ t, s \in \mathbb{R}.$$

Then it is easy to see that

$$\hat{\theta}_t\mathbb{E}^{\tilde{T}} = \mathbb{E}^{\tilde{T}}.$$

Ergodicity of invariant expectations

Theorem 15

Assume the Markov chain T_t has an invariant expectation $\tilde{T}$. Let $\hat{X}$ be the canonical processes on the canonical dynamical system $(\hat{\Omega}, \hat{\mathcal{D}}, \hat{\theta}_t, \mathbb{E}^{\tilde{T}})$. Assume for any $\varphi \in L_b(\mathcal{B}(\mathbb{R}^d))$, $|\varphi(\hat{X}(0))|^2$ have no mean-uncertainty. Then $\tilde{T}$ is ergodic if and only if the following statement is true:

if $T_t\varphi = \varphi$, $T_t(-\varphi) = -\varphi$, $\varphi \in L_b(\mathcal{B}(\mathbb{R}^d))$ for any $t \geq 0$, then φ is constant, $\tilde{T}$-q.s..

Proposition 16

Consider G-Brownian motion on the unit circle $S^1 = [0, 2\pi]$ with normal distribution $N(0, [\underline{\sigma}^2 t, \overline{\sigma}^2 t])$, where $\overline{\sigma}^2 \geq \underline{\sigma}^2 > 0$. The following results hold:
(i) The stationary process $\hat{X}$ defined in (8) has a continuous modification $\tilde{X}$.
(ii) For each $\varphi \in L_b(\mathcal{B}(S^1))$, $\varphi(\tilde{X}(0))$ has no mean-uncertainty with respect to the invariant expectation $\tilde{\mathcal{E}}$.
(iii) There exists a weakly compact family of probability measures $\mathcal{P}$ on $(\Omega^*, \mathcal{B}(\Omega^*))$.
(iv) The invariant expectation $\tilde{\mathcal{E}}$ is strongly regular.
(v) Define for each $\xi \in \mathcal{B}(\Omega^*)$, the upper expectation $\mathbb{E}^*[\xi] := \sup_{P \in \mathcal{P}} E_P[\xi]$. Then $\mathbb{E}^*$ is strongly regular.

Ergodicity of G-Brownian motion on S^1

Theorem 17

The invariant expectation of the G-Brownian motion on the unit circle $S^1 = [0, 2\pi]$ with normal distribution $N(0, [\underline{\sigma}^2 t, \overline{\sigma}^2 t])$, where $\overline{\sigma}^2 \geq \underline{\sigma}^2 > 0$ are constant, is ergodic.

Thank you!