Mean-Field Games with Common Noise and Nonlinear SPDEs

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Main ideas taken from

V. Kolokoltsov and M. Troeva.
On the mean field games with common noise and the McKean-Vlasov SPDEs.
arXiv:1506.04594

Some related contributions, I

• O.Guéant, J-M. Lasry and P-L. Lions. Mean Field Games and Applications. Paris-Princeton Lectures on Mathematical Finance 2010. Lecture Notes in Math. 2003, Springer, Berlin, p. 205-266.

• A. Bensoussan, J. Frehse and Ph. Yam. Mean field games and mean field type control theory. Springer Briefs in Mathematics. Springer, New York, 2013.

A. Bensoussan, J. Frehse and Ph. Yam. On the interpretation of the Master Equation. Stochastic Process. Appl. 127:3 (2017), 2093 - 2137.

• P. E. Caines, "Mean Field Games", *Encyclopedia of Systems and Control*, Eds. T. Samad and J. Ballieul. Springer Reference 364780; DOI 10.1007/978-1-4471-5102-9 30-1, Springer-Verlag, London, 2014.

Some related contributions, II

- R. Carmona, F. Delarue and D. Lacker. Mean field games with common noise (2014). arXiv:1407.6181 or arXiv:1407.6181v2

- P. Cardaliaguet, F. Delarue, J.-M. Lasry and P.-L. Lions. The master equation and the convergence problem in mean field games. arXiv:1509.02505v1 [math.AP]

- Th. Kurtz and J. Xiong. Particle representations for a class of nonlinear SPDEs. Stoch.Proc. Appl. 83 (1999), 103-126.

Highlights I

Consider N agents, whose positions are governed by the system of SDEs

$$dX_t^i = b(t, X_t^i, \mu_t^N, u_t^i)\, dt + \sigma_{ind}(X_t^i) dB_t^i + \sigma_{com}(X_t^i) dW_t, \quad (1)$$

and who are trying to minimize certain integral and terminal costs, where

$$\mu_t^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{X_t^i}.$$

For simplicity all objects are one-dimensional: $x \in \mathbf{R}$ and independent standard BM $B^1, \cdots, B^N, W$.

Highlights II

• We formulate the MFG limit as a single quasi-linear deterministic infinite-dimensional partial differential second order backward equation.

• We prove that any its (regular enough) solution represents an $1/N$-Nash equilibrium for the initial N-player game.

• We use the method of stochastic characteristics to provide the link with the basic models of MFG with a major player.

• We develop two auxiliary theories of independent interest: sensitivity and regularity analysis for McKean-Vlasov SPDEs and the $1/N$-convergence rate for the propagation of chaos property of interacting diffusions.

Plan

• Formulation of the MFG consistency with common noise via a single infinite-dimensional PDE.
• Formulation of our main result, setting the link with McKean-Vlasov SPDEs and the propagation of chaos of interacting diffusions; formulation of the results on their rates of convergence as basic ingredient in the proof of the main Theorem 1.
• Independent of the above: Regularity and sensitivity of the McKean-Vlasov SPDEs, proving that the domain of the corresponding measure-valued Markov process contains an invariant sub-domain of smooth functionals.
• Consequence of the previous step: Rate of convergence $1/N$ for the propagation of chaos of interacting diffusions.

Setting, I

Dynamics of N players:

$$dX_t^i = b(t, X_t^i, \mu_t^N, u_t^i)\,dt + \sigma_{ind}(X_t^i)dB_t^i + \sigma_{com}(X_t^i)dW_t,$$

$$\mu_t^N = \frac{1}{N}\sum_{i=1}^{N}\delta_{X_t^i}.$$

Payoffs

$$V_{[t,T]}^i(x) = \mathbf{E}\left[\int_t^T J(s, X_s^i, \mu_s^N, u_s^i)\,ds + V_T(X_T^i, \mu_T^N)\right],$$

For simplicity, b is smooth in x, μ and linear in u:

$$b(t, x, \mu, u) = b_1(t, x, \mu) + b_2(t, x, \mu)u.$$

Setting, II

It is known that, for fixed common functions $u_t^i(X_t^i) = u_t(X_t^i)$ (and appropriate regularity assumptions) the system of N SDEs above is well-posed and μ_t^N converge, as $N \to \infty$, to the unique solution μ_t of the McKean-Vlasov SPDE:

$$d(\phi, \mu_t) = (L[t, \mu_t, u_t]\phi, \mu_t)\, dt + (\sigma_{com}(.)\nabla\phi, \mu_t)\, dW_t, \quad (2)$$

$$L[t, \mu_t, u_t]\phi(x) = \frac{1}{2}(\sigma_{ind}^2 + \sigma_{com}^2)(x)\frac{\partial^2\phi}{\partial x^2} + b(t, x, \mu_t, u_t(x))\frac{\partial\phi}{\partial x}$$

(written in the weak form).
Identify measures with their densities (with respect to Lebesgue), the strong form of the above is

$$d\mu_t = L'[t, \mu_t, u_t]\mu_t\, dt - \nabla(\sigma_{com}(.)\mu_t)\, dW_t,$$

$$L'[t, \mu, u_t]\nu = \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[(\sigma_{ind}^2 + \sigma_{com}^2)\mu)\right] - \frac{\partial}{\partial x}[b(t, x, \mu, u_t)\mu].$$

Setting, III

Let us mention directly that in our approach it is more convenient to work with the equivalent Stratonovich differentials representation:

$$d(\phi, \mu_t) = (L_{St}[t, \mu_t, u_t]\phi, \mu_t)\, dt + (\sigma_{com}(.)\nabla\phi, \mu_t) \circ dW_t,$$

$$L_{St}[t, \mu_t, u_t]\phi(x) = \frac{1}{2}\sigma_{ind}^2(x)\frac{\partial^2\phi}{\partial x^2}$$
$$+[b(t, x, \mu_t, u_t(x)) - \frac{1}{2}\sigma_{com}\sigma'_{com}(x)]\frac{\partial\phi}{\partial x}.$$

