The 2D Euler equations with random initial conditions

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Plan of the talk

- Review of results on deterministic 2D Euler equations
- Albeverio-Cruzeiro in the framework of more classical results
- White noise initial conditions
- Weak vorticity formulation
- Point vortex approximation
- Main results and perspectives

The 2D Euler equations

To simplify the exposition, let us consider the equations on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.
Euler equations for the pair $(u, p) =$ (velocity,pressure) read

$$\begin{aligned}\partial_t u + u \cdot \nabla u + \nabla p &= 0 \\ \operatorname{div} u &= 0.\end{aligned}$$

The vorticity $\omega = \partial_2 u_1 - \partial_1 u_2$ satisfies

$$\partial_t \omega + u \cdot \nabla \omega = 0.$$

We shall always consider the vorticity formulation.
Main formal invariants (among others):

$$\text{kinetic energy} = \frac{1}{2} \int_{\mathbb{T}^2} |u(x)|^2 \, dx$$

$$\text{enstrophy} = \int_{\mathbb{T}^2} \omega(x)^2 \, dx.$$

Review of results on 2D Euler equations

- existence and uniqueness, when $\omega_0 \in L^\infty$ (Wolibner, Yudovich)
- existence for $\omega_0 \in L^p$, $p \geq 2$ (uniqueness is open) (velocity $u \in W^{1,p}$)
- existence for $\sim$positive measures $\omega_0\left(dx\right)$ of class H^{-1} (Delort) (velocity $u \in L^2$)
- when
$$\omega_0\left(dx\right) = \sum \omega_i \delta_{X_0^i}$$
(which belongs to $H^{-1-} := \bigcap_{\epsilon>0} H^{-1-\epsilon}$; velocity $u \notin L^2$) local existence and uniqueness, global solutions for a.e. initial configuration (Marchioro-Pulvirenti)

Albeverio-Cruzeiro (CMP '90) result of existence for μ-a.e. $\omega_0 \in H^{-1-}$ (μ described below) may be seen as a natural continuation of this investigation.

Albeverio-Cruzeiro solutions as limit of more classical ones

Recall:

- existence and uniqueness, when $\omega_0 \in L^\infty$,
- existence for $\omega_0 \in L^p$, $p \geq 2$
- existence for $\sim$positive measures $\omega_0(dx)$ of class H^{-1}
- existence and uniqueness for a.e. $\omega_0(dx) = \sum \omega_i \delta_{X_0^i}$.

Nikolai Tzvetkov posed me the following question: are Albeverio-Cruzeiro solution the limit of more classical solutions?
By "classical solutions" he meant solutions with $\omega_0 \in L^\infty$. This question seems to be extremely difficult.
I will show they are limit of point vortices.

PDE formulation of Albeverio-Cruzeiro theory

Originally Albeverio-Cruzeiro theory has been formulated using *Fourier analysis*. For dispersive equations this is a very natural approach but in the framework of fluid dynamics it is not common.

We have made an effort to use classical fluid dynamic tools, to formulate and prove Albeverio-Cruzeiro result.

The main tool is taken from Delort (also Shochet, Poupaud, Di Perna and Majda, and others).

Delort studied the case when ω is a measure. He used a trick, called *weak vorticity formulation*, to deal with measure-valued vorticity. We shall show that the same trick allows one to treat Albeverio-Cruzeiro theory.

White noise initial conditions

We consider random initial conditions and precisely we assume that ω_0 is a *white noise* on $\mathbb{T}^2$.

White noise on $\mathbb{T}^2$ is by definition a distributional-valued stochastic process $\omega_0 : \Xi \to C^\infty\left(\mathbb{T}^2\right)'$ (here $(\Xi, \mathcal{F}, P)$ is a probability space) such that

$$\mathbb{E}\left[\langle \omega_0, \phi \rangle \langle \omega_0, \psi \rangle\right] = \langle \phi, \psi \rangle$$

for all $\phi, \psi \in C^\infty\left(\mathbb{T}^2\right)$. In more heuristic terms,

$$\mathbb{E}\left[\omega_0(x)\,\omega_0(y)\right] = \delta(x - y).$$

It will turn out that the solutions constructed below is a white noise *at every time* (similarly to the stochastic Burgers equation of KPZ theory).

