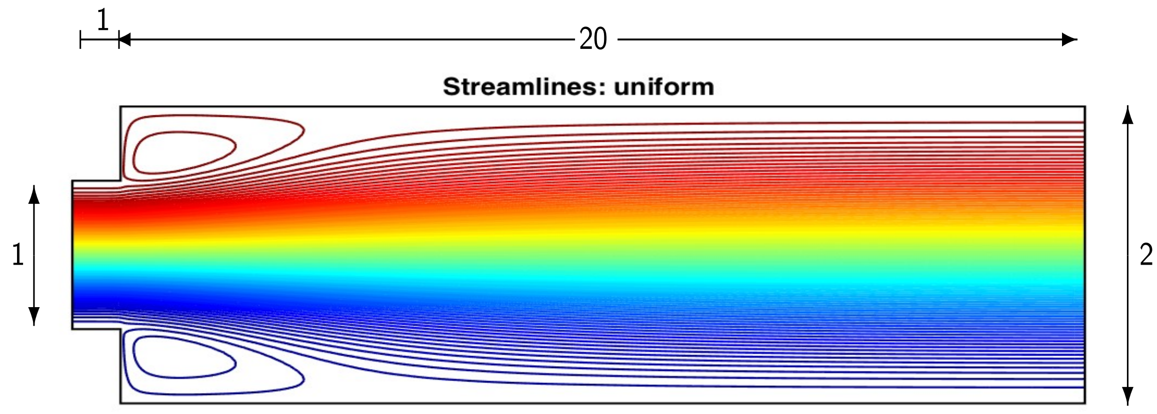
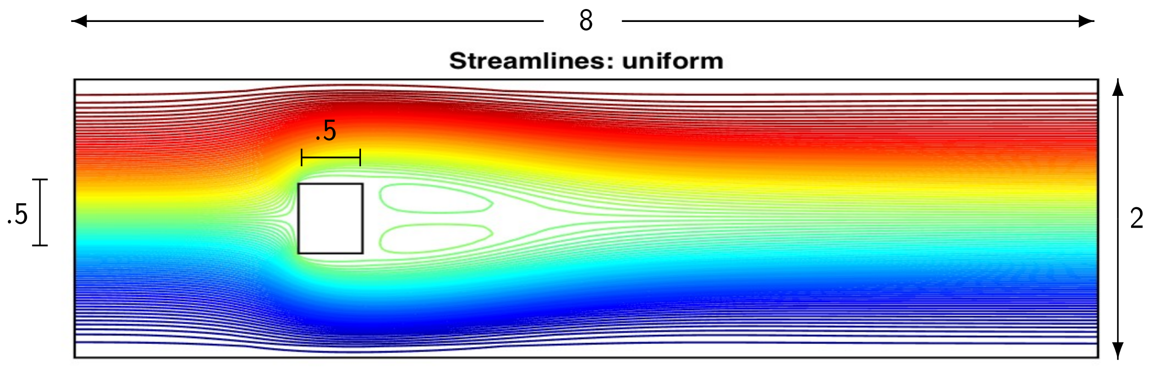
Comparison: Eigenvalue Analysis and Simulation in Time

Two benchmark problems:

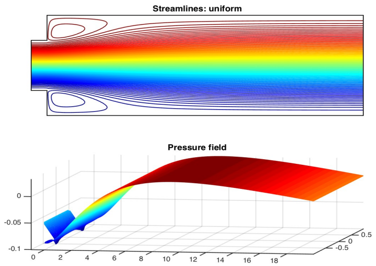
(1) Flow in expanding step
Critical viscosity $\nu \approx 1/220.5$
Real rightmost eigenvalue
Pitchfork bifurcation

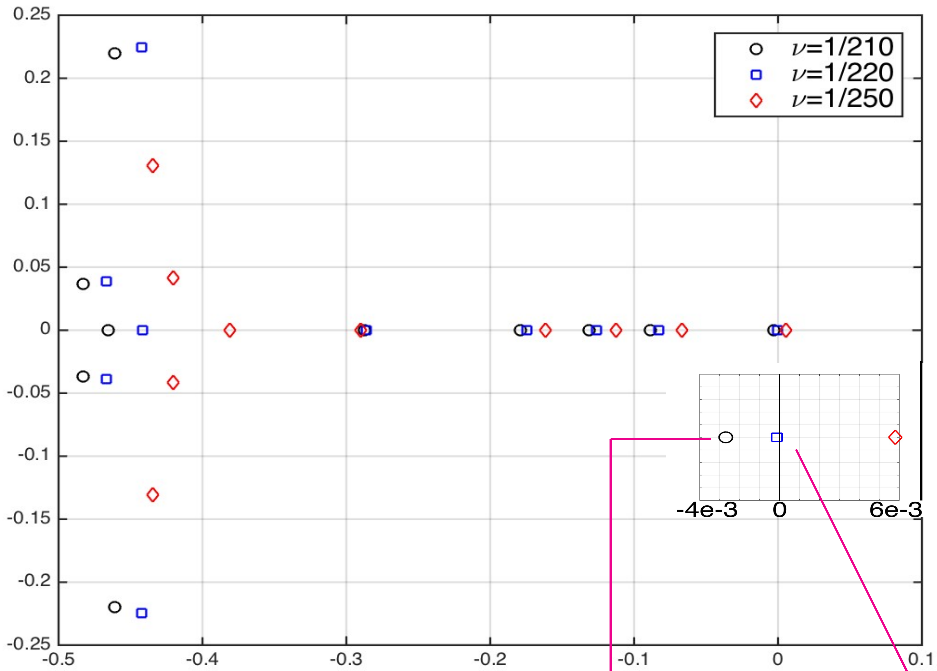


(2) Flow around square obstacle
Critical viscosity $\nu \approx 1/186$
Complex conjugate rightmost eigenvalues, Hopf bifurcation