Setting, IV

For fixed N, if all players, apart from the first one, are using the same control $u_{com}(t, x, \mu)$, the optimal payoff for the first player is found from the HJB equation for the above diffusion (where we denote X^1 by x):

$$\frac{\partial V}{\partial t} + \inf_u \left[b(t, x, \mu, u) \frac{\partial V}{\partial x} + J(t, x, \mu, u) \right] + \frac{1}{2}(\sigma_{ind}^2 + \sigma_{com}^2)(x) \frac{\partial^2 V}{\partial x^2}$$

$$+ \sum_{j \neq 1} b(t, x_j, \mu, u_{com}(t, x_j, \mu)) \frac{\partial V}{\partial x_j} + \frac{1}{2}(\sigma_{ind}^2 + \sigma_{com}^2)(x_j) \frac{\partial^2 V}{\partial x_j^2}$$

$$+ \sum_{j \neq 1} \sigma_{com}(x) \sigma_{com}(x_j) \frac{\partial^2 V}{\partial x_1 \partial x_j} + \sum_{1 < i < j} \sigma_{com}(x_i) \sigma_{com}(x_j) \frac{\partial^2 V}{\partial x_i \partial x_j} = 0.$$

Functionals on particles positions and on measures

Identification of symmetric functions f on X^N with the functionals $F = F_f$ on $\mathcal{P}_N(\mathbf{R})$ via the equation

$$f(x_1, \cdots, x_N) = F_f[(\delta_{x_1} + \cdots + \delta_{x_N})/N].$$

allows one to recalculate the equations on f in terms of $F = F_f$ on measures by using the the following formulas for the differentiation of functionals on measures: for $\mu = h(\delta_{x_1} + \cdots + \delta_{x_N})$ with $h = 1/N$

$$\frac{\partial}{\partial x_j} F(\mu) = h \frac{\partial}{\partial x_j} \frac{\delta F(\mu)}{\delta \mu(x_j)},$$

$$\frac{\partial^2}{\partial x_j^2} F(\mu) = h \frac{\partial^2}{\partial x_j^2} \frac{\delta F(\mu)}{\delta \mu(x_j)} + h^2 \frac{\partial^2}{\partial y \partial z} \left. \frac{\delta^2 F(\mu)}{\delta \mu(y) \delta \mu(z)} \right|_{y=z=x_j},$$

$$\frac{\partial^2}{\partial x_i \partial x_j} F(\mu) = h^2 \frac{\partial^2}{\partial x_i \partial x_j} \frac{\delta^2 F(\mu)}{\delta \mu(x_i) \delta \mu(x_j)}, \quad i \neq j.$$

Setting, V

Hence, in the limit $(\delta_{x_1} + \cdots + \delta_{x_N})/N \to \mu_t$, the above HJB turns to the limiting HJB equation

$$\frac{\partial V}{\partial t} + \inf_u \left[b(t,x,\mu,u)\frac{\partial V}{\partial x} + J(t,x,\mu,u) \right] + \frac{1}{2}(\sigma_{ind}^2 + \sigma_{com}^2)(x)\frac{\partial^2 V}{\partial x^2}$$

$$+ \int_{\mathbf{R}} \left([b(t,.,\mu,u_{com}(t,.,\mu))\nabla + \frac{1}{2}(\sigma_{com}^2(.) + \sigma_{ind}^2(.))\nabla^2] \frac{\delta V}{\delta \mu(.)} \right)(y)$$

$$+ \frac{1}{2} \int_{\mathbf{R}^2} \sigma_{com}(y)\sigma_{com}(z) \frac{\partial^2}{\partial y \partial z} \frac{\delta^2 V}{\delta \mu(y) \delta \mu(z)} \mu(dy)\mu(dz).$$

$$+ \int \sigma_{com}(x)\sigma_{com}(y) \frac{\partial^2}{\partial x \partial y} \frac{\delta V(x,\mu)}{\delta \mu(y)} \mu(dy) = 0.$$

If J is convex, the infimum here is achieved on the single point

$$\hat{u}_{ind}(t,x,\mu) = \left(\frac{\partial J}{\partial u} \right)^{-1} \left(b_2(t,x,\mu_t) \frac{\partial V}{\partial x} \right).$$

MFG consistency, I

The difference with the games without common noise: for games without noise, one expects to get a deterministic curve μ_t in the limit of large N, so that in the limit, each player should solve a usual optimization problem for a diffusion in $\mathbf{R}$. Here the limit is stochastic, and thus even in the limit the optimization problem faced by each player is an optimization with respect to an infinite-dimensional, in fact measure-valued, process.

As a result: instead of a pair of coupled forward-backward equations we have now one single infinite-dimensional HJB equation. Namely, for any curve $u_{com}(t,x,\mu)$ (defining Λ_{lim}), we should solve the above HJB with a given terminal condition leading to the optimal control $\hat{u}_{ind}$. The key *MFG consistency requirement* is now

$$\hat{u}_{ind}(t,x,\mu) = u_{com}(t,x,\mu).$$

MFG consistency, II

Equivalently, the MFG consistency can be encoded into a single quasi-linear deterministic infinite-dimensional partial differential second order backward equation on the function $V(t, x, \mu)$:

$$\frac{\partial V}{\partial t}(t, x\mu) + \left[b(t, x, \mu, u) \frac{\partial V}{\partial x} + J(t, x, \mu, u) \right] + \frac{1}{2}(\sigma_{ind}^2 + \sigma_{com}^2)(x) \frac{\partial^2 V}{\partial x^2}$$

$$+ \frac{1}{2} \int_{\mathbf{R}^2} \sigma_{com}(y) \sigma_{com}(z) \frac{\partial^2}{\partial y \partial z} \frac{\delta^2 V(t, x, \mu)}{\delta \mu(y) \delta \mu(z)} \mu(dy) \mu(dz)$$

$$+ \int \left(\left[b(t, ., \mu, u) \nabla + \frac{1}{2}(\sigma_{ind}^2 + \sigma_{com}^2)(.) \nabla^2 \right] \frac{\delta V(t, x, \mu)}{\delta \mu(.)} \right) (y) \mu(dy$$

$$+ \int \sigma_{com}(x) \sigma_{com}(y) \frac{\partial^2}{\partial x \partial y} \frac{\delta V(x, \mu)}{\delta \mu(y)} \mu(dy) = 0,$$

where everywhere

$$u(t, x, \mu) = (\partial J / \partial u)^{-1} (b_2(t, x, \mu) \frac{\partial V}{\partial x}(t, x, \mu))$$

