Parabolic estimates and Poisson process

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Introduction

1. As a first result we show that from Schauder or Sobolev-space estimates for the one-dimensional heat equation one gets their multidimensional analogs for equations with time-dependent coefficients with the *same* constants as in the case of the one-dimensional heat equation.

2. In particular constants in the parabolic estimates do not depend on the dimension.

3. The method is quite general and is based on using the Poisson stochastic process.

4. We can also treat equations involving non-local operators and other class of equations and systems.

5. It seems to be a challenging problem to find a purely analytic approach to proving such results.

6. I will only mention some general results of paper. Rather I will try to show how the method works and the basic idea.

Some function spaces

$C^\alpha(\mathbb{R}^d)$, $\alpha \in (0,1)$, is the space of all $f : \mathbb{R}^d \to \mathbb{R}$ for which the following norm

$$\|f\|_{C^\alpha(\mathbb{R}^d)} = \sup_{x\in\mathbb{R}^d} |f(x)| + [f]_{C^\alpha(\mathbb{R}^d)}$$

is finite, where $[f]_{C^\alpha(\mathbb{R}^d)} = \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha}$.

By $C^{2+\alpha}(\mathbb{R}^d)$ we mean the space of real-valued twice continuously differentiable functions f on $\mathbb{R}^d$ having finite norm

$$\|f\|_{C^{2+\alpha}(\mathbb{R}^d)} = \sup_{x\in\mathbb{R}^d}(|f(x)| + |Df(x)| + |D^2 f(x)|) + [D^2 f]_{C^\alpha(\mathbb{R}^d)},$$

where Df is the gradient of f and $D^2 f$ is its Hessian.

For a real-valued function $f(t,x)$, $t \in (0,T)$, $x \in \mathbb{R}^d$, write

$$f \in B_c((0,T), C_0^\infty(\mathbb{R}^d))$$

if f is a Borel bounded function, such that $f(t,\cdot) \in C_0^\infty(\mathbb{R}^d)$ for any $t \in (0,T)$; for any $n = 0, 1, ...$, the $C^n(\mathbb{R}^d)$-norms of $f(t,\cdot)$ are bounded on $(0,T)$, and the supports of $f(t,\cdot)$ belong to the same ball.

Parabolic estimates for the one dimensional heat equation

$$\boxed{\partial_t u(t,x) = D^2 u(t,x) + f(t,x), \qquad u(0,\cdot) = 0} \tag{1}$$

for $t \in (0,T)$, $x \in \mathbb{R}$. We treat the problem in the integral form:

$$u(t,x) = \int_0^t (D^2 u(s,x) + f(s,x))ds, \ \ t \in [0,T], \ x \in \mathbb{R}.$$

Fix $\alpha \in (0,1)$ and $p \in (1,\infty)$. One knows (see for instance Ladyzhenskaya-Solonnikov-Uraltseva 1968):

if $f \in B_c((0,T), C_0^\infty(\mathbb{R}))$, then there is a unique solution $u(t,x)$ such that u is continuous in $[0,T] \times \mathbb{R}$; $u(t,\cdot) \in C^{2+\alpha}(\mathbb{R})$, for any $t \in [0,T]$, and

$$\sup_{t\in[0,T]} \|u(t,\cdot)\|_{C^{2+\alpha}(\mathbb{R})} \leqslant N_0(T,\alpha) \sup_{t\in(0,T)} \|f(t,\cdot)\|_{C^\alpha(\mathbb{R})},$$

furthermore:

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} |u(t,x)| \leqslant T \sup_{(t,x)\in(0,T)\times\mathbb{R}} |f(t,x)|, \tag{2}$$

$$\sup_{t\in[0,T]} [D^2 u(t,\cdot)]_{C^\alpha(\mathbb{R})} \leqslant N_0(\alpha) \sup_{t\in(0,T)} [f(t,\cdot)]_{C^\alpha(\mathbb{R})}, \tag{3}$$

$$\|D^2 u\|^p_{L_p((0,T)\times\mathbb{R})} \leqslant N_p \|f\|^p_{L_p((0,T)\times\mathbb{R})}, \tag{4}$$

where L_p-spaces are defined with respect to Lebesgue measure and $N_0(\alpha), N_p$ are some constants. The previous (2), (3) and (4) are *parabolic estimates.*

I Theorem in [Krylov, P.]

Let $a(t) = (a^{ij}(t))$ be a $d \times d$ symmetric matrix-valued locally bounded Borel measurable function on $(0, T)$ such that

$$a^{ij}(t)\lambda^i\lambda^j \geqslant |\lambda|^2, \quad t \in (0, T),\ \lambda \in \mathbb{R}^d.$$

For any $f \in B_c((0, T), C_0^\infty(\mathbb{R}^d))$ there exists a unique continuous in $[0, T] \times \mathbb{R}^d$ solution $u(t, x)$ of the equation

$$\partial_t u(t, x) = a^{ij}(t)D_{ij}u(t, x) + f(t, x), \quad u(0, \cdot) = 0$$

in $(0, T) \times \mathbb{R}^d$ such that, for any $t \in [0, T]$, $u(t, \cdot) \in C^{2+\alpha}(\mathbb{R}^d)$ and, for any $i, j = 1, ..., d$ and unit vector $l \in \mathbb{R}^d$, we have:

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |u(t, x)| \leqslant T \sup_{(t,x)\in(0,T)\times\mathbb{R}^d} |f(t, x)| \quad \text{(Max. Principle)},$$

$$\sup_{t\in[0,T]} [D_{ij}u(t, \cdot)]_{C^\alpha(\mathbb{R}^d)} \leqslant N'(\alpha)N_0(\alpha) \sup_{t\in(0,T)} [f(t, \cdot)]_{C^\alpha(\mathbb{R}^d)}, \tag{5}$$

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} [D_l^2 u(t, x + l\,\cdot)]_{C^\alpha(\mathbb{R})} \leqslant N_0(\alpha) \sup_{(t,x)\in(0,T)\times\mathbb{R}^d} [f(t, x + l\,\cdot)]_{C^\alpha(\mathbb{R})}, \tag{6}$$

$$\|D_l^2 u\|^p_{L_p((0,T)\times\mathbb{R}^d)} \leqslant N_p \|f\|^p_{L_p((0,T)\times\mathbb{R}^d)}, \tag{7}$$

where $N_0(\alpha)$, N_p are the previous one-dimensional constants.