Eigenvalues for step problem

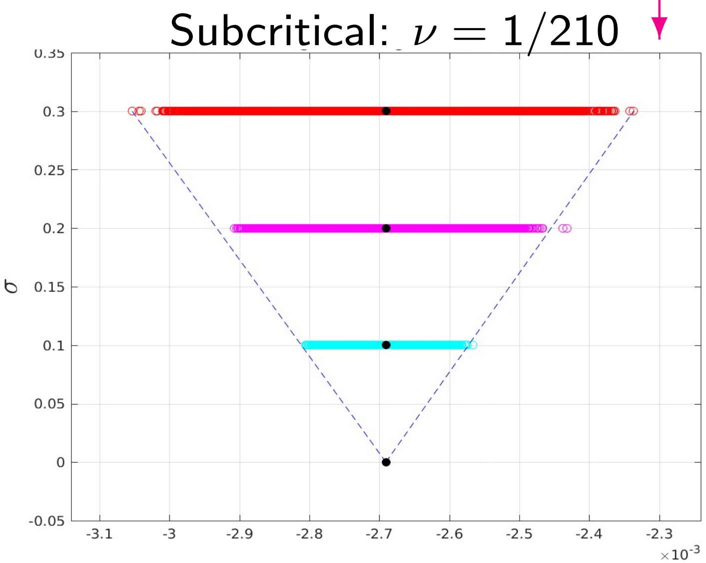


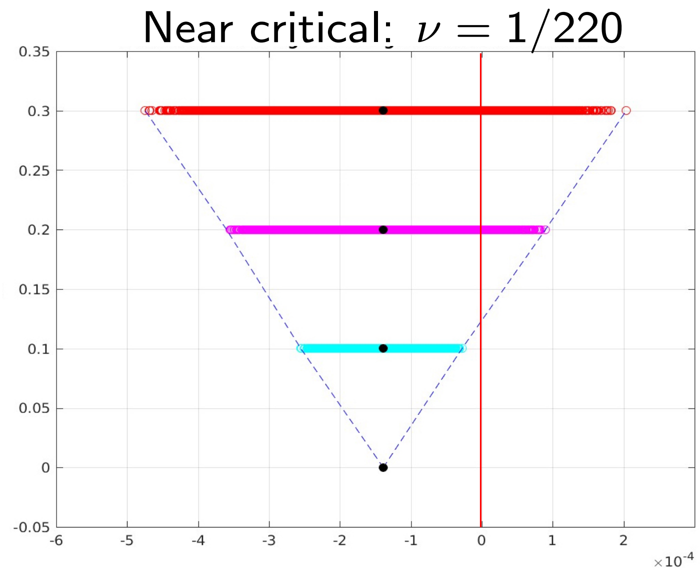


ν	λ
$1/210$	-2.7×10^{-3}
$1/220$	-1.4×10^{-4}
$1/250$	5.8×10^{-3}

Perturbed eigenvalues

For this:
Solve 760 perturbed eigenvalue problems
Sample surrogate
1M samples, ∼5 min





Simulation in Time

Experiment: Simulate laboratory scenario

1. Start from quiescent state, integrate to steady state
   Done using adaptive stabilized trapezoidal rule
   (Gresho, Griffiths, Silvester)

2. Perturb the velocity and continue the integration until either
   – flow returns to steady state, or
   – something else happens

Assessed using

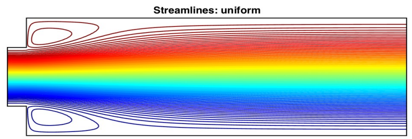
Acceleration $a(t) = \sqrt{\int_{\mathcal{D}} \left(\frac{\partial \vec{u}_h}{\partial t}\right)^2}$, small if velocity $\vec{u}_h$ is steady

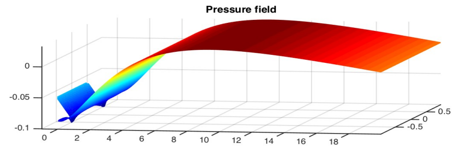
Mean vorticity $\omega(t) = \int_{\mathcal{D}} \nabla \times \vec{u}_h(\cdot, t) = \int_{\partial\mathcal{D}_{\mathcal{N}}} u_y(\cdot, t)\,\mathrm{ds}$,

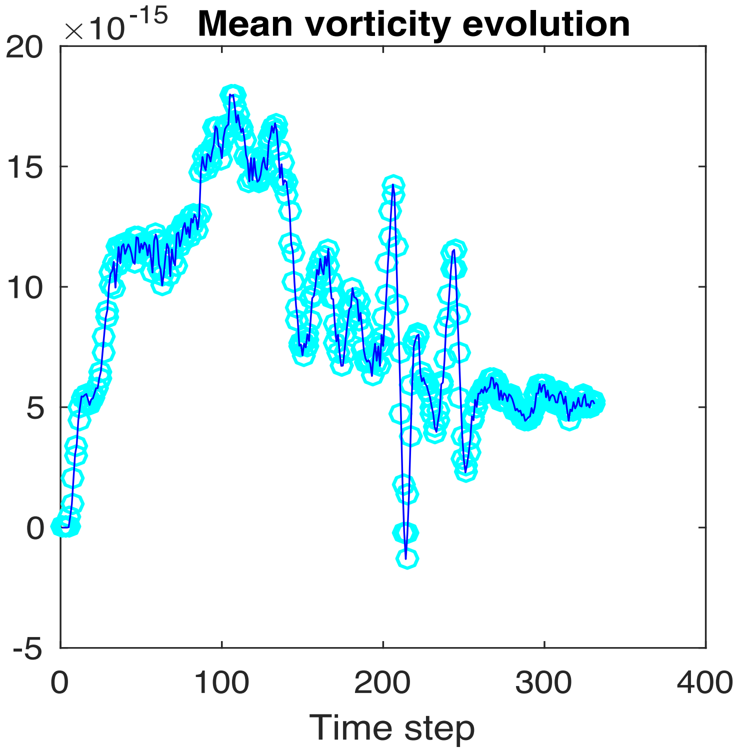
avg vertical velocity at outflow, 0 for reflectionally symmetric flow

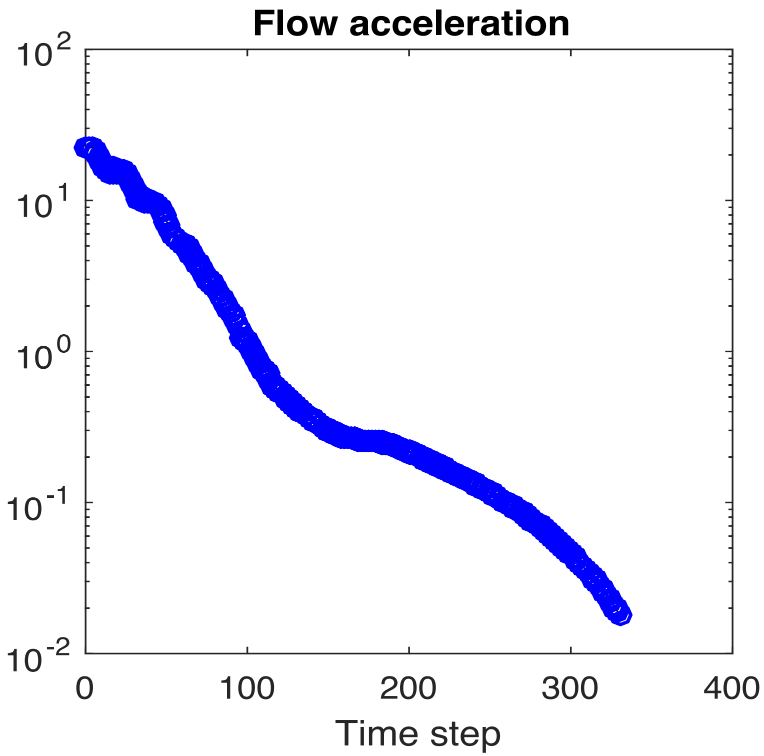
Preliminary: What happens for *supercritical* viscosity, $\nu = 1/250$?