MFG consistency, III

If

$$J(t,x,\mu,u) = \frac{1}{2}u^2, \quad b_2(t,x,\mu) = 1,$$

this equation simplifies to

$$\frac{\partial V}{\partial t} + \left[b\left(t,x,\mu,\frac{\partial V}{\partial x}\right)\frac{\partial V}{\partial x} + \frac{1}{2}\left(\frac{\partial V}{\partial x}\right)^2 \right] + \frac{1}{2}(\sigma_{ind}^2 + \sigma_{com}^2)(x)\frac{\partial^2 V}{\partial x^2}$$

$$+\frac{1}{2}\int_{\mathbf{R}^2} \sigma_{com}(y)\sigma_{com}(z)\frac{\partial^2}{\partial y \partial z}\frac{\delta^2 V(t,x,\mu)}{\delta\mu(y)\delta\mu(z)}\mu(dy)\mu(dz)$$

$$+\int \left(\left[b(t,.,\mu,\nabla V)\nabla + \frac{1}{2}(\sigma_{ind}^2 + \sigma_{com}^2)(.)\nabla^2 \right] \frac{\delta V(t,x,\mu)}{\delta\mu(.)} \right)(y)\mu(d$$

$$+\int \sigma_{com}(x)\sigma_{com}(y)\frac{\partial^2}{\partial x \partial y}\frac{\delta V(x,\mu)}{\delta\mu(y)}\mu(dy) = 0.$$

More general: Argmax in convex Hamiltonians

$$H(p) = \max_{x \in X}(xp - U(x)), \quad p \in \mathbf{R}^d,$$

$$\hat{x}(p) = argmax\,(xp - U(x)) = ?$$

Theorem *Let X be a convex compact with a smooth boundary and $U(x)$ a strictly convex twice continuously differentiable function, so that*

$$\left(\frac{\partial^2 U}{\partial x^2}(x)\xi, \xi\right) \geq a(\xi, \xi)$$

for a constant a and all x and ξ, and such that $x = 0$ is the point of the global minimum of U. Then $\hat{x}(p) : \mathbf{R}^d \to X$ is a well defined (globally) Lipshitz continuous function.

Link with the usual MFG, I

To link with the usual MFG, let us notice that for the case without common noise, our basic equation above turns to

$$\frac{\partial V}{\partial t}(t,x,\mu)+\left[b\left(t,x,\mu,\frac{\partial V}{\partial x}\right)\frac{\partial V}{\partial x}+\frac{1}{2}\left(\frac{\partial V}{\partial x}\right)^2\right]+\frac{1}{2}\sigma_{ind}^2(x)\frac{\partial^2 V}{\partial x^2}$$

$$+\int\left(\left[b(t,.,\mu,\nabla V)\nabla+\frac{1}{2}\sigma_{ind}^2(.)\nabla^2\right]\frac{\delta V(t,x,\mu)}{\delta\mu(.)}\right)(y)\mu(dy)=0,$$

giving a single-equation approach to usual MFG.

Link with the usual MFG, II

Simple (abstract) explanation of the link between usual MFG and common noise: if (x_t, μ_t) is a controlled Markov process (not necessarily measure-valued), optimal payoff is defined via the corresponding HJB on a function $V(x, \mu)$ (corresponds to our general common noise case).
If the evolution of the coordinate μ_t is deterministic and does not depend on x and its control, one can (alternatively and equivalently) first solve this deterministic equation on μ (usual forward part of the basic MFG) and then substitute it in the basic HJB to get the equation on $V(t, x)$ only, with μ_t included in the time dependence (usual backward part of the basic MFG).
This decomposition into forward-backward system is not available in general.

Three basic questions of MFG in our setting

The MFG methodology suggests that for large N the optimal behavior of players arises from the control $\hat{u}$ satisfying the basic consistency condition.

To justify this claim one is confronted essentially with the 3 problems:

MFG1): Prove well-posedness of (or at least the existence of the solution to) the main infinite-dimensional HJB;

MFG2): Analyze the Nash equilibria of the initial N-player game and prove that these equilibria (or at least their subsequence) converge, as $N \to \infty$, to a solution of the MFG consistency equation; assess the convergence rates;

MFG3): Show that a solution to the consistency problem provides a profile of symmetric strategies $\hat{u}_t(x)$, which is an ϵ-Nash of the N-player game; estimate the error $\epsilon(N)$.

Our main objective

Question MFG3) with the error estimate of order $\epsilon(N) \sim 1/N$. Our approach is based on interpreting (by means of Ito's formula) the common noise as a kind of binary interaction of agents (in addition to the usual mean-field interaction of the standard situation without common noise) and then reducing the problem to the sensitivity analysis for McKean-Vlasov SPDE.

Regularity spaces, I

Standard:
Let $C^k = C^k(\mathbf{R})$ denote the Banach space of functions with all derivatives up to order k bounded continuous, L_1, L_∞ the space of integrable and bounded measurable functions on $\mathbf{R}$, $H_1^1(X)$ the Sobolev space of integrable functions such that its generalized derivative is also integrable.

Special (exotic):
Let $C^{k\times k}(\mathbf{R}^2)$ denote the space of functions f on $\mathbf{R}^2$ such that the partial derivatives

$$\frac{\partial^{\alpha+\beta} f}{\partial x^\alpha \partial y^\beta} \quad \alpha, \beta : \alpha \leq k, \beta \leq k,$$

belong to $C(\mathbf{R}^2)$.