Idea of proof when $a^{ij}(t) = \delta_{ij}$

We are considering in $(0, T) \times \mathbb{R}^d$:

$$\partial_t v(t,x) = \Delta v(t,x) + f(t,x), \quad u(0,\cdot) = 0 \tag{8}$$

and show that parabolic estimates hold true with the same one-dimensional constants.

No analytic methods are available up to now.

We use the **Poisson process** and random PDEs

Take a sequence $\tau_1 = \tau_1(\omega), \tau_2 = \tau_2(\omega), \ldots$ of independent random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ with common exponential distribution with parameter $\lambda > 0$, so that $P(\tau_n > t) = e^{-\lambda t}$ for $t \geqslant 0$ and $n = 1, 2.....$ Define

$$\sigma_0 = 0, \quad \sigma_n = \sum_{i=1}^{n} \tau_i, \quad n = 1, 2, \ldots, \quad \pi_t = \pi_t(\omega) = \sum_{n=1}^{\infty} I_{\sigma_n \leqslant t}$$

(where $I_{\sigma_n \leqslant t}$ denotes the indicator function of the event $\{\sigma_n \leqslant t\}$). We see that π_t is the number of consecutive sums of τ_i which lie on $[0, t]$.

The counting process π_t is known as a Poisson process with parameter λ.

The Poisson process

For $0 \leqslant s \leqslant t < \infty$ and $k = 0, 1, \ldots$ it holds that

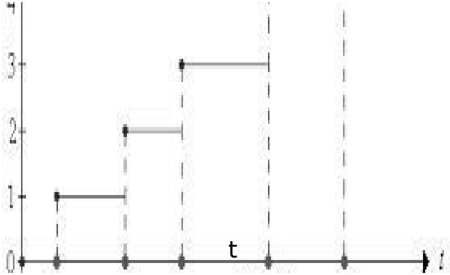
$$P(\pi_t - \pi_s = k) = \frac{[\lambda(t-s)]^k}{k!} e^{-\lambda(t-s)},$$

$\lambda > 0$ and moreover, for any $t > s \geqslant 0$, $\pi_t - \pi_s$ is independent of the σ-algebra generated by all π_r, when $r \in [0, s]$.

Let $\pi_{s-} = \lim_{t \uparrow s} \pi_t$, $s > 0$.

Let $h \in \mathbb{R}$. We do first some elementary computations related to the generator of $h\pi_t$ (below in the picture $h = 1$):

$$\mathcal{A}f(x) = \lambda(f(x+h) - f(x)), \quad x \in \mathbb{R}.$$



Generator of $h\pi_t$ with parameter $\lambda > 0$

Let $u_0 \in C_b(\mathbb{R})$ be a bounded continuous function. Let $\omega \in \Omega$ such that $n = \pi_t(\omega)$. We have, for $x \in \mathbb{R}$, $t > 0$, omitting ω,

$$u_0(x + h\pi_t) - u_0(x)$$

$$= u_0(x + h\pi_{\sigma_n -} + h) - u_0(x + h\pi_{\sigma_n -}) + u_0(x + h\pi_{\sigma_{n-1} -} + h) - u_0(x + h\pi_{\sigma_{n-1} -})$$

$$\ldots + u_0(x + h\pi_{\sigma_1 -} + h) - u_0(x + h\pi_{\sigma_1 -})$$

$$= \sum_{k=1}^{n} \left(u_0(x + h\pi_{\sigma_k -} + h) - u_0(x + h\pi_{\sigma_k -})\right)$$

$$= \sum_{\sigma_k \leqslant t} \int_0^t \left(u_0(x + h\pi_{s-} + h) - u_0(x + h\pi_{s-})\right) \delta_{\sigma_k}(ds)$$

$$= \int_0^t \left(u_0(x + h\pi_{s-} + h) - u_0(x + h\pi_{s-})\right) d\pi_s \quad \text{(Lebesgue-Stieltjes integral).}$$

Applying expectation:

$$E\left[\int_0^t \left(u_0(x + h\pi_{s-} + h) - u_0(x + h\pi_{s-})\right) d\pi_s\right] = \lambda \int_0^t E\left[u_0(x + h\pi_s + h) - u_0(x + h\pi_s)\right] ds$$

Set $v_t(x) = v(t, x) = E[u_t(x + h\pi_t)]$. Then

$$v_t(x) - u_0(x) = \lambda \int_0^t \left[v_s(x + h) - v_s(x)\right] ds,$$

i.e., $\partial_t v_t(x) = \lambda(v_t(x + h) - v_t(x))$ □

The proof for $\partial_t v(t,x) = \Delta v(t,x) + f(t,x)$ in $(0,T) \times \mathbb{R}^d$

We consider $d = 2$. The general case comes from induction. Thus we need to pass from parabolic estimates with $d = 1$ to estimates with $d = 2$.