Answer: Steady-state solution is *nearly* found

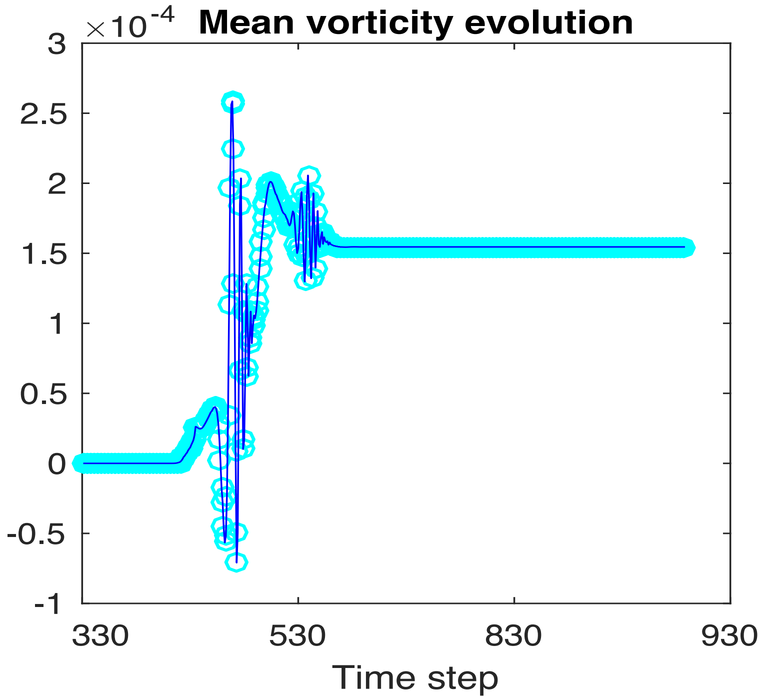


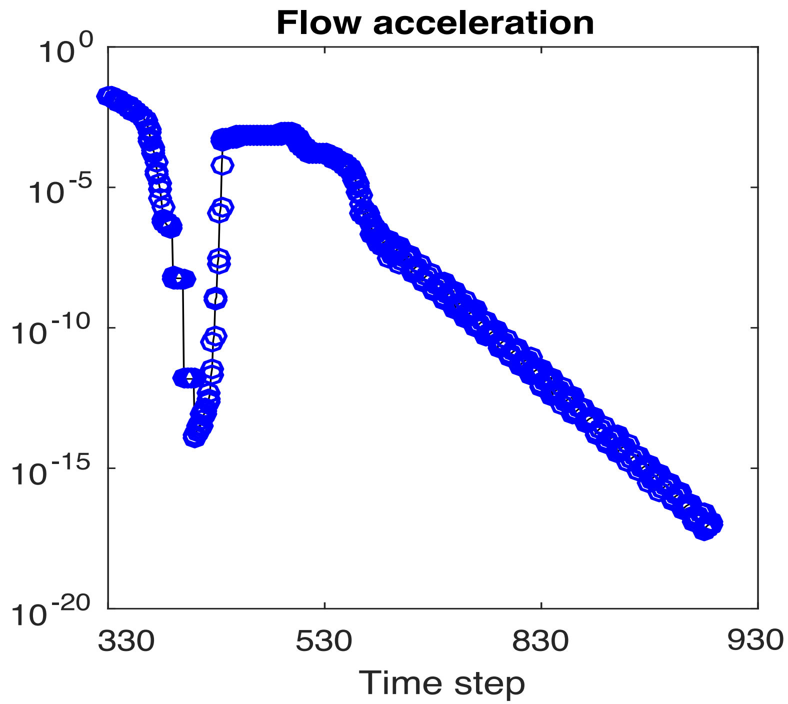




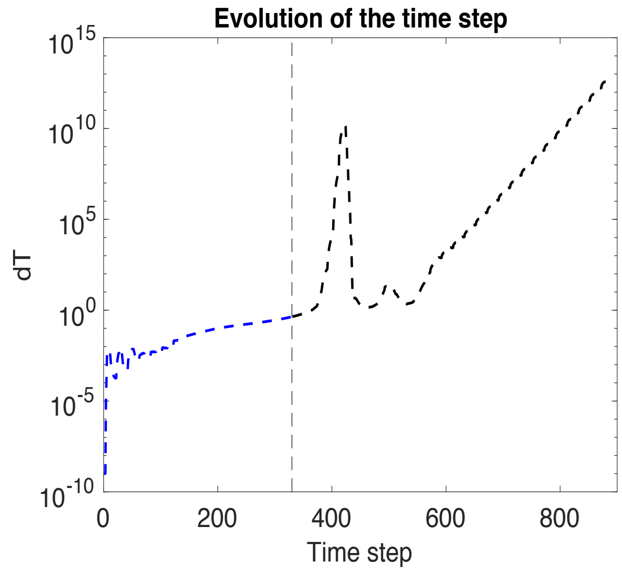


What happens next, after interrupt, w/o perturbation?



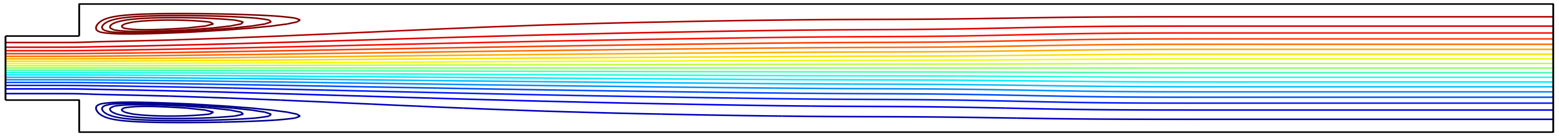


Additional insight from automatic time stepping:

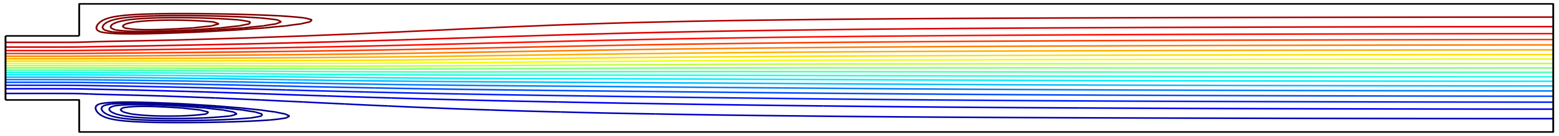


Solution obtained: symmetry breaking

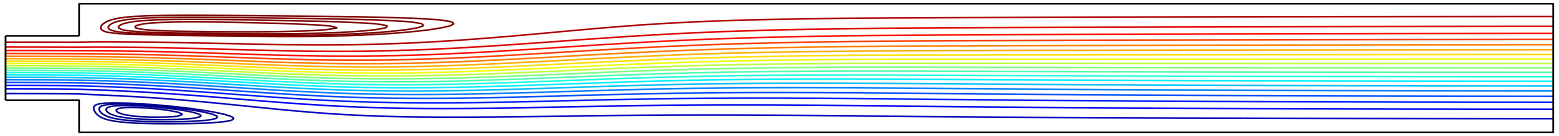
Stationary streamlines: time step = 340



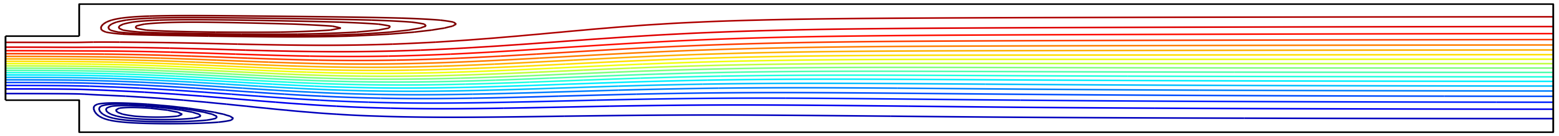
Stationary streamlines: time step = 430



Stationary streamlines: time step = 530

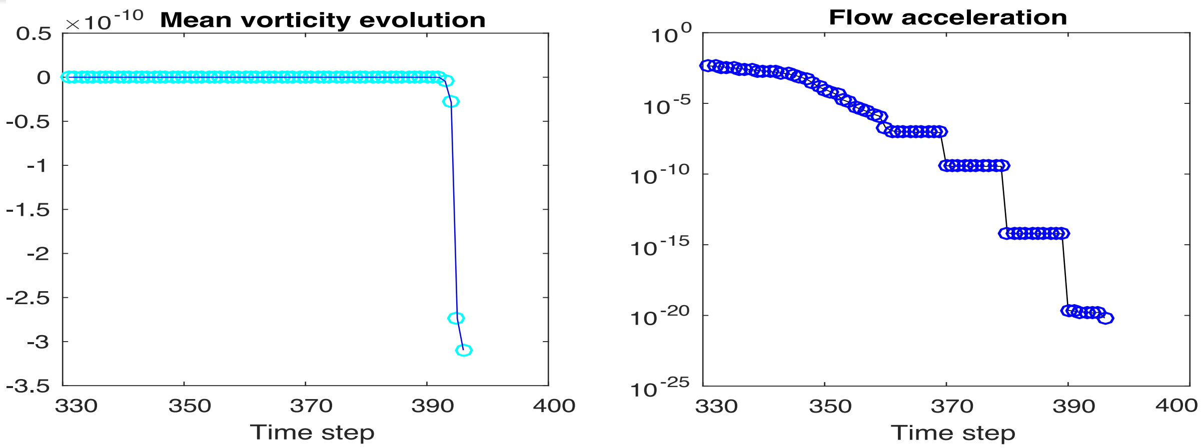


Stationary streamlines: time step = 885

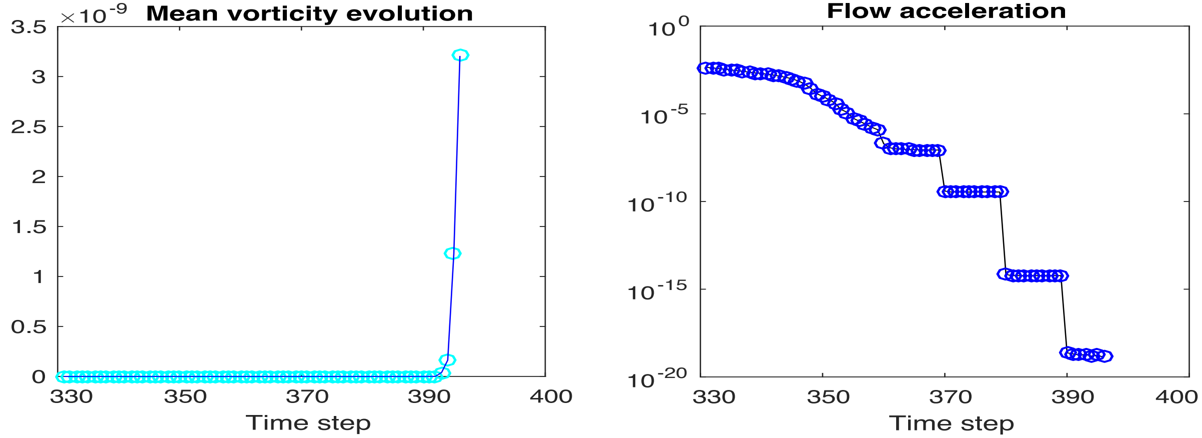


