Global Solutions to Stochastic Reaction-Diffusion Equations

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In this talk, I will present some recent results on the global existence of solutions to stochastic reaction-diffusion(/stochastic heat) equations with super-linear drift and multiplicative noise.

This talk is based on the joint work with Robert Dalang and Davar Khoshnevisan.

Introduction

Let ξ denote the space-time white noise on $\mathbb{R}_+ \times [0\,,1]$, and consider the parabolic stochastic partial differential equation

$$\dot{u}(t\,,x) = \tfrac{1}{2}u''(t\,,x) + b(u(t\,,x)) + \sigma(u(t\,,x))\xi(t\,,x), \qquad (1)$$

$t > 0$, $x \in (0,1)$, subject to the homogeneous Dirichlet boundary condition,

$$u(t\,,0) = u(t\,,1) = 0 \qquad \text{for all } t > 0,$$

and the initial condition $u(0,x) = u_0(x)$, $x \in [0,1]$. Throughout, $\sigma : \mathbb{R} \to \mathbb{R}$ is assumed to be a nonrandom and measurable function, and $b : \mathbb{R} \to \mathbb{R}$ is assumed to be nonrandom and measurable. We assume further that the initial function $u_0 : [0\,,1] \to \mathbb{R}$ is always nonrandom.

Introduction

It is well known that if, in addition, b, σ have at most linear growth—that is if $|b(z)|, |\sigma(z)| = O(|z|)$ as $|z| \to \infty$—then any local solution of (1) is a global one. A few years ago, Bonder and Groisman proved the following interesting complement.

Theorem.[Bonder and Groisman] Suppose, in addition, that σ is a nonzero constant, b is a nonnegative convex function, and satisfies either $\int_1^\infty \mathrm{d}z/b(z) < \infty$ or $\int_{-\infty}^{-1} \mathrm{d}z/b(z) < \infty$, or both, and the initial function u_0 is nonnegative, continuous on $[0\,,1]$, and vanishes on $\{0\,,1\}$. Then there exists an almost surely finite random time τ such that

$$\int_0^1 |u(t\,,x)|^2\,\mathrm{d}x = \infty \qquad \text{for every } t > \tau.$$

Introduction

Our goal is to prove that the preceding result of Bonder–Groisman is in a certain sense optimal. In fact, we introduce two rather different methods which show that, under two different sets of natural conditions, if $|b(z)| = O(|z| \log |z|)$ then the solution to (1) does not blow up at finite time.

Part I: L^2-setting

Let us first recall the definition of a $\mathbb{L}^2$-solution.
Definition 1. Let τ be a stopping time. A $L^2[0,1]$-valued continuous, adapted random field $\{u(t,\cdot), t \in [0,\tau)\}$ is called a solution to equation (1) if for every test function $\phi \in C_0^2(0,1)$,

$$\begin{aligned}\int_0^1 u(t,x)\phi(x)dx &= \int_0^1 u_0(x)\phi(x)dx + \frac{1}{2}\int_0^t\int_0^1 u(s,x)\phi''(x)dx \\ &+ \int_0^t\int_0^1 b(u(s,x))\phi(x)dx \\ &+ \int_0^t\int_0^1 \sigma(u(s,x))\phi(x)\xi(dsdx) \qquad (2)\end{aligned}$$

a.s. for all $t \in [0,\tau)$.

Part I: L^2-setting

Our first result is stated as follows.

Theorem 1. Suppose that $u_0 \in L^2[0\,,1]$, $\sigma : \mathbb{R} \to \mathbb{R}$ is bounded, and $|b(z)| = O(|z|\log|z|)$ as $|z| \to \infty$. Then, any $\mathbb{L}^2$-solution u of (1) does not blow up in finite time.