I step. Take a function $f(t,x,y)$ in $B_c((0,T), C_0^\infty(\mathbb{R}^2))$ and for each $\omega \in \Omega$ and $y \in \mathbb{R}$ solve:

$$\partial_t u(t,x,y,\omega) = D_x^2 u(t,x,y,\omega) + f(t,x,y - h\pi_t(\omega)) \tag{9}$$

with zero initial data, where $h \in \mathbb{R}$ is a parameter. We often do not indicate the dependence on ω. Moreover, we also drop the dependence on h.

There exists a unique solution $u(t,x,y)$, depending on y, h and ω as parameters, such that main estimates (2), (3), and (4) hold for each ω, h and $y \in \mathbb{R}$ *with the same constants* if we replace $u(t,x)$ and $f(t,x)$ with $u(t,x,y)$ and $f(t,x,y - h\pi_t)$, respectively.

Furthermore, since $f \in B_c((0,T), C_0^\infty(\mathbb{R}^2))$, one can prove that $u(t,x,y)$ *is uniformly continuous with respect to* y *uniformly with respect to* ω, t, h*, and* x

Let us see what equation is verified by

$$u(t,x,y + h\pi_t)$$

By considering $u(t,x,y+h\pi_t)$ on each interval $[\sigma_n,\sigma_{n+1})$ on which $h\pi_t$ is constant, one easily derives that

$$u(t,x,y+h\pi_t) = \int_0^t [D_x^2 u(s,x,y+h\pi_s) + f(s,x,y)]\,ds + \int_{(0,t]} g(s,x,y)\,d\pi_s \tag{10}$$

$$= \int_0^t [D_x^2 u(s,x,y+h\pi_s) + f(s,x,y)]\,ds + \sum_{\sigma_n \leqslant t} g(\sigma_n,x,y),$$

where

$$g(s,x,y) = u(s,x,y+h+h\pi_{s-}) - u(s,x,y+h\pi_{s-}) \tag{11}$$

is the jump of $u(t,x,y+h\pi_t)$ as a function of t at moment s if π_t has a jump at s.

Recall $\pi_{s-} = \lim_{t\uparrow s}\pi_t$, $s>0$.
For instance, if $t\in[\sigma_1,\sigma_2)$ we have

$$u(t,x,y+h\pi_t) = u(t,x,y+h) = \int_{\sigma_1}^t [D_x^2 u(s,x,y+h) + f(s,x,y)]\,ds$$

$$+u(\sigma_1,x,y+h) - u(\sigma_1,x,y) + \int_0^{\sigma_1} [D_x^2 u(s,x,y) + f(s,x,y)]\,ds.$$

Let

$$v(t,x,y) := E[u(t,x,y+h\pi_t)].$$

We get, for any $t \in (0,T)$, $x,y \in \mathbb{R}$,

$$v(t,x,y) = \int_0^t \big(D_x^2 v(s,x,y) + \lambda[v(s,x,y+h) - v(s,x,y)] + f(s,x,y)\big)ds.$$

II step. *Let $f \in B_c((0,T), C_0^\infty(\mathbb{R}^2))$, $h \in \mathbb{R}$ and $\lambda > 0$. Then there exists a unique bounded continuous function $v(t,x,y)$, $t \in [0,T]$, $x,y \in \mathbb{R}$, satisfying*

$$\partial_t v(t,x,y) = D_x^2 v(t,x,y) + \lambda[v(t,x,y+h) - v(t,x,y)] + f(t,x,y) \qquad (12)$$

for $t \in (0,T)$, $x,y \in \mathbb{R}$, with zero initial condition and such that $v(t,\cdot,y) \in C^{2+\alpha}(\mathbb{R})$ for any $t \in (0,T)$, $y \in \mathbb{R}$ and

$$\sup_{(t,y)\in[0,T]\times\mathbb{R}} \|v(t,\cdot,y)\|_{C^{2+\alpha}(\mathbb{R})} \leqslant N_0(T,\alpha) \sup_{(t,y)\in(0,T)\times\mathbb{R}} \|f(t,\cdot,y)\|_{C^\alpha(\mathbb{R})}.$$

Furthermore with $N_0(\alpha)$ and N_p as in (3) and (4):

$$\sup_{(t,z)\in[0,T]\times\mathbb{R}^2} |v(t,z)| \leqslant T \sup_{(t,z)\in(0,T)\times\mathbb{R}^2} |f(t,z)|,$$

$$\sup_{(t,y)\in[0,T]\times\mathbb{R}} [D_x^2 v(t,\cdot,y)]_{C^\alpha(\mathbb{R})} \leqslant N_0(\alpha) \sup_{(t,y)\in(0,T)\times\mathbb{R}} [f(t,\cdot,y)]_{C^\alpha(\mathbb{R})}, \qquad (13)$$

$$\|D_x^2 v\|^p_{L_p((0,T)\times\mathbb{R}^2)} \leqslant N_p \|f\|^p_{L_p((0,T)\times\mathbb{R}^2)}$$

Recall that by uniqueness $v(t,x,y) := E[u(t,x,y+h\pi_t)]$.

