Two benchmark problems:

(1) Flow in expanding step
   Critical viscosity $\nu \approx 1/220.5$
   Real rightmost eigenvalue
   Pitchfork bifurcation

(2) Flow around square obstacle
   Critical viscosity $\nu \approx 1/186$
   Complex conjugate rightmost eigenvalues, Hopf bifurcation
Eigenvalues for step problem

Perturbed eigenvalues

For this:
Solve 760 perturbed eigenvalue problems
Sample surrogate
1M samples, ~5 min

Subcritical: \( \nu = 1/210 \)

Near critical: \( \nu = 1/220 \)

<table>
<thead>
<tr>
<th>( \nu )</th>
<th>( \lambda )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/210</td>
<td>(-2.7 \times 10^{-3})</td>
</tr>
<tr>
<td>1/220</td>
<td>(-1.4 \times 10^{-4})</td>
</tr>
<tr>
<td>1/250</td>
<td>(5.8 \times 10^{-3})</td>
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</table>
**Experiment:** Simulate laboratory scenario

1. Start from quiescent state, integrate to steady state
   - Done using adaptive stabilized trapezoidal rule
     (Gresho, Griffiths, Silvester)

2. Perturb the velocity and continue the integration until either
   - flow returns to steady state, or
   - something else happens

Assessed using

\[
\text{Acceleration } a(t) = \sqrt{\int_{D} \left( \frac{\partial \tilde{u}_h}{\partial t} \right)^2}, \text{ small if velocity } \tilde{u}_h \text{ is steady}
\]

\[
\text{Mean vorticity } \omega(t) = \int_{D} \nabla \times \tilde{u}_h(\cdot, t) = \int_{\partial D_N} u_y(\cdot, t) \, ds,
\]

avg vertical velocity at outflow, 0 for reflectionally symmetric flow
Preliminary: What happens for *supercritical* viscosity, $\nu = 1/250$?

Answer: Steady-state solution is *nearly* found

![Mean vorticity evolution](image1)

![Flow acceleration](image2)
What happens next, after interrupt, w/o perturbation?

Additional insight from automatic time stepping:
Solution obtained: symmetry breaking

Stationary streamlines: time step = 340

Stationary streamlines: time step = 430

Stationary streamlines: time step = 530

Stationary streamlines: time step = 885
Repeat experiment for subcritical $\nu = 1/210$

Long-term behavior, no perturbation

Long-term behavior, perturbation #1 (benign)

Long-term behavior, perturbation #2 (lively)
Display these results differently:

- Mean vorticity evolution on log scale:
  - No perturbation
  - Benign perturbation
  - Lively perturbation

- Flow acceleration:
  - No perturbation
  - Benign perturbation
  - Lively perturbation
Repeat experiment for near critical $\nu=1/220$

Long-term behavior, no perturbation

Long-term behavior, perturbation #1 (benign)

Long-term behavior, perturbation #2 (lively)
Summarizing these results, for flow in expanding step:

- Transient iteration is consistent with perturbation analysis
  - Instability for near-critical parameter is displayed
  - Flow for sub-critical (but barely so) parameter is stable but slight leanings to instability can be observed
- Symmetry-breaking for super-critical parameter
- Effects can also seen in time step choices made by a good integrator
Problem Statement
Use of Surrogate Models
Comparison: Eigenvalue Analysis and Simulation in Time
Flow in expanding step
Flow around obstacle

Eigenvalues for obstable problem

\[ \nu = 1/175 \]
\[ \nu = 1/185.6 \]
\[ \nu = 1/250 \]

\[
\begin{array}{cc}
\nu & \text{Re}(\lambda) \\
1/175 & -2.9 \times 10^{-2} \\
1/185.6 & -3.0 \times 10^{-4} \\
1/200 & 3.7 \times 10^{-2} \\
\end{array}
\]

Perturbed eigenvalues
For this:
Solve 760 perturbed eigenvalue problems
Sample surrogate
100K samples, \( \sim 1 \text{ min} \)
Simulation for obstacle, super-critical $\nu = 1/200$ after interrupt
Highlights of evolution for three parameters, with no perturbation

Super-critical, \( \nu = 1/200 \)

Sub-critical, \( \nu = 1/175 \)

Near-critical, \( \nu = 185.6 \)
Impact of perturbation (lively)

Subcritical parameter \( \nu = 1/175 \)

Near-critical parameter \( \nu = 1/185.6 \)
Summarizing these results, for flow around obstacle:

- Transient iteration is again consistent with perturbation analysis
  - For sub-critical parameter, performance with perturbation is like that for no perturbation
  - For near-critical parameter, performance with perturbation is like that for super-critical regime

- Results affected by delicacy of stability analysis
  - Some instability is seen even for subcritical parameters
    Caused by truncation error in transient iteration

For both benchmark problems:

- New relatively cheap method for finding pseudospectra is predictive of behavior of simulation in time
- Refined understanding of simulation in time near stability limit