Sketch of the proof

Sketch of the proof. We will appeal to the logarithmic Sobolev inequality of Gross [2] in the following form: For every $\varepsilon \in (0\,,1)$ and infinitely-differentiable functions $h : [0\,,1] \to \mathbb{R}$ that vanish continuously on $\{0\,,1\}$,

$$\int_0^1 |h(x)|^2 \log|h(x)|\,\mathrm{d}x \leqslant \varepsilon\|h'\|^2_{\mathbb{L}^2} + \tfrac{1}{4}\log(1/\varepsilon)\|h\|^2_{\mathbb{L}^2} + \|h\|^2_{\mathbb{L}^2}\log\left(\|h\|^2_{\mathbb{L}^2}\right),$$

where $0\log 0 := 0$.

Sketch of the proof

For every $R > 0$, consider the stopping times,

$$\tau(R) := \inf \{t > 0 : \|u(t)\|_{\mathbb{L}^2} > R\} \quad \text{and let} \quad \tau := \lim_{R \to \infty} \tau(R).$$

Our goal is to prove that $\mathrm{P}\{\tau = \infty\} = 1$. We will do this by proving that $\mathrm{P}\{\tau < T\} = 0$ for every positive constant T.

Sketch of the proof

For every constant $R > 0$ consider the following stochastic PDE with random forcing and no reaction term:

$$\dot{v}_R(t\,,x) = \tfrac{1}{2}v_R''(t\,,x) + \sigma(u(t\wedge\tau(R)\,,x))\xi(t\,,x), \quad [0 < t < \tau,\ 0 \leqslant x \leqslant 1]. \tag{3}$$

The solution process $t \mapsto v_R(t)$ satisfies the following random integral equation:

$$v_R(t\,,x) = \int_{(0,t)\times[0,1]} G_{t-s}(x\,,y)\sigma(u(s\wedge\tau(R)\,,y))\,\xi(\mathrm{d}s\,\mathrm{d}y). \tag{4}$$

Here, the function $G : (0\,,\infty) \times [0\,,1]^2 \to \mathbb{R}_+$ denotes the heat kernel. We will use $\mathcal{G} := \{\mathcal{G}_t\}_{t\geqslant 0}$ to denote the corresponding heat semigroup.

Sketch of the proof

Define, for every fixed $R > 0$,

$$d_R := u - v_R.$$

We may observe that d_R solves the following random heat equation: For all $t \in [0\,, \tau)$,

$$\dot{d}_R(t) = \tfrac{1}{2} d_R''(t) + b\left(v_R(t) + d_R(t)\right), \tag{5}$$

subject to $d_R(0) = u_0$.
Next we consider the Lyapunov function,

$$\Phi(r) := \exp\left(\int_0^r \frac{\mathrm{d}z}{1 + z \log_+ z}\right),$$

defined for every $r > 0$.

Sketch of the proof

Choose and fix some $T > 0$. Since σ is a bounded measurable function, we can show that

$$A := \sup_{R>0} \mathrm{E}\left(\sup_{t\in[0,T]} \sup_{x\in[0,1]} |v_R(t\,,x)| \right) < \infty. \tag{6}$$

Consider the stopping time

$$\tau_M(R) := \inf\left\{ t > 0 : \sup_{x\in[0,1]} |v_R(t\,,x)| > M \right\} \qquad \text{for every } M > 0.$$

It follows from (6) and the Chebyshev inequality that

$$\sup_{R>0} \mathrm{P}\left\{\tau_M(R) < T\right\} \leqslant \frac{A}{M}. \tag{7}$$

Sketch of the proof

Temporarily define two random space-time functions D and V as

$$D(t) := d_R\,(t \wedge \tau(R) \wedge \tau_M(R))\,, \quad V(t) := v_R\,(t \wedge \tau(R) \wedge \tau_M(R))$$

for $0 \leqslant t \leqslant T$, all the time suppressing the dependence of D and V on $(R\,,M)$, as well as on the spatial variable $x \in [0\,,1]$.