Regularity spaces, II

Recall: $F(\mu)$ on $\mathcal{M}^{sign}(\mathbf{R}^d)$, the variational derivative is defined as

$$\frac{\delta F}{\delta \mu(x)}[\mu] = \frac{d}{dh}|_{h=0} F(\mu + h\delta x).$$

Here $\mathcal{M}^{sign}(\mathbf{R}^d)$ is the Banach space of signed measures on $\mathbf{R}^d$, $\mathcal{M}_\lambda^{sign}(\mathbf{R}^d)$ its subset of total variation bounded by λ. We abbreviate $\mathcal{M}^{sign} = \mathcal{M}^{sign}(\mathbf{R})$, $\mathcal{M}_\lambda^{sign} = \mathcal{M}_\lambda^{sign}(\mathbf{R})$. Let $C^k(\mathcal{M}_\lambda^{sign})$ denote the space of functionals such that the kth order variational derivatives are well defined and represent continuous functions. It is a Banach space with the norm

$$\|F\|_{C^k(\mathcal{M}_\lambda^{sign})} = \sum_{j=0}^{k} \sup_{x_1,\cdots,x_j,\mu\in\mathcal{M}_\lambda^{sign}} \left| \frac{\delta^j F}{\delta\mu(x_1)\cdots\delta\mu(x_j)} \right|.$$

Regularity spaces, III

Let $C^{k,l}(\mathcal{M}_\lambda^{sign})$ denote the subspace of $C^k(\mathcal{M}_\lambda^{sign})$ such that all derivatives up to order k have continuous bounded derivatives up to order l as functions of their spatial variables. It is a Banach space with the norm

$$\|F\|_{C^{k,l}(\mathcal{M}_\lambda^{sign})} = \sum_{j=0}^{k} \sup_{\mu \in \mathcal{M}_\lambda^{sign}} \left\| \frac{\delta^j F}{\delta\mu(.) \cdots \delta\mu(.)}[\mu] \right\|_{C^l(\mathbf{R}^j)} .$$

Finally, let $C^{2,k\times k}(\mathcal{M}_\lambda^{sign})$ be the space of functionals with the norm

$$\|F\|_{C^{2,k\times k}(\mathcal{M}_\lambda^{sign})} = \sup_{\mu \in \mathcal{M}_\lambda^{sign}} \left\| \frac{\delta^2 F}{\delta\mu(.)\delta\mu(.)} \right\|_{C^{k\times k}(\mathbf{R}^2)} .$$

As we are interested mostly in probability measures, we shall usually tacitly assume $\lambda = 1$ for these spaces.

Regularity spaces, IV

As the derivatives of measures are not always measures (say, the derivative of δ_x is δ'_x), to study the derivatives of the nonlinear evolutions one needs the spaces dual to the spaces of smooth functions. Namely, for a generalized function (distribution) ξ on $\mathbf{R}^d$ we say that it belongs to the space $[C^k(\mathbf{R}^d)]'$ if the norm

$$\|\xi\|_{[C^k(\mathbf{R}^d)]'} = \sup_{\phi:\|\phi\|_{C^k(\mathbf{R}^d)}\le 1} |(\xi, \phi)|$$

is finite. Exam. 1:

$$\|\delta_x^{(k)}\|_{[C^k(\mathbf{R})]'} = 1.$$

Exam. 2: For functions, $[C(\mathbf{R}^d)]'$-norm coincides with the L_1 norm.
Exam. 3: The spatial derivative of the heat kernel has L_1-norm of order $t^{-1/2}$ for small t, but its $[C'(\mathbf{R}^d)]'$-norm is uniformly bounded.

Main Theorem

Theorem 1. Let $V(t,x,\mu)$ be a solution to the main MFG problem.
Assumption A): Assume $\sigma_{ind}, \sigma_{com} \in C^3(\mathbf{R})$ and are positive functions never approaching zero. Assume

$$b(t,x,.,\hat{u}_t(x,.)) \in (C^{2,1\times 1} \cap C^{1,2})(\mathcal{M}_1^{sign}),$$

$$b(t,.,\mu,\hat{u}_t(.,\mu)) \in C^2, \quad \frac{\partial b}{\partial x}(t,x,.,\hat{u}(t,x,.)) \in C^{1,0}(\mathcal{M}_1^{sign})$$

with bounds uniform with respect to all variables.
Assumption B): Assume $J(t,x,\mu,u(t,x,\mu))$ and $V(t,x,\mu)$ belong to $(C^{2,1\times 1} \cap C^{1,2})(\mathcal{M}_1^{sign}(\mathbf{R}))$ as functions of μ, belong to C^2 as functions of x and $\frac{\partial J}{\partial x}(x,.) \in C^{1,1}(\mathcal{M}_1^{sign}(\mathbf{R}))$, $\frac{\partial V}{\partial x}(x,.) \in C^{1,1}(\mathcal{M}_1^{sign}(\mathbf{R}))$.
Then the profile of symmetric strategies $\hat{u}_t(x,\mu)$ given by V is an ϵ-Nash equilibrium of the N-player game.