Repeat experiment for **subcritical** $\nu = 1/210$

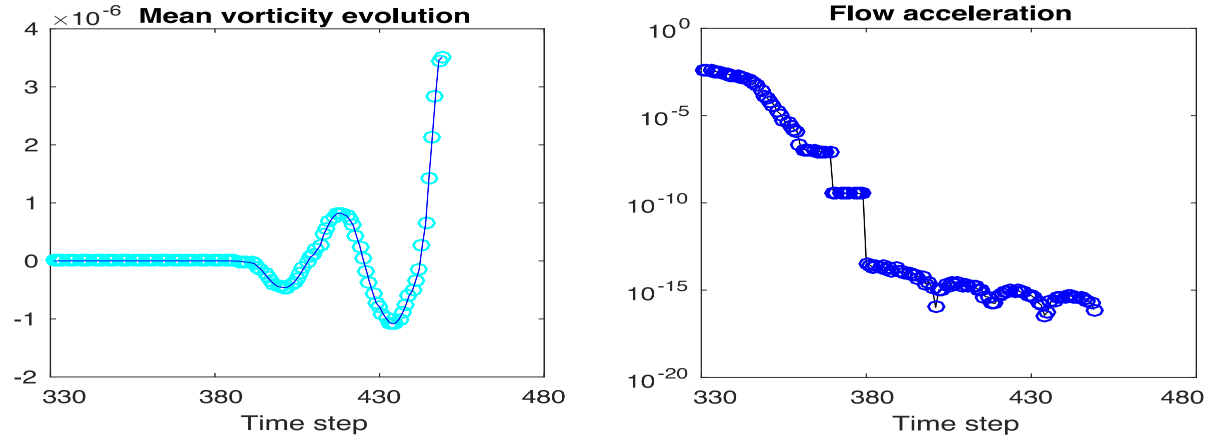
Long-term behavior, no perturbation



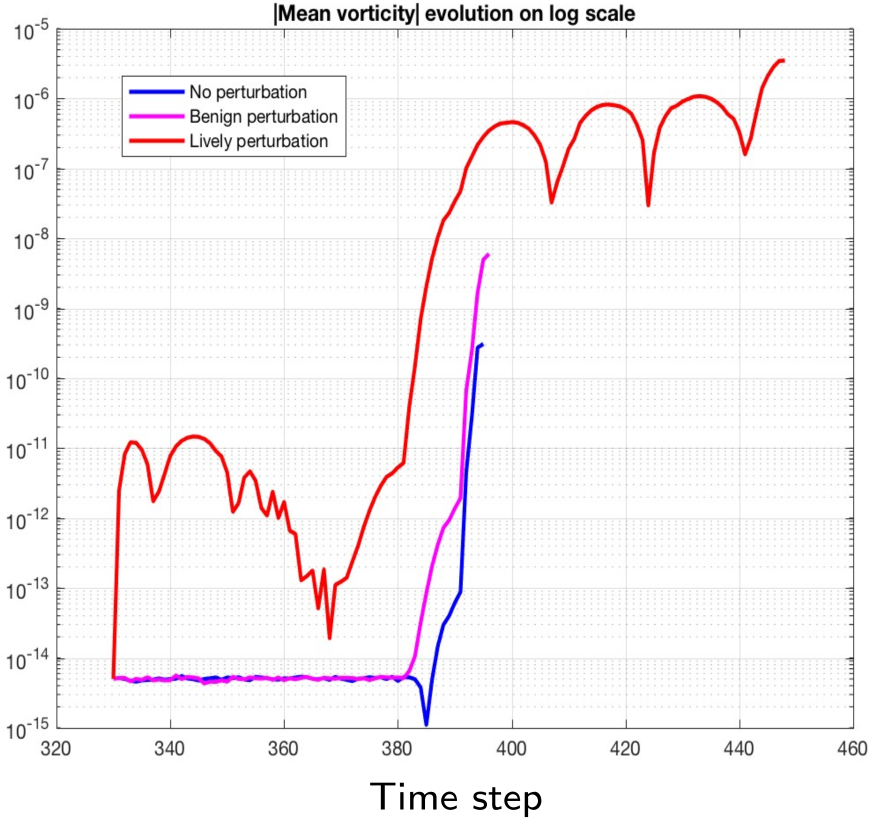
Long-term behavior, perturbation #1 (benign)

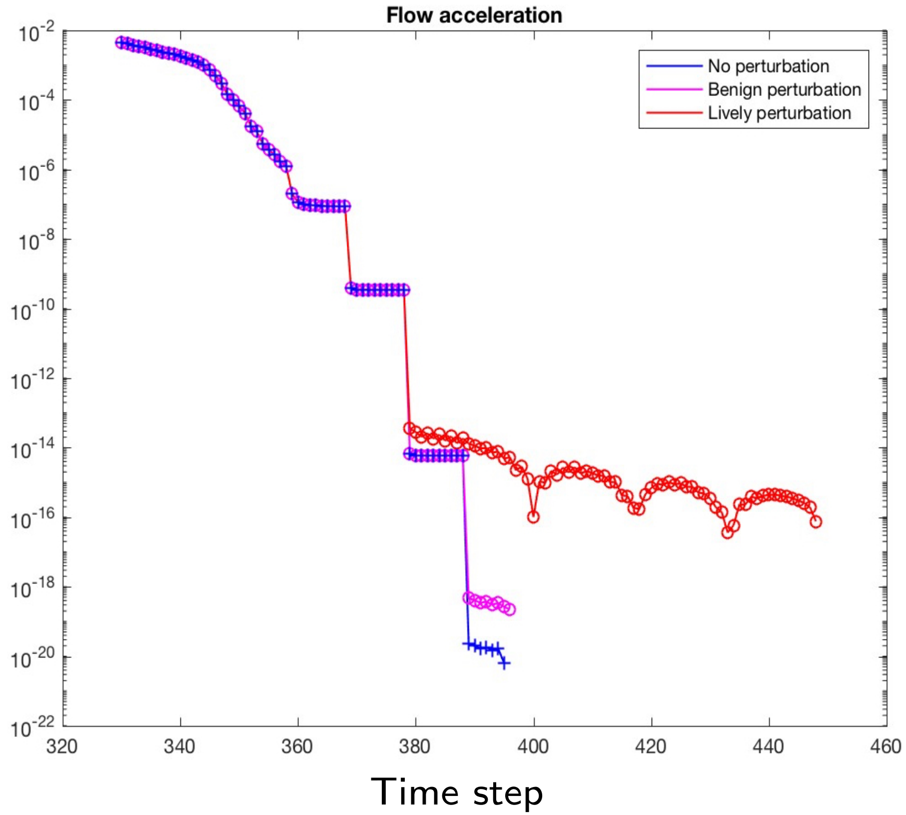


Long-term behavior, perturbation #2 (lively)



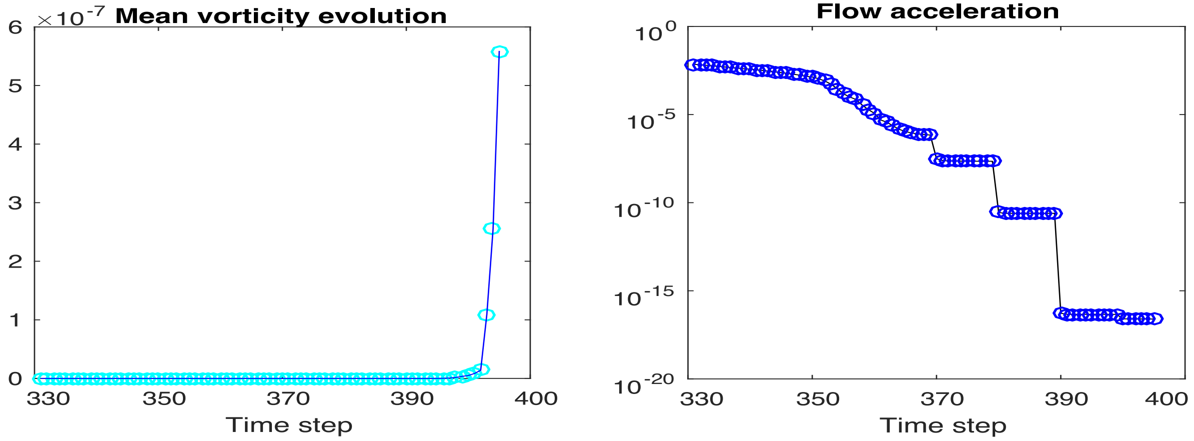
Display these results differently:



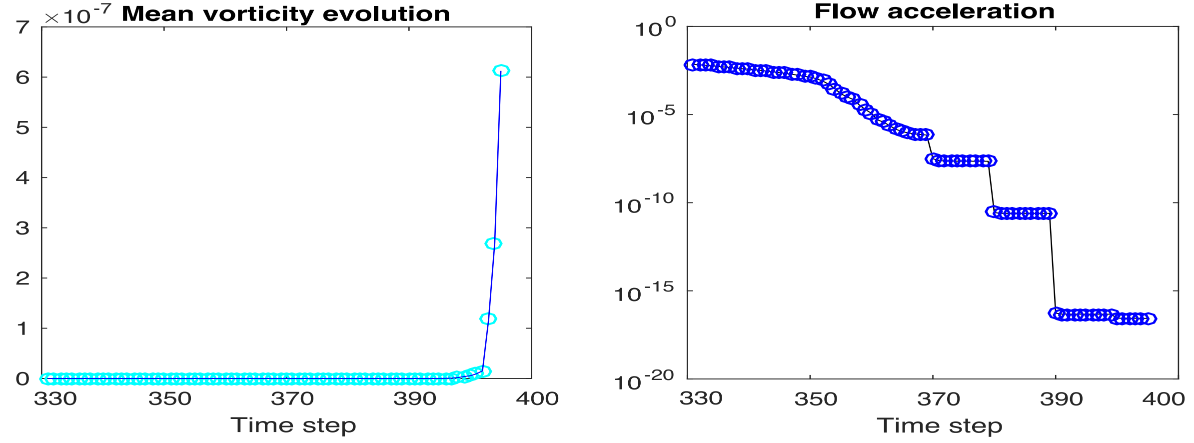


Repeat experiment for **near critical** ν=1/220

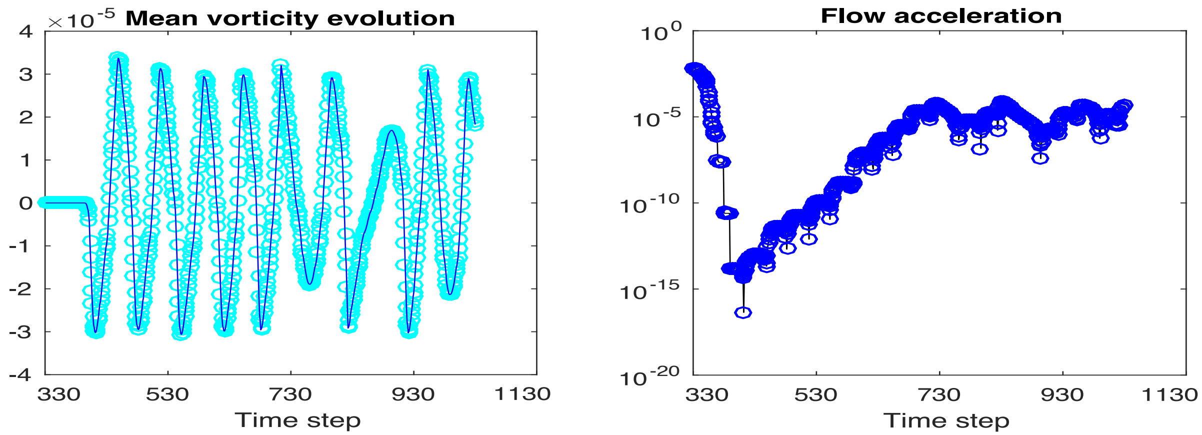
Long-term behavior, no perturbation



Long-term behavior, perturbation #1 (benign)



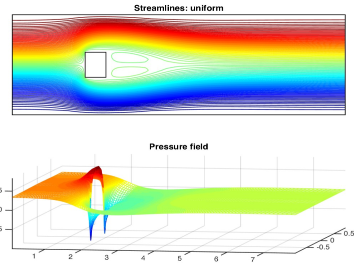
Long-term behavior, perturbation #2 (lively)

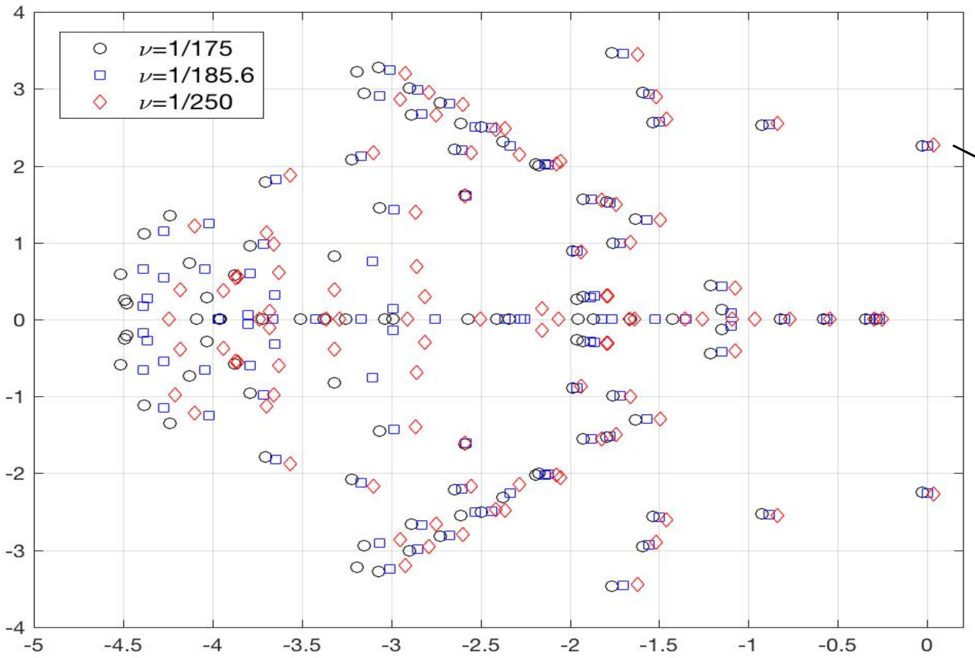


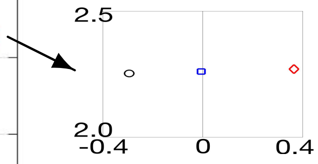
Summarizing these results, for flow in expanding step:

- Transient iteration is consistent with perturbation analysis
  – Instability for near-critical parameter is displayed
  – Flow for sub-critical (but barely so) parameter is stable but slight leanings to instability can be observed
- Symmetry-breaking for super-critical parameter
- Effects can also seen in time step choices made by a good integrator

Eigenvalues for obstable problem







ν	Re(λ)
$1/175$	-2.9×10^{-2}
$1/185.6$	-3.0×10^{-4}
$1/200$	3.7×10^{-2}

Perturbed eigenvalues

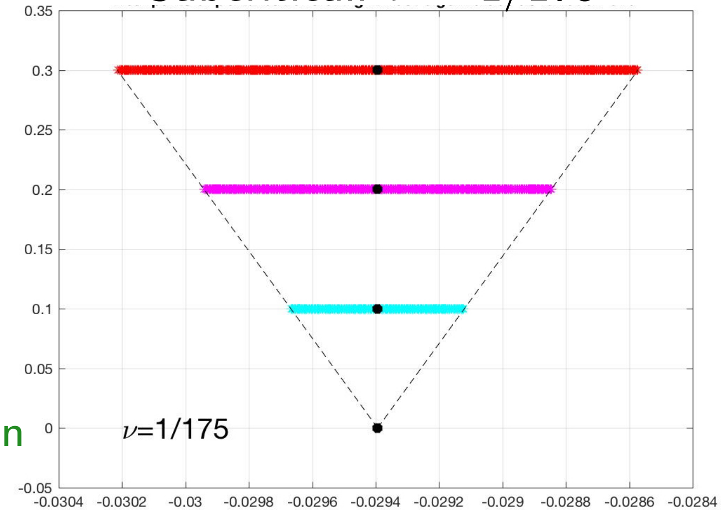
For this:

Solve 760 perturbed eigenvalue problems

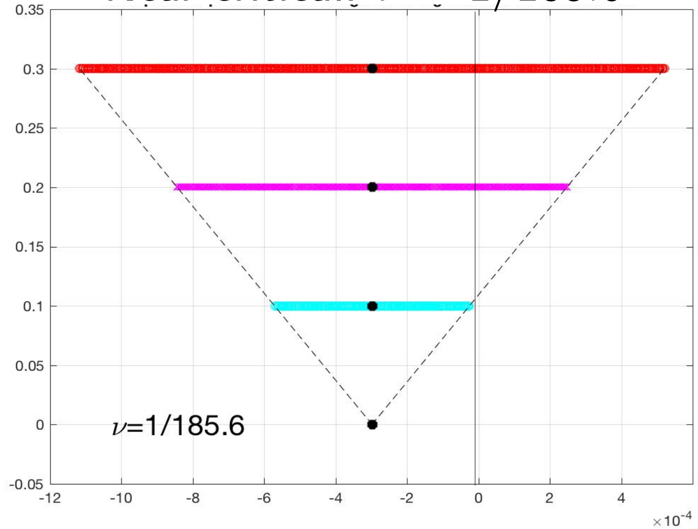
Sample surrogate

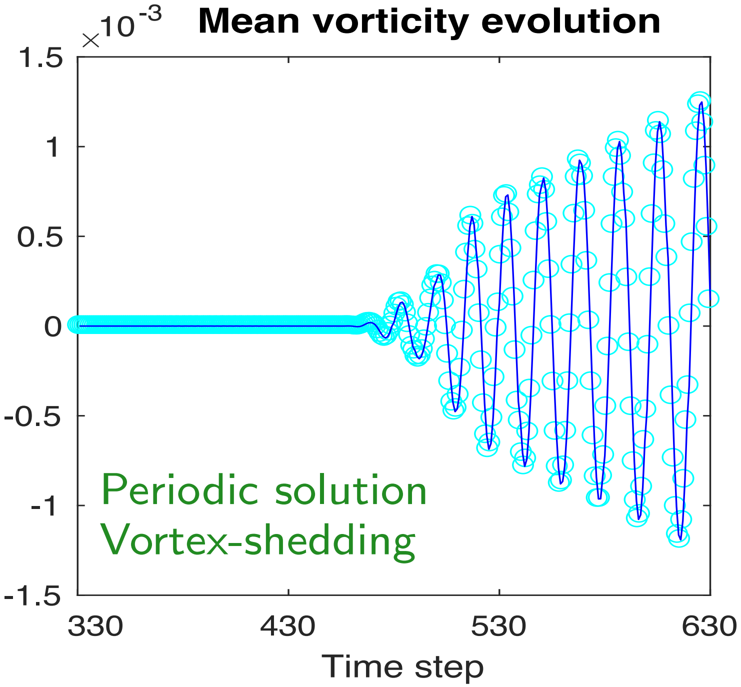
100K samples, ~1 min

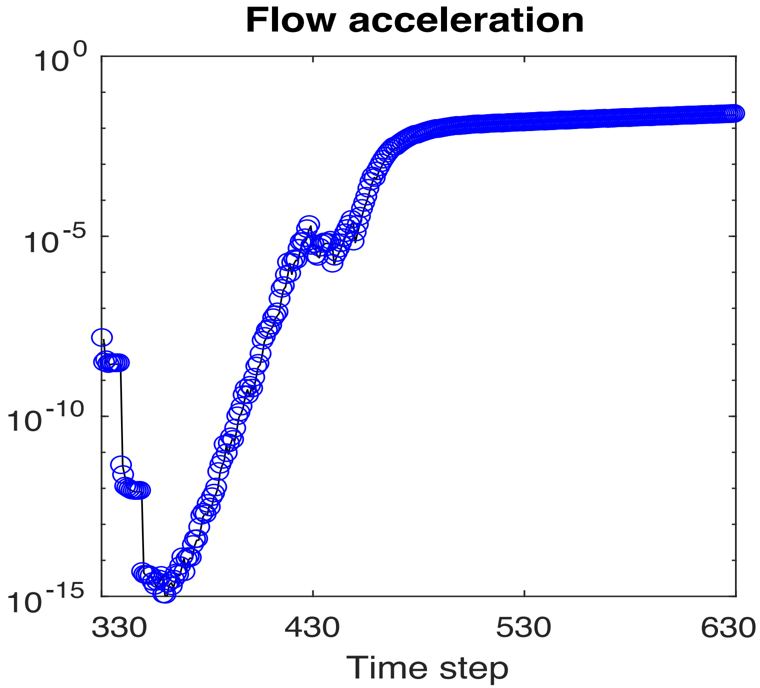
Subcritical: $\nu = 1/175$



Near critical: $\nu = 1/185.6$



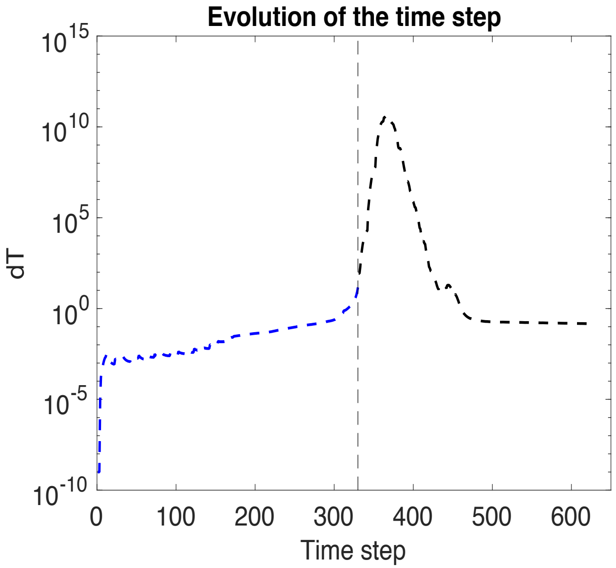




Simulation for obstacle,

super-critical $\nu = 1/200$