The enstrophy measure

Let us call *enstrophy measure* the Gaussian Gibbs measure heuristically defined on vorticity fields "$\omega \in L^2\left(\mathbb{T}^2\right)$" as

$$\mu(d\omega) = \frac{1}{Z} \exp\left(-\frac{1}{2}\int_{\mathbb{T}^2} \omega^2 dx\right) d\omega.$$

It is rigorously defined as the unique Gaussian measure μ on $H^{-1-}\left(\mathbb{T}^2\right)$ such that

$$\int_{H^{-1-}} \langle \omega, \phi \rangle \langle \omega, \psi \rangle \mu(d\omega) = \langle \phi, \psi \rangle$$

for all $\phi, \psi \in C^\infty\left(\mathbb{T}^2\right)$; $\langle \omega, \phi \rangle$ denotes the dual pairing w.r.t. $L^2\left(\mathbb{T}^2\right)$. It is the law of white noise.

This measure is supported on $H^{-1-}\left(\mathbb{T}^2\right)$ but

$$\begin{aligned} \mu\left(H^{-1}\left(\mathbb{T}^2\right)\right) &= 0 \\ \mu\left(\mathcal{M}\left(\mathbb{T}^2\right)\right) &= 0. \end{aligned}$$

Weak vorticity formulation

We need to give a meaning to the nonlinear term of the equation

$$\partial_t \omega + u \cdot \nabla \omega = 0$$

when ω is a white noise. Trivial integration by parts on test functions $\phi \in C^\infty\left(\mathbb{T}^2\right)$

$$\int_{\mathbb{T}^2} \omega(x)\, u(x) \cdot \nabla \phi(x)\, dx \qquad \text{(formal notation)}$$

is not sufficient, since u is not regular enough (u is not even L^2).
Remark. It may seem there is an analogy with KPZ theory, but in fact here it is much easier.

Weak vorticity formulation

First, using Biot-Savart formula $u(x) = \int_{\mathbb{T}^2} K(x-y)\,\omega(y)\,dy$ (where $|K(x)| \sim \frac{1}{|x|}$ near $x = 0$) we rewrite

$$\int_{\mathbb{T}^2} \omega(x)\,u(x) \cdot \nabla\phi(x)\,dx = \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} K(x-y) \cdot \nabla\phi(x)\,\omega(x)\,\omega(y)\,dxdy.$$

The function $K(x-y) \cdot \nabla\phi(x)$ is still not regular enough. Then we symmetrize:

$$\begin{aligned} &= \frac{1}{2}\int_{\mathbb{T}^2}\int_{\mathbb{T}^2} K(x-y)\,(\nabla\phi(x) - \nabla\phi(y))\,\omega(x)\,\omega(y)\,dxdy \\ &=: \frac{1}{2}\int_{\mathbb{T}^2}\int_{\mathbb{T}^2} H_\phi(x,y)\,\omega(x)\,\omega(y)\,dxdy. \end{aligned}$$

The function $H_\phi(x,y)$ is bounded, smooth outside the diagonal, discontinuous along the diagonal. Can we integrate $H_\phi(x,y)$ against $\omega(x)\,\omega(y)\,dxdy$?

Delort, in his study of measure-valued solutions of class H^{-1}, proved that this is possible and allows one to prove global existence of solutions.

Weak formulation of the Euler equations for white noise

We pose the following preliminary question. Assume $\omega : \Xi \to C^{\infty}\left(\mathbb{T}^2\right)'$ is a white noise. Can we give a meaning to

$$\int_{\mathbb{T}^2}\int_{\mathbb{T}^2} H_\phi\left(x,y\right)\omega\left(x\right)\omega\left(y\right) dxdy?$$

Being $\omega \in H^{-1-}\left(\mathbb{T}^2\right)$, we have at least

$$\omega \otimes \omega \in H^{-2-}\left(\mathbb{T}^2 \times \mathbb{T}^2\right) \text{ with probability one}$$

The question is: can we define

$$\left\langle \omega \otimes \omega, H_\phi \right\rangle$$

in spite of the fact that H_ϕ is not of class $H^{2+}\left(\mathbb{T}^2 \times \mathbb{T}^2\right)$?