Statistical mechanics ingredient, I

Let us explain our strategy for proving the main Theorem. For any N and a fixed common strategy $u_t(x,\mu)$, solutions $(X_1,\cdots,X_N)_{s,t}(x_1,\cdots,x_N)$ to the initial system of SDEs on $t\in[0,T]$ with the initial condition

$$(X_1,\cdots,X_N)_{s,s}(x_1,\cdots,x_N)=(x_1,\cdots,x_N)$$

at time s define a backward propagator (also referred in the literature as a flow or as a two-parameter semigroup) $U_N^{s,t}=U_N^{s,t}[u(.)]$, $0\le s\le t\le T$, of linear contractions on the space $C_{sym}(\mathbf{R}^N)$ of symmetric functions via the formula

$$(U_N^{s,t}f)(x_1,\cdots,x_N)=\mathbf{E}f(X_1,\cdots,X_N)_{s,t}(x_1,\cdots,x_N).$$

The corresponding dual forward propagator $V_N^{t,s}=(U_N^{s,t})'$ acts on probability measures on $\mathbf{R}^N$ and is defined by the equation

$$(f,V_N^{t,s}\mu)=(U_N^{s,t}f,\mu).$$

Statistical mechanics ingredient, II

By the standard inclusion

$$(x_1, \cdots, x_N) \to \frac{1}{N}(\delta_{x_1} + \cdots + \delta_{X_N})$$

the set $\mathbf{R}^N$ is mapped to the set $\mathcal{P}_N(\mathbf{R})$ of normalized sums of N Dirac's measures, so that $U_N^{s,t}$, $V_N^{t,s}$ can be considered as propagators in $C(\mathcal{P}_N(\mathbf{R}))$ and $\mathcal{P}(\mathcal{P}_N(\mathbf{R}))$ respectively.
On the other hand, for a fixed function $u_t(x, \mu)$, the solution of the limiting McKean-Vlasov SPDE

$$d(\phi, \mu_t) = (L[t, \mu_t, u_t]\phi, \mu_t)\, dt + (\sigma_{com}(.)\nabla\phi, \mu_t)\, dW_t,$$

specifies a stochastic process, a diffusion, on the space of probability measures $\mathcal{P}(\mathbf{R})$ defining the backward propagator $U^{s,t} = U^{s,t}[u(.)]$ on $C(\mathcal{P}(\mathbf{R}))$:

$$(U^{s,t}f)(\mu) = \mathbf{E}f(\mu_{s,t}(\mu)),$$

where $\mu_{s,t}(\mu)$ is the solution to the SPDE above at time t with a given initial condition μ at time $s \le t$.

Statistical mechanics ingredient, III

Theorem 2 (Main theorem on the interacting diffusions).
Assume $\sigma_{ind}, \sigma_{com} \in C^3(\mathbf{R})$ and are positive functions never approaching zero. Assume b is in C^2 as a function of the space variable and

$$b(t,x,.,u(t,x,.)) \in (C^{2,1\times 1} \cap C^{1,2})(\mathcal{M}_1^{sign}(\mathbf{R})),$$
$$\frac{\partial b}{\partial x}(t,x,.,u(t,x,.)) \in C^{1,0}(\mathcal{M}_1^{sign}(\mathbf{R})).$$

with bounds uniform with respect to all variables, for $0 \le s \le t \le T$. Then, for any $\mu \in \mathcal{P}_N(\mathbf{R})$ and $F \in (C^{2,1\times 1} \cap C^{1,2})(\mathcal{M}_1^{sign}(\mathbf{R}))$

$$\|(U^{s,t} - U_N^{s,t})F(\mu)\|_{C(\mathcal{M}_1^{sign}(\mathbf{R}))}$$
$$\le \frac{C(T)}{N}\left(\|F\|_{C^{2,1\times 1}(\mathcal{M}_1^{sign}(\mathbf{R}))} + \|F\|_{C^{1,2}(\mathcal{M}_1^{sign}(\mathbf{R}))}\right).$$

This result belongs to the statistical mechanics of interacting diffusions.

Statistical mechanics with a tagged particle, I

This result is not sufficient for us, as we have to allow one of the agent to behave differently from the others. To tackle this case we shall considered the corresponding problem with a tagged agent.
Consider the Markov process on pairs $(X_t^{1,N}, \mu_t^N)[u^{ind}(.), u^{com}(.)]$, where u^{ind}, u^{com} are some U-valued functions $u_t^{ind}(x,\mu), u_t^{com}(x,\mu)$, $(X_t^{1,N}, \cdots, X_t^{N,N})$ solves our initial system (1) under the assumptions that the first agent uses the control $u_t^{ind}(X_t^{1,N}, \mu_t^N)$ and all other agents $i \neq 1$ use the control $u_t^{com}(X_t^{i,N}, \mu_t^N)$, and $\mu_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,N}}$. (Instead of our μ_t one can use $\tilde{\mu}_t^N = \frac{1}{N}\sum_{i=2}^N \delta_{X_t^{i,N}}$, which does not include X_t^1.)

Statistical mechanics with a tagged particle, II

The corresponding tagged propagators $U^{s,t}_{N,tag} = U^{s,t}_{N,tag}[u^{ind}(.), u^{com}(.)]$ and $U^{s,t}_{tag} = U^{s,t}_{tag}[u^{ind}(.), u^{com}(.)]$:

$$(U^{s,t}_{N,tag}F)(x,\mu) = \mathbf{E}F(X^{1,N}_t, \mu^N_t)[u^{ind}(.), u^{com}(.)](x,\mu),$$

where $\mu = \frac{1}{N}\sum_{j=1}^N \delta_{x_j}$ is the position of the process at time s and where $x = x_1$;

$$(U^{s,t}_{tag}F)(x,\mu) = \mathbf{E}F(X^1_t, \mu_t)[u^{ind}(.), u^{com}(.)](x,\mu),$$

where the process $(X^1_t, \mu_t)[u^{ind}(.), u^{com}(.)](x,\mu)$ with the initial data x, μ at time s is the solution to the system of stochastic equations

$$dX^1_t = b(t, X^1_t, \mu_t, u^{ind}_t(X^1_t, \mu_t)) + \sigma_{ind}(X^1_t)dB^1_t + \sigma_{com}(X^1_t)dW_t,$$

$$d(\phi, \mu_t) = (L[t, \mu_t, u^{com}_t(., \mu_t)]\phi, \mu_t)\, dt + (\sigma_{com}(.)\nabla\phi, \mu_t)\, dW_t.$$

Statistical mechanics with a tagged particle, III

Theorem 3 (Interacting diffusions with a tagged particle). Under the assumptions of Theorem 2 (with both u_t^{com}, u_t^{ind} satisfying these assumptions), let $F(x, \mu)$, $x \in \mathbf{R}$, belongs to the space $(C^{2,1\times 1} \cap C^{1,2})(\mathcal{M}_1^{sign}(\mathbf{R}))$ as a function of μ, $F \in C^2(\mathbf{R})$ as a function of x and $\frac{\partial F}{\partial x}(x, .) \in C^{1,1}(\mathcal{M}_1^{sign}(\mathbf{R}))$. Then, for any $\mu \in \mathcal{P}_N(\mathbf{R})$

$$\|(U_{tag}^{s,t} - U_{N,tag}^{s,t})F\|_{C(\mathbf{R}\times\mathcal{M}_1^{sign}(\mathbf{R}))} \leq \frac{C(T)}{N}.$$

Theorem 2 is a particular case of Theorem 3.