Sketch of the proof

We are able to justify the use of the chain rule to get that for every $t \in [0\,,\,T]$,

$$\begin{aligned} \|D(t)\|_{\mathbb{L}^2}^2 &= \|u_0\|_{\mathbb{L}^2}^2 - 2\int_0^t \left\|D'(s)\right\|_{\mathbb{L}^2}^2 \,\mathrm{d}s \\ &\quad +2\int_0^t \langle b(V(s)+D(s))\,,\,D(s)\rangle_{\mathbb{L}^2}\,\mathrm{d}s. \end{aligned} \tag{8}$$

A second application of the chain rule yields

$$\begin{aligned} \Phi\left(\|D(t)\|_{\mathbb{L}^2}^2\right) = \Phi\left(\|u_0\|_{\mathbb{L}^2}^2\right) &- 2\int_0^t \Phi'\left(\|D(s)\|_{\mathbb{L}^2}^2\right)\left\|D'(s)\right\|_{\mathbb{L}^2}^2 \,\mathrm{d}s \\ &+ 2\int_0^t \Phi'\left(\|D(s)\|_{\mathbb{L}^2}^2\right)\langle b(V(s)+D(s))\,,\,D(s)\rangle_{\mathbb{L}^2}\,\mathrm{d}s. \end{aligned} \tag{9}$$

Sketch of the proof

Using the growth condition of the drift b we can show that

$$\langle b(V(s)+D(s))\,,D(s)\rangle_{\mathbb{L}^2}\leqslant \bar{C}\left\{\|D(s)\|^2_{\mathbb{L}^2\log\mathbb{L}}+\|D(s)\|^2_{\mathbb{L}^2}+1\right\},$$

uniformly for all $s\in[0\,,T]$, where $\bar{C}$ is a non-random and finite constant, and depends only on $(C_b\,,M)$. Thus, we may apply the logarithmic Sobolev inequality to get

$$\begin{aligned}&\langle b(V(s)+D(s))\,,D(s)\rangle_{\mathbb{L}^2}\leqslant\|D'(s)\|^2_{\mathbb{L}^2}\\&+c_*\left\{\|D(s)\|^2_{\mathbb{L}^2}+\|D(s)\|^2_{\mathbb{L}^2}\log_+\left(\|D(s)\|^2_{\mathbb{L}^2}\right)+1\right\},\end{aligned}$$

uniformly for all $s\in[0\,,T]$, where c_* is a non-random and finite constant, and depends only on $(C_b\,,M)$.

Sketch of the proof

We can deduce the following from (9):

$$\Phi\left(\|D(t)\|_{\mathbb{L}^2}^2\right) \leqslant \Phi\left(\|u_0\|_{\mathbb{L}^2}^2\right) + C\int_0^t \Phi'\left(\|D(s)\|_{\mathbb{L}^2}^2\right) \times \left\{1 + \|D(s)\|_{\mathbb{L}^2}^2 \log_+\left(\|D(s)\|_{\mathbb{L}^2}^2\right)\right\} \mathrm{d}s. \tag{10}$$

But $\Phi'(r)[1 + r\log_+ r] = \Phi(r)$ for all $r \geqslant 0$. Therefore, the preceding inequality implies that

$$\Phi\left(\|D(t)\|_{\mathbb{L}^2}^2\right) \leqslant \Phi\left(\|u_0\|_{\mathbb{L}^2}^2\right) + C\int_0^t \Phi\left(\|D(s)\|_{\mathbb{L}^2}^2\right) \mathrm{d}s,$$

uniformly for all $t \in [0\,,T]$, where the implied constant is non-random and finite, and depends only on $(C_b\,,M\,,T)$. It follows from Gronwall's inequality, that $\sup_{t\in[0,T]} \Phi(\|D(t)\|_{\mathbb{L}^2}^2)$ is a.s. bounded from above by a non-random finite number $B(C_b\,,M\,,T)$, that depends only on $(C_b\,,M)$, whence

$$\sup_{R>0} \mathrm{E}\left[\Phi\left(\|d(T\wedge\tau(R)\wedge\tau_M(R)\|_{\mathbb{L}^2}^2\right)\right] \leqslant B(C_b\,,M\,,T). \tag{11}$$