Let us only check

$$\|D_x^2 v\|^p_{L_p((0,T)\times\mathbb{R}^2)} \leqslant N_p \|f\|^p_{L_p((0,T)\times\mathbb{R}^2)}.$$

We compute, using also Jensen inequality and the Fubini theorem,

$$\begin{aligned}
\|D_x^2 v\|^p_{L_p} &= \int_{[0,T]\times\mathbb{R}^2} \Big|E\Big[D_x^2 u(t,x,y+h\pi_t)\Big]|^p dt dx dy \\
&\leqslant \int_0^T dt \int_{\mathbb{R}^2} E\Big[|D_x^2 u(t,x,y+h\pi_t)|^p\Big] dx dy \\
&= \int_0^T dt \int_{\mathbb{R}^2} E\Big[|D_x^2 u(t,x,z|^p\Big] dx dz \\
&= \int_{\mathbb{R}} dz \int_0^T dt \int_{\mathbb{R}} E\Big[|D_x^2 u(t,x,z|^p\Big] dx \\
&\leqslant N_p \int_{\mathbb{R}} dz \int_0^T dt \int_{\mathbb{R}} |f(t,x,z|^p dx.
\end{aligned}$$

We have also used invariance by translation of the Lebesgue measure.

III step. By repeating the above argument, we see that

$$w(t,x,y) := Ev(t,x,y - h\pi_t)$$

satisfies

$$\partial_t w(t,x,y) = D_x^2 w(t,x,y) + \lambda[w(t,x,y+h) - 2w(t,x,y) + w(t,x,y-h)] + f(t,x,y) \tag{14}$$

and admits the same parabolic estimates as before (with the same constants)

Then we take $\lambda = h^{-2}$ in (13) and let $h \downarrow 0$.

By using Ascoli-Arzela, one can show that the solutions $w = w_h$ of (13) *with $\lambda = h^{-2}$ converge to a function $v(t,x,y)$, that is infinitely differentiable with respect to (x,y) for any t with any derivative continuous and bounded on $[0,T] \times \mathbb{R}^2$, (equals zero for $t = 0$);* it satisfies

$$\partial_t v(t,x,y) = \Delta_{xy} v(t,x,y) + f(t,x,y) \tag{15}$$

in $(0,T) \times \mathbb{R}^2$ and for which all the parabolic estimates hold true with the same constants.

Bounded continuous in $[0,T] \times \mathbb{R}^2$ solutions of (14) having continuous second-order derivatives with respect to (x,y) and vanishing at $t = 0$ are unique, and we get that, for any such solution the previous parabolic estimates hold true with the same constants.

IV step. Take a unit vector $l_1 \in \mathbb{R}^2$ and a unit vector $l_2 \in \mathbb{R}^2$ orthogonal to l_1. Let S be an orthogonal transformation of $\mathbb{R}^2$ such that $Se_i = l_i$, $i = 1, 2$, where e_1, e_2 is the standard basis in $\mathbb{R}^2$, and set $f(t, xe_1 + ye_2) = f(t, x, y)$, $v(t, xe_1 + ye_2) = v(t, x, y)$,

$$S(x,y) = xl_1 + yl_2, \quad g(t,x,y) = f(t, S(x,y)), \quad w(t,x,y) = v(t, S(x,y)).$$

Since the Laplacian is rotation invariant, we have

$$\partial_t w(t,x,y) = \Delta w(t,x,y) + g(t,x,y)$$

and, since g is as regular as f, we conclude by defining

$$K = \sup_{(t,y)\in(0,T)\times\mathbb{R}} \ \sup_{x_1,x_2\in\mathbb{R}, x_1\neq x_2} \frac{|g(t,x_1,y) - g(t,x_2,y)|}{|x_1 - x_2|^\alpha}$$

that

$$\sup_{(t,y)\in[0,T]\times\mathbb{R}} \ \sup_{x_1\neq x_2} \frac{|D_x^2 w(t,x_1,y) - D_x^2 w(t,x_2,y)|}{|x_1 - x_2|^\alpha} \leqslant N_0(\alpha)K. \tag{16}$$

Observe that

$$D_x^2 w(t,x,y) = (D_{l_1}^2 v)(t, S(x,y)) = (D_{l_1}^2 v)(t, xl_1 + yl_2),$$

where $D_l^2 = l^i l^j D_{ij}$ and $D_i = \partial/\partial x^i$, $\quad D_{ij} = D_i D_j$.

Therefore, the left-hand side of (15) equals

$$\sup_{(t,y)\in[0,T]\times\mathbb{R}} \sup_{x,\nu,\mu\in\mathbb{R},\mu\neq\nu} \frac{|D^2_{l_1}v(t,\mu l_1 + x l_1 + y l_2) - D^2_{l_1}v(t,\nu l_1 + x l_1 + y l_2)|}{|\mu-\nu|^\alpha}$$

$$= \sup_{(t,z)\in[0,T]\times\mathbb{R}^2} \sup_{\mu\neq\nu} \frac{|D^2_{l_1}v(t,\mu l_1 + z) - D^2_{l_1}v(t,\nu l_1 + z)|}{|\mu-\nu|^\alpha}.$$

Similarly the right-hand side of (15) is transformed and we get that for the bounded continuous in $[0,T]\times\mathbb{R}^2$ solution v of (14) having continuous second-order derivatives with respect to (x,y) and vanishing at $t=0$ and any unit vector $l\in\mathbb{R}^2$:

$$\sup_{(t,z)\in[0,T]\times\mathbb{R}^2} \sup_{\mu\neq\nu} \frac{|D^2_{l}v(t,\mu l + z) - D^2_{l}v(t,\nu l + z)|}{|\mu-\nu|^\alpha}$$

$$\leqslant N_0(\alpha) \sup_{(t,z)\in(0,T)\times\mathbb{R}^2} \sup_{\mu\neq\nu} \frac{|f(t,\mu l + z) - f(t,\nu l + z)|}{|\mu-\nu|^\alpha}.$$

Since the Jacobian of the above $S(x,y)$ equals one, for any unit vector $l\in\mathbb{R}^2$

$$\int_0^T \int_{\mathbb{R}^2} |D^2_l v(t,z)|^p \, dzdt \leqslant N_p \int_0^T \int_{\mathbb{R}^2} |f(t,z)|^p \, dzdt. \quad \square$$

A remark on the previous proof

We have considered $w = w_h$

$$\partial_t w(t,x,y) = D_x^2 w(t,x,y)$$

$$+ \frac{1}{h^2}[w(t,x,y+h) - 2w(t,x,y) + w(t,x,y-h)] + f(t,x,y) \tag{17}$$

One can apply the finite-difference operators with respect to (x,y) of any order to (16); these operators are obtained by compositions of the **first order difference operators** like

$$\delta_{r,i}v(z) = r^{-1}[v(z+re_i) - v(z)], \quad i = 1,2,$$

where e_i is the ith basis vector and $r > 0$.