Theorem 1 is deduced (roughly) as follows: Since the evolutions $U_{tag}^{s,t}$ and $U_{N,tag}^{s,t}$ are close to each other, the corresponding optimal policies of the tagged agent should also be close (detail - next slide).

Theorems 2 and 3 follow from the analysis of McKean-Vlasov SPDE – our last theme below.

Main Theorem from Ths 2 and 3

Let u_1 be any adaptive control of the first player and V_1 the corresponding payoff in the game of N players, where all other players are using $u_{com}(t, x, \mu)$ arising from a solution to the main MFG consistency. Then $V_1 \geq V_2$, where V_2 is obtained by playing optimally in N player game. By Theorem 3,

$$|V_2 - V_{2,lim}| \leq C/N,$$

where $V_{2,lim}$ is obtained by playing u_2 in the limiting game. But $V_{2,lim} \geq V$, where V is the optimal payoff for the first player in the limiting game of two players, where the second, measure-valued, player uses u_{com}. Consequently,

$$V_1 \geq V_2 \geq V_{2,lim} - \frac{C}{N} \geq V - \frac{C}{N},$$

completing the proof.

New start: McKean-Vlasov SPDE

For a function $v(t, x)$, $t \geq 0, x \in \mathbf{R}$, let us consider the McKean-Vlasov SPDE

$$dv = L_t(v)\, dt + \Omega v \circ dW_t, \tag{3}$$

where W_t is a one-dimensional Brownian motion,

$$\Omega v(x) = A(x)\frac{\partial v}{\partial x} + B(x)v(x),$$

$$L_t(v) = \frac{1}{2}\sigma^2(x)\frac{\partial^2 v}{\partial x^2} + b(t, x, [v])\frac{\partial v}{\partial x} + c(t, x, [v])v,$$

with some functions $A(x), B(x), \sigma(x)$ and the functions b, c depending in a smooth way on a function (or measure) v.

McKean-Vlasov SPDE, II

In the equation above, $\circ$ denotes the Stratonovich differential. From the usual rule $Y \circ dX = Y\,dX + \frac{1}{2} dY\,dX$, one can rewrite it as an equation with Ito's differential of the similar kind:

$$dv = L_t(v)\,dt + \Omega v\,dW_t + \frac{1}{2}\Omega^2 v\,dt.$$

Our objective is to study the well-posedness of the McKean-Vlasov SPDE in various classes of regular and generalized functions and more importantly its sensitivity with respect to initial conditions.

McKean-Vlasov SPDE, III

Main assumptions: $\sigma \in C^2(\mathbf{R}), B \in C^2(\mathbf{R}), A \in C^3(\mathbf{R})$ and

$$0 < \sigma_1 \leq \sigma(x) \leq \sigma_2, \quad 0 < A_1 \leq A(x) \leq A_2;$$

and

$$\max\left(\|b(t,.,[v])\|_{C^1(\mathbf{R})}, \|c(t,.,[v])\|_{C(\mathbf{R})}\right) \leq b_1,$$

$$\sup_{\|v\|_{\mathcal{M}(\mathbf{R})}\leq\lambda} \max\left(\left\|\frac{\delta b(t,y,[v])}{\delta v(.)}\right\|_{C^1(\mathbf{R})}, \left\|\frac{\delta c(t,y,[v])}{\delta v(.)}\right\|_{C(\mathbf{R})}\right) \leq C(\lambda),$$

$$\sup_{\|v\|_{\mathcal{M}(\mathbf{R})}\leq\lambda} \left\|\frac{\delta}{\delta v(.)}\frac{\partial b(t,y,[v])}{\partial y}\right\|_{C(\mathbf{R})} \leq C(\lambda).$$

Method of stochastic characteristics

Our basic approach will be the method of stochastic characteristics (generally developed by Kunita, but here in its simplest, direct form, available for one-dimensional noise).

This method allows one to turn equation with stochastic differential into a non-stochastic equation of the second order, but with random coefficients.

For mean-field games, this is the reduction of models with common noise to models with a major player (as developed by Caines et al).

Namely, for $A(x), B(x) \in C^1(\mathbf{R})$, the operator

$$\Omega v(x) = A(x)\frac{\partial v}{\partial x} + B(x)v(x),$$

generates the contraction group $e^{t\Omega}$ in $C(\mathbf{R})$, so that $e^{t\Omega}v_0(x)$ is the unique solution to the equation

$$\frac{\partial v}{\partial t} = \Omega v$$

Method of stochastic characteristics, II

Explicitly,

$$e^{t\Omega} v_0(x) = v_0(Y(t,x))G(t,x), \quad t \in \mathbf{R},$$

where $Y(t,x)$ is the unique solution to the ODE $\dot{Y} = -A(Y)$ with the initial condition $Y(0,x) = x$ and

$$G(t,x) = \exp\{\int_0^t B(Y(s,x))\, ds\}.$$

Method of stochastic characteristics, III

Since the product-rule of calculus is valid for the Stratonovich differentials, making the change of unknown function v to $g = \exp\{-\Omega W_t\} v$ rewrites the McKean-Vlasov SPDE in terms of g as (deterministic equation with random coefficients)

$$\dot{g}_t = \tilde{L}_t[W](g_t) = \exp\{-\Omega W_t\} L_t(\exp\{\Omega W_t\} g_t).$$

Since the operators $e^{t\Omega}$ form a bounded semigroup in $L_1(\mathbf{R})$, as well as in $C^k(\mathbf{R})$ and $C^k_\infty(\mathbf{R})$ whenever $A, B \in C^k(\mathbf{R})$, equations (**??**) and our initial McKean-Vlasov SPDE (3) are equivalent in the strongest possible sense.