Sketch of the proof

On the other hand, we can show that

$$\|d(T \wedge \tau(R) \wedge \tau_M(R))\|_{\mathbb{L}^2} \geqslant R - M$$

a.s. on the event $\{\tau(R) \leqslant T \leqslant \tau_M(R)\}$, whence

$$\Phi\left(\|d(T \wedge \tau(R) \wedge \tau_M(R))\|^2_{\mathbb{L}^2}\right) \geqslant \Phi\left((R - M)^2\right)$$

a.s. on $\{\tau(R) \leqslant T \leqslant \tau_M(R)\}$ as long as $R > M$.

Sketch of the proof

Combine this with (11) to see that

$$\mathrm{P}\left\{\tau(R) \leqslant T \leqslant \tau_M(R)\right\} \leqslant \frac{B(C_b\,, M\,, T)}{\Phi\left((R-M)^2\right)} \qquad \text{for all } R > M > 0.$$

Sketch of the proof

The preceding inequality and (7) together show that

$$\mathrm{P}\left\{\tau(R) \leqslant T\right\} \leqslant \frac{B(C_b, M, T)}{\Phi\left((R-M)^2\right)} + \frac{A}{M},$$

for all $R > M$. We first let $R \to \infty$ and then $M \to \infty$ in order to see that

$$\mathrm{P}\{\tau < T\} = \lim_{R\to\infty} \mathrm{P}\left\{\tau(R) < T\right\} = 0.$$

Since $T > 0$ is arbitrary, this proves the theorem.

Part II: L^∞-setting

In this second part, I will introduce another main result in the L^∞-setting under a different set of conditions. The approach is also quite different. First let us recall the definition of the solution.

Definition 2. A *random field solution* to (1) is a jointly measurable and adapted space-time process $u := \{u(t,x)\}_{(t,x)\in\mathbb{R}_+\times[0,1]}$ such that, for all $(t,x)\in\mathbb{R}_+\times[0,1]$,

$$u(t,x) = (\mathcal{G}_t u_0)(x) + \int_{(0,t)\times(0,1)} G_{t-s}(x,y) b(u(s,y))\,\mathrm{d}s\,\mathrm{d}y + \int_{(0,t)\times(0,1)} G_{t-s}(x,y)\sigma(u(s,y))\,\xi(\mathrm{d}s\,\mathrm{d}y),$$

almost surely, where $\{\mathcal{G}_t\}_{t\geqslant 0}$ and G are respectively the heat semigroup and heat kernel for the Dirichlet Laplacian.

For every globally Lipschitz function $f : \mathbb{R} \to \mathbb{R}$, there are constants $c(f)$ and $L(f)$ such that

$$|f(z)| \leqslant c(f) + L(f)|z|, \qquad \text{for all } z \in \mathbb{R}. \tag{12}$$

One possibility is to take $c(f) = |f(0)|$ and $L(f) = \text{Lip}(f)$, but often, $L(f)$ can be chosen strictly smaller than $\text{Lip}(f)$

Here is our second main result:

Theorem 2. Suppose that:

- $u_0 \in \cup_{0<\alpha\leqslant 1}\mathbb{C}_0^\alpha$;
- b and σ are locally Lipschitz functions such that $|b(z)| = O(|z|\log|z|)$ as $|z| \to \infty$; and
- $|\sigma(z)| = o\left(|z|(\log|z|)^{1/4}\right)$ as $|z| \to \infty$.