By the Maximum Principle and the fact that any derivative of any order of f is in $B_c((0,T), C_0^\infty(\mathbb{R}^2))$, we conclude that any finite-difference of any order of w_h is bounded on $\mathbb{R}^2$ uniformly with respect to t and h.

It follows that w_h is infinitely differentiable with respect to (x,y) and any derivative of any order is bounded on $[0,T] \times \mathbb{R}^2$.

Then equation (16) itself shows that these derivatives are Lipschitz continuous in t.

Thus, the family w_h is equi-Lipschitz in each compact set of $[0, T] \times \mathbb{R}^2$ and the same holds for any derivative with respect to (x, y) of w_h.

We can apply the Arzelà-Ascoli theorem on $[0, T] \times \{|(x, y)| \leqslant R\}$, $R \in (0, \infty)$, along with any derivative with respect to (x, y) of w_{h_n} and $\partial_t w_{h_n}$.

Writing (16) in the integral form and passing to the limit as $n \to \infty$, we conclude that there exists a continuous function $v(t, x, y)$ in $[0, T] \times \mathbb{R}^2$, which is infinitely differentiable with respect to (x, y) with any derivative bounded on $[0, T] \times \mathbb{R}^2$.

Hence, the equation

$$\partial_t u(t, x, y) = \Delta_{x,y} u(t, x, y) + f(t, x, y)$$

holds in integral form on $(0, T) \times \mathbb{R}^2$. □

On the proof for $\partial_t v(t,x) = Tr(a(t)D_x^2 v(t,x)) + f(t,x)$

We have to solve

$$\partial_t v(t,x) = Tr(a(t)D_x^2 v(t,x)) + f(t,x), \quad v(0,\cdot) = 0 \ \text{ or}$$

$$\partial_t v(t,x) = \triangle v(t,x) + Tr(c(t)D_x^2 v(t,x)) + f(t,x), \quad \text{with } c(t) = a(t) - I$$

We start from

$$\partial_t v(t,x) = \triangle v(t,x) + f(t,x) \tag{18}$$

Let $h \in \mathbb{R}$ and consider the unit vector $e_1 \in \mathbb{R}^d$. We define

$$b_t = \int_0^t \sqrt{c(r)}\, e_1 d\pi_r = \sum_{\sigma_k \leqslant t} \sqrt{c(\sigma_k)}\, e_1.$$

If we replace $f(t,x)$ with $f(t,x-hb_t)$, for each ω, in eq. (17), one derives that $u(t,x+hb_t)$ satisfies

$$u(t,x+hb_t) = \int_0^t [\triangle u(s,x+hb_s) + f(s,x)]\, ds + \int_{(0,t]} g(s,x)\, d\pi_s,$$

where

$$g(s,x) := u(s,x+h\sqrt{c(s)}\, e_1 + hb_{s-}) - u(s,x+hb_{s-}).$$

Let

$$v(t,x) = Eu(t, x + hb_t).$$

Then we arrive at

$$\partial_t v(t,x) = \triangle v(t,x) + \lambda[v(t, x + h\sqrt{c(t)}\, e_1) - v(t,x)] + f(t,x).$$

After that we solve

$$\partial_t w(t,x) = \triangle w(t,x) + \lambda[w(t, x + h\sqrt{c(t)}\, e_1) - w(t,x)] + f(t, x + hb_t)$$

and repeating the previous arguments we conclude that for each $h > 0$ there exists a unique solution $u_h(t,x)$ on $[0,T] \times \mathbb{R}^d$ to

$$\partial_t u_h(t,x) = \triangle u_h(t,x) + f(t,x)$$
$$+h^{-2}[u_h(t, x + h\sqrt{c(t)}\, e_1) - 2u_h(t,x) + u_h(t, x - h\sqrt{c(t)}\, e_1)]$$

in $(0,T) \times \mathbb{R}^d$ with zero initial condition and for which all estimates claimed in the theorem hold true. Passing to the limit as before we get

$$\partial_t w(t,x) = \triangle w(t,x) + \langle D^2 w(t,x) \sqrt{c(t)}\, e_1,\ \sqrt{c(t)}\, e_1 \rangle$$

By adding other terms we arrive at

$$\partial_t w(t,x) = \triangle w(t,x) + \sum_{k=1}^{d} \langle D^2 w(t,x) \sqrt{c(t)}\, e_k,\ \sqrt{c(t)}\, e_k \rangle \quad \square$$

An example from [Krylov-P.]