Main lemma

Lemma

If $\omega : \Xi \to C^{\infty}\left(\mathbb{T}^2\right)'$ *is a white noise and* $f \in H^{2+}\left(\mathbb{T}^2 \times \mathbb{T}^2\right)$ *is symmetric, then* $\int f(x,x)\,dx = \mathbb{E}\left[\langle \omega \otimes \omega, f \rangle\right]$ *and*

$$\mathbb{E}\left[\left|\langle \omega \otimes \omega, f \rangle - \int f(x,x)\,dx\right|^2\right] = 2\int\int f^2(x,y)\,dxdy.$$

Let us see the formal proof (becomes rigorous by smoothing the WN)

$$\begin{aligned} & \mathbb{E}\left[\left|\int\int f(x,y)\,\omega(x)\,\omega(y)\,dxdy\right|^2\right] \\ = & \int\int\int\int f(x,y)\,f\left(x',y'\right)\mathbb{E}\left[\omega(x)\,\omega(y)\,\omega\left(x'\right)\omega\left(y'\right)\right]dxdydx'dy' \end{aligned}$$

Proof of the main lemma

$$
\begin{aligned}
& E\left[\omega(x)\,\omega(y)\,\omega\left(x'\right)\omega\left(y'\right)\right] \\
= \;& \delta(x-y)\,\delta\left(x'-y'\right)+\delta\left(x-x'\right)\delta\left(y-y'\right)+\delta\left(x-y'\right)\delta\left(x'-y\right)
\end{aligned}
$$

by Gaussian rules for moments (Isserlis-Wick theorem). Hence

$$
\begin{aligned}
& \int\int\int\int f(x,y)\, f\left(x',y'\right)\mathbb{E}\left[\omega(x)\,\omega(y)\,\omega\left(x'\right)\omega\left(y'\right)\right] dxdydx'dy' \\
= \;& \int\int\int\int f(x,y)\, f\left(x',y'\right)\delta(x-y)\,\delta\left(x'-y'\right) dxdydx'dy' + \ldots \\
= \;& \int\int f(x,x)\, f\left(x',x'\right) dxdx' + \ldots
\end{aligned}
$$

and the proof becomes a simple computation.

Consequence of the main lemma

Theorem

Let $\omega : \Xi \to C^{\infty}\left(\mathbb{T}^2\right)'$ *be a white noise and* $\phi \in C^{\infty}\left(\mathbb{T}^2\right)$ *be given. Assume that* $H_{\phi}^{n} \in H^{2+}\left(\mathbb{T}^2 \times \mathbb{T}^2\right)$ *are symmetric and approximate* H_{ϕ} *in the following sense:*

$$\begin{aligned} \lim_{n\to\infty} \int\int \left(H_{\phi}^{n} - H_{\phi}\right)^2 (x,y)\, dxdy &= 0 \\ \lim_{n\to\infty} \int H_{\phi}^{n}(x,x)\, dx &= 0. \end{aligned}$$

Then the sequence of r.v.'s $\left\langle \omega \otimes \omega, H_{\phi}^{n} \right\rangle$ *is a Cauchy sequence in mean square. We denote its limit by*

$$\left\langle \omega \otimes \omega, H_{\phi} \right\rangle.$$

The limit is the same when $\lim_{n\to\infty} \int\int \left(H_{\phi}^{n} - \widetilde{H}_{\phi}^{n}\right)^2 (x,y)\, dxdy = 0$.