Then, the SPDE (1) has a unique random field solution in $L^\infty[0\,,1]$. In fact, u has a continuous modification that satisfies

$$\sup_{t\in[0,T]}\sup_{x\in[0,1]}|u(t\,,x)| < \infty \qquad \text{a.s., for all } T \in (0\,,\infty). \tag{13}$$

Some uniform bounds

To prove Theorem 2, we first establish some rather precise moment bounds of the solution of the stochastic heat equation in the classical case that

$$b \text{ and } \sigma \text{ are globally Lipschitz continuous.}$$

We consider only the case that

$$L(b) \geqslant 4L(\sigma)^4 > 0. \tag{14}$$

Some uniform bounds

Proposition 3. The following logical implication is valid:

$$u_0 \in \bigcup_{0<\alpha\leqslant 1} \mathbb{C}_0^\alpha \quad \Longrightarrow \quad \mathrm{P}\left\{ u(t) \in \bigcup_{0<\alpha\leqslant 1} \mathbb{C}_0^\alpha \text{ for all } t > 0 \right\} = 1.$$

Some uniform bounds

Set

$$\mathcal{M}_1 := c(b) + c(\sigma); \quad \mathcal{M}_2 := L(b) + L(\sigma); \text{ and}$$

$$\mathcal{M}_3 := \|u_0\|_{\mathbb{L}^\infty} + \frac{c(b)}{L(b)} + \frac{c(\sigma)}{L(\sigma)}.$$

Some uniform bounds

Proposition 4. Choose and fix $\alpha \in (0\,,1]$. There exists a finite universal constant A—independent of $(b\,,\sigma)$—such that

$$\sup_{0\leqslant x<x'\leqslant 1} \mathrm{E}\left(\left|\frac{u(t\,,x)-u(t\,,x')}{|x'-x|^{\alpha\wedge(1/2)}}\right|^k\right) \leqslant A^k\left(\|u_0\|^k_{\mathbb{C}_0^\alpha} + k^{k/2}\mathcal{M}_1^k + k^{k/2}\mathcal{M}_2^k\,\mathcal{M}_3^k\,\mathrm{e}^{AkL(b)t}\right), \quad (15)$$

uniformly for all $u_0 \in \mathbb{C}_0^\alpha$, $t \geqslant 0$, and $k \in [2\,,\sqrt{L(b)}/L(\sigma)^2]$.

Some uniform bounds

Proposition 5. Fix $T_0 > 0$. Choose and fix some $\alpha \in (0\,,1]$, and define $\mu := \min(\frac{1}{4}\,,\frac{1}{2}\alpha)$. Then there exists a finite constant A—independent of $(b\,,\sigma)$—such that

$$\begin{aligned} &\sup_{x\in[0,1]} \mathrm{E}\left(\left|\frac{u(T\,,x)-u(t\,,x)}{(T-t)^{\mu}}\right|^k\right) \\ \leqslant\;\; & A^k\left(\|u_0\|_{\mathbb{C}_0^\alpha}^k + k^{k/2}\left[\mathcal{M}_1^k + \mathcal{M}_2^k\mathcal{M}_3^k\mathrm{e}^{AkL(b)(T)}\right]\right), \quad (16)\end{aligned}$$

for all $u_0 \in \mathbb{C}_0^\alpha$, $0 \leqslant t < T \leqslant T_0$, and $k \in [2\,,\sqrt{L(b)}/L(\sigma)^2]$.

Some uniform bounds

The following estimate plays a key role.

Proposition 6. Let $u = \{u(t\,, x)\}_{t \geqslant 0, x \in [0,1]}$ denote the solution of SHE, and define $\varpi := \max(12\,, 6/\alpha)$ and fix $T_0 > 0$. If $u_0 \in \mathbb{C}_0^\alpha$ for some $\alpha \in (0\,, 1]$ and $\sqrt{\mathrm{Lip}(b)} > \varpi \mathrm{Lip}(\sigma)^2$, then there exists a finite constant A—independent of $L(b)$, $L(\sigma)$—such that for all $T \in [0, T_0]$,

$$\mathrm{E}\left(\sup_{t \in [0,T]} \sup_{x \in [0,1]} |u(t\,, x)|^k \right) \leqslant A^k (1 \vee T)^{k(1 + \frac{\alpha}{2} \wedge \frac{1}{4})} \Big(\|u_0\|_{\mathbb{C}_0^\alpha}^k + k^{k/2} \mathcal{M}_1^k + k^{k/2} \mathcal{M}_2^k \mathcal{M}_3^k \, \mathrm{e}^{AkL(b)T} \Big)\,, \tag{17}$$

uniformly for all $k \in \Big(\varpi\,, \sqrt{L(b)}/L(\sigma)^2\Big]$.