Let $d = 2$, $\alpha \in (0,1)$, and $L_t = \Delta$. We know that for any

$$f \in B_c((0,T), C_0^\infty(\mathbb{R}^2))$$

the equation (we write $f(t,x) = f_t(x)$)

$$u_t(x) = \int_0^t [\Delta u_s(x) + f_s(x)]\, ds, \quad t \leqslant T, x \in \mathbb{R}^2,$$

has a unique continuous solution such that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^2} |u_t(x)| + \sup_{t\in[0,T]} \int_{\mathbb{R}^2} |u_t(x)|\, dx$$

$$\leqslant N_0 \Big[\int_0^T \int_{\mathbb{R}^2} |f_t(x)|\, dxdt + \sup_{(t,x)\in[0,T]\times\mathbb{R}^2} |f_t(x)| \Big], \qquad (19)$$

$$\sup_{t\in[0,T]} [D_l^2 u_t]_{C^\alpha(\mathbb{R}^2)} \leqslant N_\alpha \sup_{t\in[0,T]} [f_t]_{C^\alpha(\mathbb{R}^2)}, \ \forall l : |l| = 1, \qquad (20)$$

where N_0 and N_α are some constants.

We can prove that the equation

$$u_t(x) = \int_0^t [\Delta u_s(x) + M u_s(x) + f_s(x)]\, ds,$$

where

$$M\phi(x) = (x^2)^2 D_{11}\phi(x) - 2x^1x^2 D_{12}\phi(x) + (x^1)^2 D_{22}\phi(x) \\ -x^1 D_1\phi(x) - x^2 D_2\phi(x) \qquad (21)$$

has a continuous solution, which satisfies estimates (18) and (19) (with the same N_0 and N_α).

It seems that this is an unexpected new result. □

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The general results

Let W be a set consisting of real-valued (Borel) measurable functions $u = u_t = u_t(x) = u(t,x)$ on $[0,T] \times \mathbb{R}^d$.

Let $\mathcal{G}$ be a commutative group of affine volume-preserving transformations of $\mathbb{R}^d$. If $g, h \in \mathcal{G}$ by gh we mean the composition of the two transformations.

If $f(x)$ is a function on $\mathbb{R}^d$ and $g \in \mathcal{G}$, we define $(gf)(x) = f(gx)$, where gx is the image of x under mapping g.

By $B((0,T),\mathcal{G})$ we denote the set of bounded measurable $\mathcal{G}$-valued functions on $(0,T)$, $B(\mathbb{R}^d)$ is the set of Borel bounded functions on $\mathbb{R}^d$, $\mathcal{B}([0,T] \times \mathbb{R}^d$ is the Borel σ-field in $[0,T] \times \mathbb{R}^d$.
Fix a constant $K \in [0,\infty)$.

Hypothesis (1)

(i) For any $u \in W$ we have $\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |u_t(x)| \leqslant K$.

(ii) (Convexity of W.) If $(\Omega, \mathcal{F}, P)$ is a probability space and $u(\omega) = u_t(\omega, x)$ is an $\mathcal{F} \times \mathcal{B}([0,T] \times \mathbb{R}^d)$-measurable function such that $u(\omega) \in W$ for any ω, then the function $E[u_t(x)]$ belongs to W.

(iii) ("Shift" invariance of W.) For $u \in W$ and any bounded measurable $\mathcal{G}$-valued function g_t given on $[0,T]$, the function $u_t(g_t x)$ is in W.

Example

Fix a constant $K_0 \in (0, \infty)$ and let W be the set of Borel functions on $[0, T] \times \mathbb{R}$ satisfying, for each $t \in [0, T]$,

$$0 \leqslant u_t(x) \leqslant 1, \quad \int_{\mathbb{R}} u_t^2(x)\, dx \leqslant K_0.$$

Then Hypothesis 1 is satisfied if $\mathcal{G}$ is the group of translations of $\mathbb{R}$.

Next, let $L := \{L_t, t \in (0, T)\}$, be a family of linear operators

$$L_t : C_0^\infty(\mathbb{R}^d) \to B(\mathbb{R}^d)$$

and take and fix

$$f \in B_c((0, T), C_0^\infty(\mathbb{R}^d)), \quad u_0 \in B(\mathbb{R}^d).$$

Hypothesis (2)

The couple (L, f) is W-regular in the following sense.

(i) ($\mathcal{G}$ and L commute.) For any $t \in (0, T)$ and $g \in \mathcal{G}$, we have $gL_t = L_t g$.
(ii) For $\zeta \in C_0^\infty(\mathbb{R}^d)$, $L_t\zeta(x) := (L_t\zeta)(x)$ is measurable with respect to (t, x) and

$$\int_{[0,T]\times\mathbb{R}^d} |L_t\zeta(x)| dtdx < \infty.$$

(iii) There is a mapping $B((0, T), \mathcal{G}) \to W$ sending $h \in B((0, T), \mathcal{G})$ into $u[h] \in W$ such that $u = u[h]$ has initial condition u_0 and satisfies

$$\partial_t u_t(x) = L_t^* u_t(x) + (h_t f_t)(x), \quad t \in [0, T], \ x \in \mathbb{R}^d \tag{22}$$

(iv) For any $h', h'' \in B((0, T), \mathcal{G})$ and $(t, x) \in [0, T] \times \mathbb{R}^d$, we have

$$|u_t[h'](x) - u_t[h''](x)| \leqslant K \int_0^t \sup_{y\in\mathbb{R}^d} |f_r(h'_r y) - f_r(h''_r y)| \, dr. \tag{23}$$

$u \in W$ satisfies (21) with initial condition u_0 if, for any $\zeta \in C_0^\infty(\mathbb{R}^d)$, $t \in [0, T]$,

$$(u_t, \zeta) := \int_{\mathbb{R}^d} u_t(x)\zeta(x)dx = (u_0, \zeta) + \int_0^t (u_s, L_s\zeta) \, ds + \int_0^t (h_s f_s, \zeta) \, ds.$$