Definition of solution

Definition

We say that a measurable map $\omega_{\cdot} : \Xi \times [0, T] \to C^{\infty}\left(\mathbb{T}^2\right)'$ is a white noise solution of Euler equations if ω_t is a white noise at every time $t \in [0, T]$ and, for every $\phi \in C^{\infty}\left(\mathbb{T}^2\right)$, $t \mapsto \langle \omega_t, \phi \rangle$ is continuous and we have the identity a.s.

$$\langle \omega_t, \phi \rangle = \langle \omega_0, \phi \rangle + \int_0^t \left\langle \omega_s \otimes \omega_s, H_\phi \right\rangle ds.$$

Generalization

Assume $\omega : \Xi \to C^\infty\left(\mathbb{T}^2\right)'$ has the property that

$$\mathbb{E}\left[\Phi\left(\omega\right)\right] = \mathbb{E}\left[\rho\left(\omega_{WN}\right)\Phi\left(\omega_{WN}\right)\right]$$

where $\omega_{WN} : \Xi \to C^\infty\left(\mathbb{T}^2\right)'$ is a white noise and $\rho : H^{-1-}\left(\mathbb{T}^2\right) \to [0, \infty)$ satisfies

$$\mathbb{E}\left[\rho^2\left(\omega_{WN}\right)\right] < \infty.$$

This is equivalent to say that the law of ω is $<<\mu$ with density ρ and $\int \rho^2 d\mu < \infty$.

Then one can prove

$$\mathbb{E}\left[\left|\langle \omega \otimes \omega, \phi\rangle - \int \phi\left(x, x\right) dx\right|\right]^2 \leq \mathbb{E}\left[\rho^2\left(\omega_{WN}\right)\right] \cdot \int\int \phi^2\left(x, y\right) dxdy.$$

This allows one to define $\left\langle H_\phi, \omega_s \otimes \omega_s\right\rangle$ as above (limit in $L^1\left(\Xi\right)$, not mean square).

Generalization

Definition

Let $\rho : [0, T] \times H^{-1-}\left(\mathbb{T}^2\right) \to [0, \infty)$ satisfy $\int \rho_t^2 d\mu \leq C$. Let $\omega_\cdot : \Xi \times [0, T] \to C^\infty\left(\mathbb{T}^2\right)'$ be a measurable map such that ω_t has law $\rho_t \mu$ for every $t \in [0, T]$. We say that ω is a $\rho-$white noise solution of Euler equations if for every $\phi \in C^\infty\left(\mathbb{T}^2\right)$, $t \mapsto \langle \omega_t, \phi \rangle$ is continuous and we have the identity a.s.

$$\langle \omega_t, \phi \rangle = \langle \omega_0, \phi \rangle + \int_0^t \langle \omega_s \otimes \omega_s, H_\phi \rangle \, ds.$$

Everything extends to $\int \rho_t^p d\mu \leq C$ for some $p \geq 1$.

Random point vortex dynamics

Consider, for every $N \in \mathbb{N}$, the finite dimensional dynamics in $\left(\mathbb{T}^2\right)^N$

$$\frac{dX_t^{i,N}}{dt} = \sum_{j=1}^{N} \frac{1}{\sqrt{N}} \xi_j K\left(X_t^{i,N} - X_t^{j,N}\right) \qquad i = 1, ..., N$$

with initial condition $\left(X_0^{1,N}, ..., X_0^{N,N}\right) \in \left(\mathbb{T}^2\right)^N$, where K is Biot-Savard kernel on $\mathbb{T}^2$, with $K(0) := 0$ to neglect self-interaction.

Theorem (Marchioro-Pulvirenti)

Given $\xi_1, \ldots, \xi_N$, for $\otimes_N Leb_{\mathbb{T}^2}$- almost every $\left(X_0^{1,N}, ..., X_0^{N,N}\right) \in \Delta_N^c$, there is a unique solution $\left(X_t^{1,N}, ..., X_t^{N,N}\right)$ with the property that $\left(X_t^{1,N}, ..., X_t^{N,N}\right) \in \Delta_N^c$ for all $t \geq 0$.