Some uniform bounds

The proof of Proposition 6 is also lengthy. It requires estimates providing precise dependence of the moment bounds of

$$E[|u(t,x) - u(s,y)|^k]$$

on the Linear growth constants of the coefficients b and σ.

Sketch of the proof of Theorem 2

For all $N \geqslant 1$ let b_N be the following truncation of the drift function:

$$b_N(z) := \begin{cases} b(z) & \text{if } |z| \leqslant N, \\ b(N) & \text{if } z > N, \\ b(-N) & \text{if } z < -N. \end{cases} \tag{18}$$

Let $\sigma_N(z)$ denote the corresponding truncation of the diffusion coefficient σ.
Consider the stochastic PDE

$$\dot{u}_N(t\,,x) = \tfrac{1}{2}u_N''(t\,,x) + b_N\left(u_N(t\,,x)\right) + \sigma_N\left(u_N(t\,,x)\right)\xi(t\,,x), \tag{19}$$

subject to $u_N(0) = u_0$. Since b_N is globally Lipschitz, the solution u_N exists for all time.

Sketch of the proof of Theorem 2

Consider also the stopping times

$$\tau_N := \inf\left\{ t > 0 : \sup_{x\in[0,1]} |u_N(t\,,x)| > N \right\},$$

where $\inf\varnothing := \infty$. One has

$$u_N(t\,,x) = u_{N+1}(t\,,x) \qquad \text{for all } t \in [0\,,\tau_N) \text{ and } x \in [0\,,1].$$

Since u_N is well defined for all time, and is a continuous function of $(t\,,x)$, this proves that $\tau_N \leqslant \tau_{N+1}$ a.s. for all $N \geqslant 1$, and therefore there exists a space-time stochastic process u such that for all $N \geqslant 1$, $u(t\,,x) = u_N(t\,,x)$ for all $x \in [0\,,1]$ and $t \in [0\,,\tau_N)$. Consider the stopping time

$$\tau_\infty = \lim_{N\uparrow\infty} \tau_N.$$

The aim is to show that $\tau_\infty = \infty$ a.s.

Sketch of the proof of Theorem 2

The proof is divided into several steps. We first assume that the drift b in (1) has the following special form: There exist two constants $\vartheta_1, \vartheta_2 \in \mathbb{R}$ such that $\vartheta_2 \neq 0$ and

$$\tilde{b}(z) = \vartheta_1 + \vartheta_2 |z| \log_+ |z| \qquad \text{for all } z \in \mathbb{R}, \tag{20}$$

where we recall $\log_+(a) := \log(a \vee \mathrm{e})$ for all $a \geqslant 0$.

Sketch of the proof of Theorem 2

Define

$$\tilde{b}_N(z) := \vartheta_1 + \vartheta_2(|z| \wedge N) \log_+ (|z| \wedge N),$$

for all $N \geqslant 3$. We can take

$$L(\tilde{b}_N) = \vartheta_2(\log N). \tag{21}$$

Sketch of the proof of Theorem 2

For every fixed integer $N \geqslant 3$, the following stochastic PDE is well posed for all time:

$$\dot{U}_N(t\,,x) = \tfrac{1}{2} U_N''(t\,,x) + \tilde{b}_N\left(U_N(t\,,x)\right) + \sigma_N\left(U_N(t\,,x)\right)\xi(t\,,x),$$

valid for all $t > 0$ and $x \in [0\,,1]$, subject to $U_N(0) \equiv u_0$.
Define

$$\tau_N^{(1)} := \inf\left\{ t > 0 : \sup_{x\in[0,1]} |U_N(t\,,x)| > N \right\},$$

where $\inf \varnothing := \infty$. As an important part of the proof, we need to show that

$$\tau_\infty^{(1)} := \lim_{N\nearrow\infty} \tau_N^{(1)} = \infty \qquad \text{a.s.} \tag{22}$$