Theorem

Assume Hypotheses (1) and (2). For any $g^{(1)}, ..., g^{(n)} \in B((0,T), \mathcal{G})$ *and* $\lambda_1, ..., \lambda_n \geqslant 0$*, the couple, consisting of the family of operators* $\hat{L}_t$*, such that*

$$\hat{L}_t^* = L_t^* + \sum_{i=1}^{n} \lambda_i (g_t^{(i)} - 1), \tag{24}$$

where 1 stands for the operation of multiplying by one, and f*, is W-regular.*

Now we add another assumption:

Hypothesis (3)

For any sequence $u^k \in W$ and a bounded function $u = u(t,x) = u_t(x)$, $(t,x) \in [0,T] \times \mathbb{R}^d$, such that

$$\int_{\mathbb{R}^d} u_t^k(x) \zeta(x)\, dx \to \int_{\mathbb{R}^d} u_t(x) \zeta(x)\, dx$$

for any $t \in [0,T]$ and $\zeta \in C_0^\infty(\mathbb{R}^d)$, there exists $w \in W$ such that $w_t = u_t$ (a.e.) on $\mathbb{R}^d$ for any $t \in [0,T]$.

Let $\mathfrak{N}$ be a subset of the space of affine transformations of $\mathbb{R}^d$ and suppose that

$$\mathcal{G} = \{e^{t\nu} : t \in \mathbb{R}, \nu \in \mathfrak{N}\}, \tag{25}$$

where by $e^{t\nu}$ we mean a transformation $g(t)$ defined as a unique solution of the equation

$$g(t) = 1 + \int_0^t \nu g(s)\, ds. \tag{26}$$

We keep the assumption that $\mathcal{G}$ is a commutative group of volume-preserving transformations.

With any $\nu \in \mathfrak{N}$ we associate an operator M_ν acting on smooth functions $\phi : \mathbb{R}^d \to \mathbb{R}$ by the formula

$$M_\nu \phi(x) = \frac{d^2}{(d\varepsilon)^2} \phi(e^{\varepsilon\nu} x)\Big|_{\varepsilon=0} = (\nu x)^i (\nu x)^j D_{ij}\phi(x) + (\nu^2 x - \nu 0)^i D_i \phi(x).$$

Example (1)

Let l be a unit vector in $\mathbb{R}^d$ and define a transformation $\nu = \nu_l$ by $\nu_l x \equiv l$ on $\mathbb{R}^d$. Then (25) becomes

$$g(t)x = x + \int_0^t \nu g(s)x\,ds = x + \int_0^t l\,ds = x + tl.$$

Observe that in this example, for smooth ϕ, we have $M_\nu \phi(x) = D_l^2 \phi(x)$.

Thus, if $\mathfrak{N} = \{\nu_l : l \in \mathbb{R}^d, |l| = 1\}$, then $\mathcal{G}$ is the set of shifts of $\mathbb{R}^d$ and $\mathcal{G}$ is a commutative group.

Example (2)

Let $\nu x = Qx$, where Q is a skew-symmetric $d \times d$-matrix as in the previous example (see (20))

Then $g_t x = e^{t\nu} x = (\exp[tQ])x$, where $\exp[tQ]$ is an orthogonal matrix. In this example, for smooth ϕ,

$$M_\nu \phi(x) = (Qx)^i (Qx)^j D_{ij} \phi(x) + (Q^2 x)^i D_i \phi(x).$$

A further result

Let W, $\mathcal{G}$, L, u_0, and f satisfy Hypotheses (1), (2) and (3). with $\mathcal{G}$ from (24).

Then, for any $\mu^{(1)}, ..., \mu^{(n)} \in B((0,T), \mathfrak{N})$ equation (21) with

$$L_t^* + \sum_{i=1}^{n} M_{\mu_t^{(i)}}$$

in place of L_t^* and initial condition u_0 has a solution in W.

To apply this result to Example (2) we fix the datum f and denote by A_0 and A_α the right-hand sides of (18) and (19), respectively. Then introduce

$$W = \{u \in B([0,T] \times \mathbb{R}^2) : u_t \in C^{2+\alpha}(\mathbb{R}^d),\ t \in [0,T], \sup_{(t,x) \in [0,T] \times \mathbb{R}^2} |u_t(x)|$$

$$+ \sup_{t \in [0,T]} \int_{\mathbb{R}^2} |u_t(x)|\, dx \leqslant A_0, \sup_{t \in [0,T]} [D_l^2 u_t]_{C^\alpha(\mathbb{R}^2)} \leqslant A_\alpha\ \forall l \in S_1\},$$

and let $\mathfrak{N} = \{tQ : t \in \mathbb{R}\}$, where $Q = (Q_{ij})$ is a 2×2-matrix, $Q^{ii} = 0$, $Q^{12} = 1$, $Q^{21} = -1$, $i = 1, 2$. Q is skew-symmetric and $\mathcal{G} = \{e^{tQ}; t \in \mathbb{R}\}$ is a **group of rotations** of $\mathbb{R}^2$.