Here

$$\Delta_N = \left\{ \left(x^1, ..., x^N\right) \in \left(\mathbb{T}^2\right)^N : x^i = x^j \text{ for some } i \neq j,\ i,j = 1, ..., n \right\}.$$

Random point vortex dynamics

Assume $\xi_1, \ldots, \xi_N$ are random intensities, distributed as $N(0,1)$, $X_0^{1,N}, \ldots, X_0^{N,N}$ are random and uniformly distributed, all independent of each other. Consider the measure-valued vorticity field

$$\omega_0^N = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \xi_n \delta_{X_0^n}.$$

Let us denote by Q_N its covariance operator, defined as $\langle Q_N \varphi, \psi \rangle = \mathbb{E}\left[\langle \omega_0^N, \varphi \rangle \langle \omega_0^N, \psi \rangle\right]$ for all $\varphi, \psi \in C^\infty\left(\mathbb{T}^2\right)$. We have $\mathbb{E}\left[\omega_0^N\right] = 0$ and

$$\langle Q_N \varphi, \psi \rangle = \int_{\mathbb{T}^2} \varphi(x) \psi(x) \, dx$$

(the same as white noise). One can prove that

$$\omega_0^N \overset{Law}{\rightharpoonup} \omega_{WN}$$

in $H^{-1-\delta}$ for every $\delta > 0$.

Random point vortex dynamics

Theorem

Consider the vortex dynamics with random intensities $(\xi_1, ..., \xi_N)$ *and random initial positions* $(X_0^1, ..., X_0^N)$ *as above. For a.e. value of* $(\xi_1, ..., \xi_N, X_0^1, ..., X_0^N)$ *the dynamics* $\left(X_t^{1,N}, ..., X_t^{N,N}\right)$ *is well defined in* Δ_N^c *for all* $t \geq 0$*, and the associated measure-valued vorticity* ω_t^N *satisfies the weak vorticity formulation. The stochastic process* ω_t^N *is stationary in time and space-homogeneous.*

Lemma

for all $f : \mathbb{T}^2 \times \mathbb{T}^2 \to \mathbb{R}$ *symmetric, bounded and measurable*

$$\begin{aligned} \mathbb{E}\left[\left\langle \omega_t^N \otimes \omega_t^N, f \right\rangle^2\right] &= \frac{3}{N} \int f^2(x,x)\,dx + \left(\int f(x,x)\,dx\right)^2 \\ &\quad +2 \int \int f^2(x,y)\,dxdy. \end{aligned}$$

Albeverio-Cruzeiro revisited

Theorem

i) There exists a probability space $(\Xi, \mathcal{F}, P)$ and a measurable map $\omega_\cdot : \Xi \times [0,T] \to C^\infty(\mathbb{T}^2)'$ such that $\omega_\cdot$ is a time-stationary white noise solution of Euler equations.
ii) The random point vortex system converges in law to this solution.

Theorem

Given $\rho_0 \in C_b\left(H^{-1-\delta}\left(\mathbb{T}^2\right)\right)$ for some $\delta > 0$, $\rho_0 \geq 0$, $\int \rho_0 d\mu = 1$, there exist a probability space $(\Xi, \mathcal{F}, P)$, a bounded measurable function $\rho : [0,T] \times H^{-1-}\left(\mathbb{T}^2\right) \to \left[0, \|\rho_0\|_\infty\right]$ and a measurable map $\omega_\cdot : \Xi \times [0,T] \to C^\infty(\mathbb{T}^2)'$ such that $\omega_\cdot$ is a $\rho-$white noise solution of Euler equations. It is also the limit in law of a suitable sequence of random point vortices.

Main steps in the proof (only WN case)

Let Q^N be the law of ω^N on

$$X := L^2\left(0, T; H^{-1-\delta}\left(\mathbb{T}^2\right)\right) \cap C\left(\left[0, T\right]; H^{-\gamma}\right) \tag{1}$$

where $\gamma > 3$. We prove that the family $\left\{Q^N\right\}_{N\in\mathbb{N}}$ is tight in X.
By Aubin-Lions and Ascoli-Arzelà, $Y \subset X$ is compact, where

$$Y := L^2\left(0, T; H^{-1-\delta/2}\left(\mathbb{T}^2\right)\right) \cap W^{1,2}\left(0, T; H^{-\gamma'}\left(\mathbb{T}^2\right)\right)$$

for $\gamma' < \gamma$.
Therefore, in order to prove that $\left\{Q^N\right\}_{N\in\mathbb{N}}$ is tight in X it is sufficient to prove that it is bounded in probability in Y.