Method of stochastic characteristics, IV

Concrete version of $\dot{g}_t = \tilde{L}_t[W](g_t)$ (by inspection):

$$\tilde{L}_t[W](g_t)(x) = \frac{1}{2}\tilde{\sigma}^2(x)\frac{\partial^2 g_t}{\partial x^2} + \tilde{b}(t, x, [g_t])\frac{\partial g_t}{\partial x} + \tilde{c}(t, x, [g_t])g_t,$$

with

$$\tilde{\sigma}^2(x) = \sigma^2(Y(-W_t, x))\left(\frac{\partial Y}{\partial z}(W_t, z)|_{z=Y(-W_t,x)}\right)^2,$$

$$\tilde{b}(t, x, [g]) = \left(b(t, z, [\exp\{\Omega W_t\}g])\frac{\partial Y}{\partial z}(W_t, z)\right)|_{z=Y(-W_t,x)}$$

$$+\left[\frac{1}{2}\sigma^2(z)\left(\frac{\partial^2 Y}{\partial z^2}(W_t, z) + 2\frac{\partial \ln G}{\partial z}(W_t, z)\frac{\partial Y}{\partial z}(W_t, z)\right)\right]|_{z=Y(-W_t,x)}$$

and similarly $\tilde{c}$ (Y, G are defined two slides above).

One-dimensional simplifications

The simplification arising from working in one-dimension is as follows:

$$Y(t,x) = \Phi^{-1}(t + \Phi(x)),$$

where

$$\Phi(y) = \int_0^y \frac{dz}{A(z)}.$$

Hence, under above conditions operator $\tilde{L}$ is uniformly (even with resect to the noise) elliptic and

$$\tilde{b}(t,x,[g]) \le C(T)(1+\bar{W}_T), \quad \tilde{c}(t,x,[g]) \le C(T)(1+\bar{W}_t),$$

with some constants $C(T)$ and $\bar{W}_T = \max_{t\in[0,T]} |W_t|$.
Thus on any finite interval of time $[0, T]$ the equation $\dot{g}_t = \tilde{L}_t[W]g_t$ is the usual nonlinear McKean-Vlasov diffusion equation with uniformly elliptic second order part and bounded coefficients a.s. But: use the known results for it carefully – to assess the expectation with respect to the noise W.

McKean-Vlasov SPDE regularity, I

Theorem 4. (On McKean-Vlasov SPDE)
Assume above conditions (7 slides back). Then
(i) For any $v_0 \in \mathcal{M}^{sign}(\mathbf{R})$ there exists a unique solution v_t on $[0, T]$ such that $v_t \in L_1(\mathbf{R})$ for all $t > 0$, positive whenever v_0 is positive, and

$$\mathbf{E}\|v_t\|_{L_1} \leq C_2(T)\|v_0\|_{\mathcal{M}(\mathbf{R})};$$

(ii) If $v_0 \in H_1^1$, then

$$\mathbf{E}\|g_t\|_{H_1^1} \leq C_2(T)\|g_0\|_{H_1^1};$$

(iii) The variational derivative $\xi_t(.;x)[v_0] = \frac{\delta v_t}{\delta v_0(x)}$ of the solution v_t with respect to initial data exists a.s. as a measure of finite total variation.

McKean-Vlasov SPDE: particular case

The particular case which is mostly relevant to the applications to MFG:

$$dv = L'_{t,v} v \, dt - \nabla(A(x)v) \circ dW_t, \tag{4}$$

where

$$L_{t,v}\phi = \frac{1}{2}\sigma^2(x)\frac{\partial^2 \phi}{\partial x^2} + b(t, x, [v])\frac{\partial \phi}{\partial x},$$

and $L'_{t,v}$, its dual, defined as

$$L'_{t,v}u = \frac{1}{2}\frac{\partial^2}{\partial x^2}(\sigma^2(x)u(x)) - \frac{\partial}{\partial x}(b(t, x, [v])u(x)).$$

McKean-Vlasov SPDE: sensitivity (assumptions)

Let $T > 0$, $\sigma \in C^2(\mathbf{R}), A \in C^3(\mathbf{R})$ and

$$0 < \sigma_1 \le \sigma(x) \le \sigma_2, \quad 0 < A_1 \le A(x) \le A_2,$$

$$\|b(t,.,[v])\|_{C^2(\mathbf{R})} \le b_1,$$

$$\sup_{t,y} \sup_{\|v\|_{\mathcal{M}(\mathbf{R})} \le \lambda} \left\| \frac{\delta b(t,y,[v])}{\delta v(.)} \right\|_{C^2(\mathbf{R})} \le C(\lambda),$$

$$\sup_{t,y} \sup_{\|v\|_{\mathcal{M}(\mathbf{R})} \le \lambda} \left\| \frac{\delta}{\delta v(.)} \frac{\partial b(t,y,[v])}{\partial y} \right\|_{C(\mathbf{R})} \le C(\lambda),$$

with some constants $\sigma_1, \sigma_2, A_1, A_2, b_1$ and a function $C(\lambda)$.
Then the following holds:

McKean-Vlasov SPDE: sensitivity (result), I

Theorem (i) For any $v_0 \in \mathcal{M}^{sign}(\mathbf{R})$ there exists a unique solution v_t of equation (3) on $[0, T]$ such that $v_t \in L_1(\mathbf{R})$ for all $t > 0$, positive whenever v_0 is positive, and with the norm not exceeding $\|v_0\|_{\mathcal{M}(\mathbf{R})}$ for all realization of the noise W. Moreover, $v_t \in H_1^1$ for all $t > 0$ and the following estimates hold

$$\mathbf{E}\|v_t\|_{H_1^1} \leq C(T)\|v_0\|_{H_1^1},$$

$$\mathbf{E}\|v_t\|_{H_1^1} \leq C(T)\frac{1}{\sqrt{t}}\|v_0\|_{\mathcal{M}(\mathbf{R})}.$$

McKean-Vlasov SPDE: sensitivity (result), II

Theorem (ii) The variational derivative $\xi_t(.;x)[v_0] = \frac{\delta v_t}{\delta v_0(x)}$ of the solution v_t with respect to initial data are well defined as elements of $L_1(\mathbf{R})$ for any x and $t > 0$, and their first and second derivatives with respect to x are bounded elements of the dual spaces $(C^1(\mathbf{R}))'$ and $(C^2(\mathbf{R}))'$ respectively, so that

$$\|\xi_0(.,x)\|_{L_1} \le C(T),$$

$$\|\frac{\partial}{\partial x}\xi_0(.,x)\|_{(C^1(\mathbf{R}))'} \le C(T), \quad \|\frac{\partial^2}{\partial x^2}\xi_0(.,x)\|_{(C^2(\mathbf{R}))'} \le C(T),$$

with constants $C(T)$ depending only on the norm $\|v_0\|_{\mathcal{M}(\mathbf{R})}$ and independent of the noise.

Similarly one analyzes the second derivatives

$$\eta_t(.;x_1,x_2) = \frac{d}{dh}|_{h=0}\xi_t(.;x_1)[v_0 + h\delta_x].$$

Domain of the Markov propagator generated by the McKean-Vlasov SPDEs, I

The Markov propagator defined by the solutions to the McKean-Vlasov SPDE is given on the continuous functionals of measures in the usual way:

$$U^{s,t}F(v) = \mathbf{E}F(v_t(v,[W])),$$

where v_t is the solution to the McKean-Vlasov SPDE for $t > s$ with given $v = v_s$ at time s.

The main conclusion from the sensitivity analysis:

Theorem. (On the invariant domain of the McKean-Vlasov-SPDEs propagator)

The spaces $C^{1,2}(\mathcal{M}_\lambda^{sign})$ and its intersection with $C^{2,1\times 1}(\mathcal{M}_\lambda^{sign})$ are invariant under $U^{s,t}$, so that

$$\|U^{s,t}F\|_{C^{1,2}(\mathcal{M}_\lambda^{sign})} \le C(T)\|F\|_{C^{1,2}(\mathcal{M}_\lambda^{sign})},$$

$$\|U^{s,t}F\|_{C^{2,1\times 1}(\mathcal{M}_\lambda^{sign})} \le C(T)\left(\|F\|_{C^{2,1\times 1}(\mathcal{M}_\lambda^{sign})} + \|F\|_{C^{1,2}(\mathcal{M}_\lambda^{sign})}\right).$$

The generator for the N interacting diffusions

Recall the standard inclusion

$$(x_1, \cdots, x_N) \to \frac{1}{N}(\delta_{x_1} + \cdots + \delta_{X_N})$$

that maps $\mathbf{R}^N$ to the set $\mathcal{P}_N(\mathbf{R})$. We can rewrite the generator A_N of our initial N interacting diffusions in terms of functionals on measures, via the link

$$f(x_1, \cdots, x_N) = F_f[(\delta_{x_1} + \cdots + \delta_{x_N})/N].$$

We get

$$A_N F(\mu) = \Lambda_{lim} F(\mu) + \frac{1}{N} \Lambda_{corr} F(\mu),$$

where (see next slide):

The generator for the N interacting diffusions, II

$$\Lambda_{lim} F(\mu) = \int_{\mathbf{R}} \left(B_\mu \frac{\delta F}{\delta \mu(.)} \right)(y) \mu(dy)$$

$$+ \frac{1}{2} \int_{\mathbf{R}^2} \sigma_{com}(y) \sigma_{com}(z) \frac{\partial^2}{\partial y \partial z} \frac{\delta^2 F}{\delta \mu(y) \delta \mu(z)} \mu(dy) \mu(dz),$$

$$\Lambda_{corr} F(\mu) = \frac{1}{2} \int_{\mathbf{R}} \sigma_{ind}^2(x) \, \frac{\partial^2}{\partial y \partial z} \frac{\delta^2 F(\mu)}{\delta \mu(y) \delta \mu(z)} \bigg|_{y=z=x} \mu(dx).$$

with

$$B_\mu g(x) = b(t, x, \mu, u(t, x, \mu)) \frac{\partial g}{\partial x} + \frac{1}{2} (\sigma_{com}^2(x) + \sigma_{ind}^2(x)) \frac{\partial^2 g}{\partial x^2}.$$

Sensitivity of McKean-Vlasov and Ths. 2 and 3

Thus we have an explicit expression for the limit of A_N as $N \to \infty$ and for the correction term, which are well defined for functional F from the space $\tilde{C} = C^{1,2}(\mathcal{M}^{sign}) \cap C^{2,1\times 1}(\mathcal{M}^{sign})$.
Hence we have the convergence of the generators of N-particle approximations to the generator of the process arising from the McKean-Vlasov SPDE on the space $\tilde{C}$ with the uniform rates of order $1/N$.
According to the Theorem above 'On the invariant domain of the McKean-Vlasov-SPDEs propagator', the propagator of the process generated by the McKean-Vlasov SPDE acts by bounded operators on $\tilde{C}$. Hence Ths 2 and 3 follow from the standard representation of the difference of two propagators in terms of the difference of their generators:

$$U_N^{t,r} - U^{t,r} = \int_t^r U_N^{t,s}(A_N - \Lambda_{lim})_s U^{s,r} ds$$

or similarly for the tagged particle version of it.

THANK YOU