Sketch of the proof of Theorem 2

We apply Proposition 6 to show that $\tau_\infty^{(1)}$ is greater than a positive, deterministic constant δ. In order to justify this assertion, we appeal to the Chebyshev inequality to see that for every $\varepsilon > 0$ and $N \geqslant 3$,

$$\begin{aligned} \mathrm{P}\left\{\tau_N^{(1)} < \varepsilon\right\} &= \mathrm{P}\left\{\sup_{t\in[0,\varepsilon]}\sup_{x\in[0,1]}|U_N(t\,,x)| > N\right\} \\ \leqslant \quad & N^{-k}\mathrm{E}\left(\sup_{t\in[0,\varepsilon]}\sup_{x\in[0,1]}|U_N(t\,,x)|^k\right). \end{aligned} \tag{23}$$

Next, we may apply (21) and Proposition 6 in order to see that there exist universal constants A and B such that

$$\mathrm{E}\left(\sup_{t\in[0,\varepsilon]}\sup_{x\in[0,1]}|U_N(t\,,x)|^k\right)\leqslant A^k\|u_0\|^k_{\mathbb{C}_0^\alpha}(B+\log N)^k N^{Ak\vartheta_2\varepsilon}. \tag{24}$$

Here we have used the assumption that $\sigma(z)=o(|z|(log|z|)^{\frac{1}{4}})$ as $|z|\to\infty$ in order to be able to apply Proposition 6.

Sketch of the proof of Theorem 2

In other words, we now have

$$\mathrm{P}\{\tau_N^{(1)} < \varepsilon\} \leqslant A^k \|u_0\|_{\mathbb{C}_0^\alpha}^k (B + \log N)^k N^{k(A\vartheta_2\varepsilon - 1)}, \qquad (25)$$

uniformly for all sufficiently large integers N and $\varepsilon \in (0,1)$. Provided that $\varepsilon < A^{-1}\vartheta_2^{-1}$, the right-hand side converges to 0 as $N \to \infty$, so (25) implies that $\tau_\infty^{(1)} \geqslant \varepsilon$ with probability one. This in turn proves that

$$\tau_\infty^{(1)} > \delta := \frac{1}{2}\min(A^{-1}\vartheta_2^{-1}, 1) \qquad \text{a.s.} \qquad (26)$$

As δ is independent of the initial function, we can further exploit the Markov property to prove that $\tau_\infty^{(1)} = \infty$.

Sketch of the proof of Theorem 2

Finally we prove the theorem in the general case where b is an arbitrary locally-Lipschitz function that satisfies the growth condition $|b(z)| = O(|z| \log |z|)$ as $|z| \to \infty$.
We can find $\vartheta_1 \in \mathbb{R}$ and $\vartheta_2 > 0$ such that

$$b_-(z) \leqslant b(z) \leqslant b_+(z), \qquad \text{for all } z \in \mathbb{R},$$

where

$$b_\pm(z) := \vartheta_1 \pm \vartheta_2 |z| \log_+ |z|, \qquad \text{for all } z \in \mathbb{R}.$$

Let $U_\pm(t\,, x)$ denote the solution to (1), where b is replaced by $b_\pm$.
By analogy with (18), let $b_{N,-}$ and $b_{N,+}$ be the truncations of b_- and b_+, respectively. Then

$$b_{N,-}(z) \leqslant b_N(z) \leqslant b_{N,+}(z).$$

Sketch of the proof of Theorem 2

Let u_N be the solution to (19), $U_{N,-}$ (resp. $U_{N,+}$) be the solution to (19) with b_N replaced by $b_{N,-}$ (resp. $b_{N,+}$). According to the comparison theorem, for all $(t,x) \in \mathbb{R}_+ \times [0,1]$,

$$U_{N,-}(t\,,x) \leqslant u_N(t\,,x) \leqslant U_{N,+}(t\,,x). \tag{27}$$

We have shown in Step 2 that

$$\sup_{t\in[0,T]} \sup_{x\in[0,1]} |U_\pm(t,x)| < \infty \qquad \text{for all } T > 0. \tag{28}$$