Compactness in space

From stationarity of ω_t^N

$$\mathbb{E}\left[\int_0^T \left\|\omega_t^N\right\|_{H^{-1-\delta/2}}^2 dt\right] = \int_0^T \mathbb{E}\left[\left\|\omega_t^N\right\|_{H^{-1-\delta/2}}^2\right] dt = T\mathbb{E}\left[\left\|\omega_0^N\right\|_{H^{-1-\delta/2}}^2\right]$$

$$= T\mathbb{E}\left[\left\|\frac{1}{\sqrt{N}}\sum_{n=1}^N \xi_n \delta_{X_0^n}\right\|_{H^{-1-\delta/2}}^2\right] = \frac{T}{N}\sum_{n=1}^N \mathbb{E}\left[\xi_n^2 \left\|\delta_{X_0^n}\right\|_{H^{-1-\delta/2}}^2\right] = CT.$$

Hence the family $\left\{Q^N\right\}_{N\in\mathbb{N}}$ is bounded in probability in $L^2\left(0, T; H^{-1-\delta/2}\left(\mathbb{T}^2\right)\right)$.

Compactness in time

We use the equation in its weak vorticity formulation.
For all $\phi \in C^{\infty}\left(\mathbb{T}^2\right)$, $\partial_t \left\langle \omega_t^N, \phi \right\rangle = \left\langle \omega_t^N \otimes \omega_t^N, H_\phi \right\rangle$, hence

$$\begin{aligned} \mathbb{E}\left[\left|\partial_t \left\langle \omega_t^N, \phi \right\rangle\right|^2\right] &= \mathbb{E}\left[\left|\left\langle \omega_t^N \otimes \omega_t^N, H_\phi \right\rangle\right|^2\right] \\ &\leq C \left\| H_\phi \right\|_\infty^2 \leq C \left\| D^2 \phi \right\|_\infty^2 . \end{aligned}$$

With $\phi = e_k$ we get

$$\mathbb{E}\left[\left|\partial_t \left\langle \omega_t^N, e_k \right\rangle\right|^2\right] \leq C \left|k\right|^4$$

$$\mathbb{E}\left[\int_0^T \left\|\partial_t \omega_t^N\right\|_{H^{-\gamma'}}^2 dt\right] \leq C\mathbb{E}\left[\int_0^T \sum_k \left(1+\left|k\right|^2\right)^{-\gamma'} \left|k\right|^4 dt\right] < \infty$$

for $\gamma' > 3$. The family $\left\{Q^N\right\}_{N\in\mathbb{N}}$ is bounded in probability in $W^{1,2}\left(0, T; H^{-\gamma'}\left(\mathbb{T}^2\right)\right)$.

From Prohorov theorem, there exists $\left\{Q^{N_k}\right\}_{k\in\mathbb{N}}$ which converges weakly, in X, to a probability measure Q.
A process $\omega_\cdot$ with law Q is time-stationary and ω_t is white noise for every $t \in [0, T]$.
The passage to the limit is performed using Skorohod representation theorem.
The main work is to prove that

$$E\left[\left(\left|\int_0^t \left\langle H_\phi, \omega_s^{N_k} \otimes \omega_s^{N_k} \right\rangle ds - \int_0^t \left\langle H_\phi, \omega_s \otimes \omega_s \right\rangle ds\right|\right) \wedge 1\right] \to 0.$$

Here all the detailed informations proved above are used.

Regularization by noise?