One can check the hypotheses for W and $\mathfrak{N}$, $u_0 = 0$ and Δ in place of L_t. □

A useful lemma

Lemma

Let $u \in C^{2+\alpha}(\mathbb{R}^d)$ be such that, for any unit vector $l \in \mathbb{R}^d$, we have

$$\sup_{x \in \mathbb{R}^d} [D_l^2 u(x + l\,\cdot)]_{C^\alpha(\mathbb{R})} \leqslant 1.$$

Then there exists a constant $N'(\alpha)$ such that for any $i, j = 1, ..., d$ we have

$$M := [D_{ij} u]_{C^\alpha(\mathbb{R}^d)} \leqslant N'(\alpha).$$

A hyperbolic system taken from the Evans book on PDEs [Evans]

$$\partial_t u_t^r(x) + B_j^{rk} D_j u_t^k(x) = g_t^r(x) \tag{27}$$

$r = 1, ..., m$, in $(0, T) \times \mathbb{R}^d$ with zero initial condition, where the $m \times m$ constant matrices $B_j := (B_j^{rk})$, $j = 1, ..., d$, are such that for any $\xi \in \mathbb{R}^d$, the matrix $\xi^j B_j$ has m *real* eigenvalues. Assume that $g_t(x) = (g_t^r(x))$ is an $\mathbb{R}^m$-valued measurable functions such that

$$\int_0^T \|g_t\|^2_{H^s(\mathbb{R}^d;\mathbb{R}^m)} \, dt = A < \infty,$$

where $s > m + d/2$ and $H^s(\mathbb{R}^d; \mathbb{R}^m) = W_2^s(\mathbb{R}^d; \mathbb{R}^m)$ are the usual fractional Sobolev spaces of $\mathbb{R}^m$-valued functions.

By following the proof of Theorem 5 in §7.3.3 of [Evans] one arrives at the conclusion that (26) with zero initial condition has a unique solution in class W, which consists of measurable functions $u = u_t(x)$ on $[0, T] \times \mathbb{R}^d$, such that $u_t \in C^{0,1}(\mathbb{R}^d; \mathbb{R}^m)$ (here $C^{0,1}(\mathbb{R}^d; \mathbb{R}^m)$ is the usual space of $\mathbb{R}^m$-valued Lipschitz functions on $\mathbb{R}^d$) for any $t \in [0, T]$ and

$$\|u\|_{L_2([0,T]\times\mathbb{R}^d;\mathbb{R}^m)} + \sup_{t\in[0,T]} \|u_t\|_{C^{0,1}(\mathbb{R}^d;\mathbb{R}^m)} \leqslant N'A, \tag{28}$$

where N' is a constant independent of g.

Now take a bounded measurable $d \times d$-matrix valued function $a = a_t$ which is symmetric and nonnegative for any $t \in [0, T]$.
Define $\sigma_t = a_t^{1/2}$. One knows that σ_t is also measurable and if $\sigma_t^{(i)}$ is the ith column of $\sigma(t)$, $i = 1, .., d$, then for smooth $\phi = \phi(x)$

$$a_t^{ij} D_{ij} \phi = \sum_{i=1}^{d} D^2_{\sigma_t^{(i)}} \phi$$

Therefore, by our last theorem, system (26) with the additional terms on the right-hand side $a_t^{ij} D_{ij} u_t^r(x)$ has a solution of class W.

In particular, estimate (27) holds for the solution of the new system with the *same* right-hand side. The system seems to be of unknown type.

It is worth mentioning that the fact that estimate (27) holds for the new system with a constant N' independent of a can also be obtained by closely following the proof of Theorem 5 in §7.3.3 of [Evans].

Still, what is important, we do not need to know how the initial result about (26) *was obtained*

A stochastic Poisson type process with values in $\mathcal{G}$

Take $g \in B((0,T),\mathcal{G})$, extend it to $[0,\infty)$ by setting $g_0 = 1$ and $g_t = 1$ for $t \geqslant T$, where 1 is the operator of multiplying by 1.

Define $h_t = h_t(\omega) \in \mathcal{G}$ for $t \geqslant 0$ and $\omega \in \Omega$ by

$$h_t = g_{\sigma_n} h_{\sigma_n -} \quad \text{for} \quad t \in [\sigma_n, \sigma_{n+1}), \tag{29}$$

$n = 0, 1, \ldots$, where $\sigma_0 - = 0- := 0$ and $h_0 x :\equiv x$, $x \in \mathbb{R}^d$.

In other terms,

$$h_t = \prod_{n \leqslant \pi_t} g_{\sigma_n} = \prod_{n \leqslant \pi_t} g_{\sigma_n \wedge t}.$$

Observe that the random variables $\sigma_n \wedge t$ are $\mathcal{F}_t$-measurable.

Since g_t is measurable, $g_{\sigma_n \wedge t}$ is $\mathcal{F}_t$-measurable. It follows that h_t is $\mathcal{F}_t$-measurable for each t, or, in other words, the process h_t is $\mathcal{F}_t$-adapted.

On the case $\hat{L}_t^* = L_t^* + \lambda(g_t - 1)$

Let ζ be a test function and $\hat{h} \in B((0,T), \mathcal{G})$.

We prove that for each ω and $t \in [0,T]$ we have ($h_t = h_t(\omega)$ as before)

$$(u_t[h\hat{h}], \zeta(h_t\cdot)) = (u_0, \zeta) + \int_0^t (u_s[h\hat{h}], L_s\zeta(h_s\cdot))\, ds$$

$$+ \int_0^t (h_s\hat{h}_s f_s, \zeta(h_s\cdot))\, ds$$

$$+ \int_{(0,t]} \left[(u_s[h\hat{h}], \zeta(g_s h_{s-}\cdot)) - \xi_{s-}\right] d\pi_s$$

where

$$\xi_t = (u_t[h\hat{h}], \zeta(h_t\cdot)).$$

Then introduce

$$w_t(x) = E[u_t[h(\omega)\hat{h}](h_t^{-1}(\omega)x)]$$

We find

$$w_t(x) = u_0(x) + \int_0^t [L_r^* w_r(x) + \lambda(g_r^{-1} - 1)w_r(x) + \hat{h}_r f_r(x)]\, dr.$$