For any given $(t\,,x)$, for N sufficiently large, $U_\pm(t,x) = U_{N,\pm}(t,x)$, therefore (27) implies that

$$U_-(t\,,x) \leqslant u_N(t\,,x) \leqslant U_+(t\,,x). \tag{29}$$

Sketch of the proof of Theorem 2

Recall that

$$\tau_N = \inf\{t > 0 : \sup_{x\in[0,1]} |u_N(t,x)| > N\}.$$

Then (28) and (29) imply that $\lim_{N\to\infty} \tau_N = \infty$ a.s., and we can define

$$u(t\,,x) = u_N(t\,,x), \qquad \text{for } t \in [0\,,\tau_N] \text{ and } x \in [0\,,1].$$

As above, this definition is coherent. By (29),

$$U_-(t,x) \leqslant u(t,x) \leqslant U_+(t,x), \qquad \text{for all } t \in \mathbb{R}_+ \text{ and } x \in [0,1].$$

Sketch of the proof of Theorem 2

We can show that u is the solution of the equation:

$$u(t\,,x) = (\mathcal{G}_t u_0)(x) + \int_{(0,t)\times(0,1)} G_{t-s}(x\,,y) b(u(s\,,y))\,\mathrm{d}s\,\mathrm{d}y \\ + \int_{(0,t)\times(0,1)} G_{t-s}(x\,,y)\sigma(u(s\,,y))\,\xi(\mathrm{d}s\,\mathrm{d}y),$$

and

$$\sup_{t\in[0,T]}\sup_{x\in[0,1]} |u(t\,,x)| < \infty \qquad \text{for all } T > 0,$$

References

Bonder, Julian Fernández, and Pablo Groisman. Space time white noise eliminates global solutions in reaction diffusion equations, *Physica D* **238** (2009) 209–215.

Dalang, Robert C. Extending the martingale measure stochastic integral with applications to spatially homogeneous s.p.d.e.'s, *Electron. J. Probab.* **4** (1999) no. 6, 29 pp. (electronic).

Dalang, Robert C., Davar Khoshnevisan, and Eulalia Nualart. Hitting probabilities for systems of non-linear stochastic heat equations with additive noise, *Alea* **3** (2007) 231–271.

References

- Da Prato, G. and Zabczyk, J. Stochastic equations in infinite dimensions, Cambridge University Press, 1992.
- Fang, Shizan, and Zhang, Tusheng. A study of a class of stochastic differential equations with non-Lipschitzian coefficients, *Probab. Theory Rel. Fields* **132** (2005) 356–390.
- Foondun, Mohammud, and Khoshnevisan, Davar. Intermittence and nonlinear stochastic partial differential equations *Electronic J. Probab.* Vol. 14, Paper no. 21 (2009) 548–568.

References

Garsia, Adriano M. Continuity properties of gaussian processes with multidimensional time parameter. In *Proc. Sixth Berkeley Symp. Math. Statist. and Probab.* Vol. II: Probability Theory, pages 369274, 1972.

Gross, Leonard. *Logarithmic Sobolev Inequality and Contractive Properties of Semigroups*, Lecture Notes in Math. **1563**, pp. 54–84, Springer-Verlag, 1993.

Walsh, John B. *An Introduction to Stochastic Partial Differential Equations*, in: École d'été de probabilités de Saint-Flour, XIV—1984, 265–439, Lecture Notes in Math., vol. 1180, Springer, Berlin, 1986.