Two directions:

1. regularization by random initial conditions
2. regularization by noise (additive or multiplicative)

For PDEs of dispersive type, after Bourgain, Burq, Tzvetkov, Ho and others, we know that random initial conditions is already a powerful tool to improve the deterministic theory.
For PDEs of fluid dynamics it is less clear that random initial conditions alone may have a strong effect.
Albeverio-Cruzeiro is an example of new *existence result*, due to random initial conditions. But uniqueness looks improbable.
Could noise, maybe jontly with random initial conditions, improve the theory?

2D Euler equations with stochastic transport term

Consider

$$d\omega + u \cdot \nabla \omega dt + \nabla \omega \circ dW = 0$$

$$\mathrm{div}\, u = 0, \qquad \nabla^{\perp} u = \omega$$

which originates by the substitution

$$u \longrightarrow u + \frac{\partial W}{\partial t}.$$

Here

$$W(t,x) := \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} \sigma_k(x) W_t^k$$

$$\sigma_k(x) \sim |k|^{-\alpha} e_k(x) \qquad e_k(x) \sim \frac{k^{\perp}}{|k|} e^{ik \cdot x}.$$

The exponent α corresponds to space-regularity of the noise.

No collapse of point vortices

When $\omega_0 = \sum_{i=1}^{N} \xi_i \delta_{X_0^i}$, the solution $\omega_0 = \sum_{i=1}^{N} \xi_i \delta_{X_t^i}$ of the SPDE above corresponds to the dynamics

$$dX_t^i = \sum_{j=1}^{N} \xi_j K\left(X_t^i - X_t^j\right) dt + \sum_k \sigma_k \left(X_t^i\right) dW_t^k.$$

Theorem (F.-Gubinelli-Priola SPA 2014)

There exist σ_k such that **for every** *initial condition $\left(X_0^{1,N}, ..., X_0^{N,N}\right) \in \Delta_N^c$, there is a unique solution $\left(X_t^{1,N}, ..., X_t^{N,N}\right)$ with the property that $\left(X_t^{1,N}, ..., X_t^{N,N}\right) \in \Delta_N^c$ for all $t \geq 0$.*

Random perturbations of white noise initial condition

Maybe, due to some analogy between vortex points and white noise solutions, we could expect some improvement due to noise also in the case of random i.c. with law μ. The first striking fact is:

Lemma

The enstrophy measure μ is infinitesimally invariant also for the stochastic Euler equations.

$$\begin{aligned} \mathcal{L}F(\omega) &= \frac{1}{2}\sum \langle \sigma_k \cdot \nabla\omega, \nabla_{L^2} \langle \sigma_k \cdot \nabla\omega, \nabla_{L^2} F(\omega)\rangle_{L^2} \rangle_{L^2} \\ &\quad + \left\langle \left(\nabla^{\perp}\right)^{-1} \omega \cdot \nabla\omega, \nabla_{L^2} F(\omega) \right\rangle_{L^2} \end{aligned}$$

$$\int \mathcal{L}F(\omega)\,\mu(d\omega) = 0.$$

Gradient estimates

The second remarkable fact is the gradient estimate

$$\int_0^t \int \left(\sum_k \langle \sigma_k \cdot \nabla \omega, \nabla_{L^2} \rho_s (\omega) \rangle_{L^2}^2 \right) \mu (d\omega)\, ds \leq C$$

expected to hold by the density ρ_t or for solutions of the backward Kolmogorov equation.
In other problems, gradient estimates have been a key tool to improve the deterministic theory. However, the problem here is very difficult because of:

- the degeneracy of $\mathcal{L}$
- the difficulty to control the drift by these degenerate diffusion terms.

Summary of results and questions

1. We have proved Albeverio-Cruzeiro (CMP '90) result using a classical PDE approach called *weak vorticity formulation*, plus some white noise analysis
2. We have proved that Albeverio-Cruzeiro is the *limit of random point vortices*
3. Tzvetkov question about the limit of L^∞ solutions is open
4. We have extended Albeverio-Cruzeiro result to some class of *absolutely continuous initial conditions*
5. Similarly to regularization by noise for point vortices, the effect of transport noise on Albeverio-Cruzeiro theory is under investigation.

Thank you for your attention