after interrupt

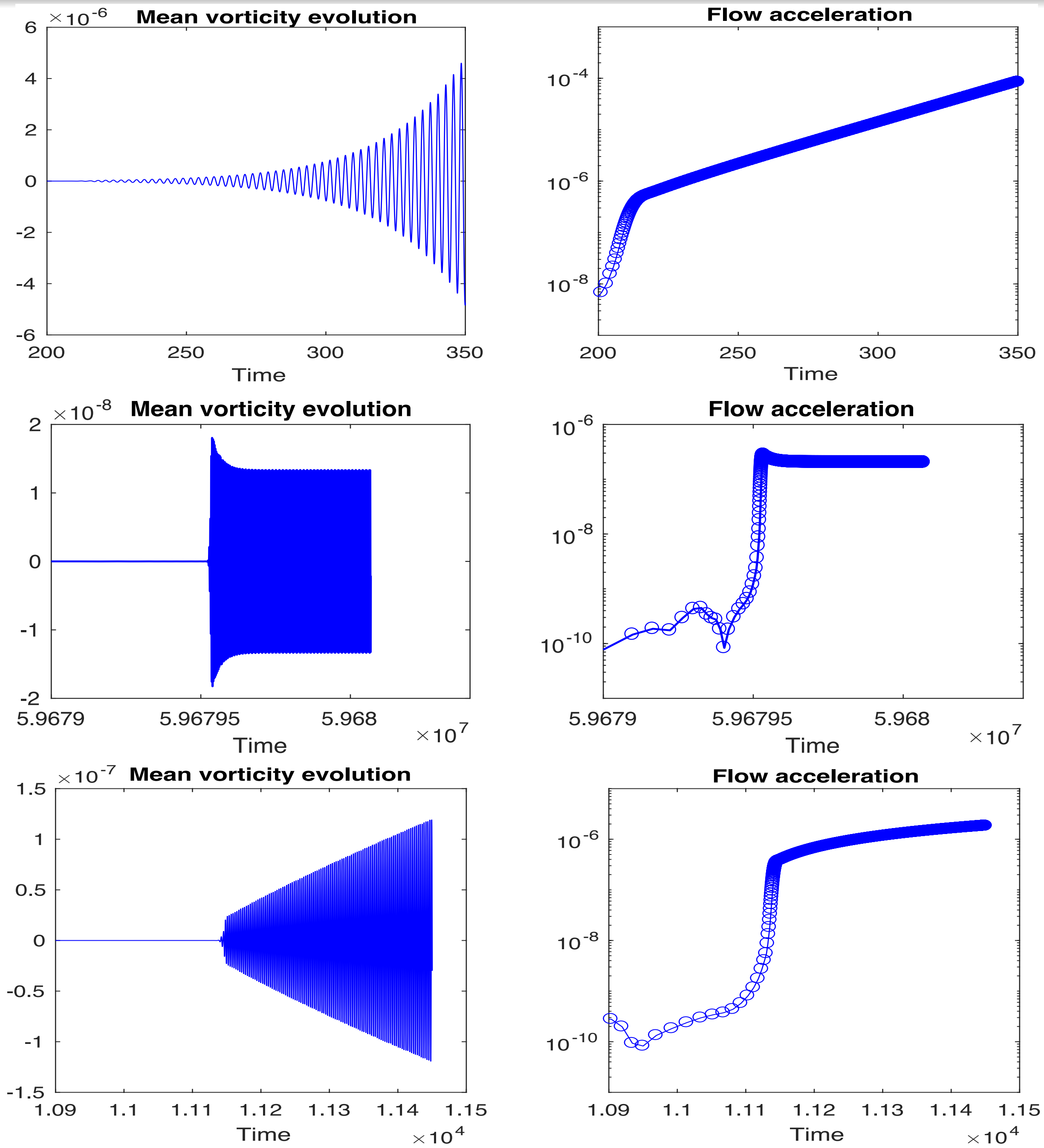


Highlights of evolution for three parameters, with **no perturbation**

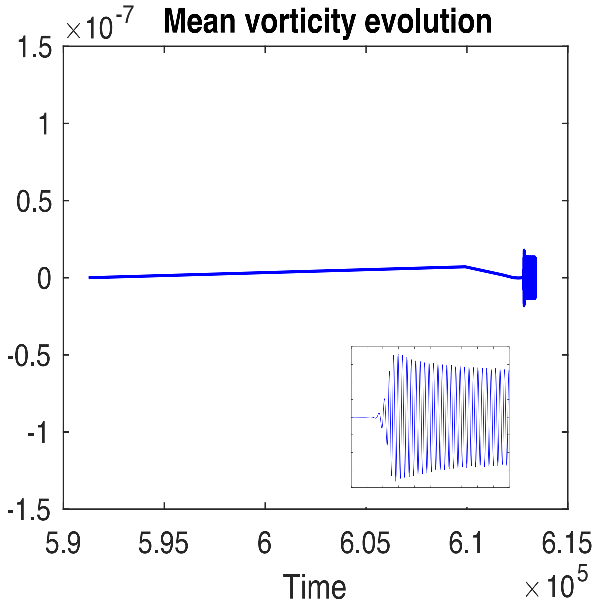
Super-critical, $\nu = 1/200$

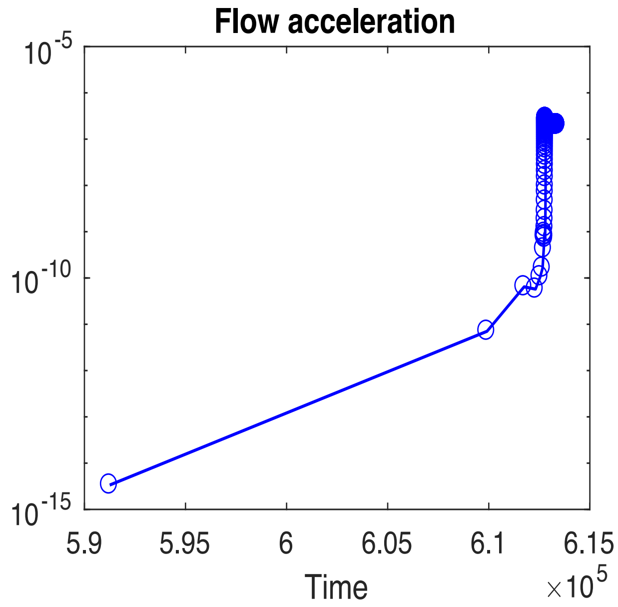
Sub-critical, $\nu = 1/175$

Near-critical, $\nu = 185.6$

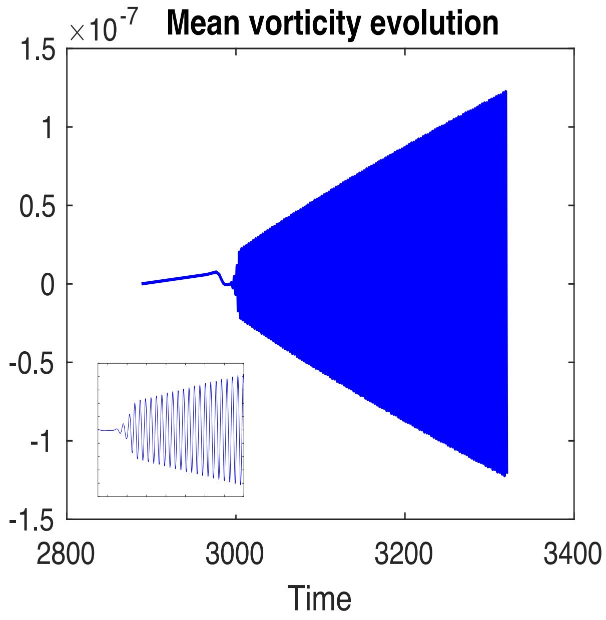


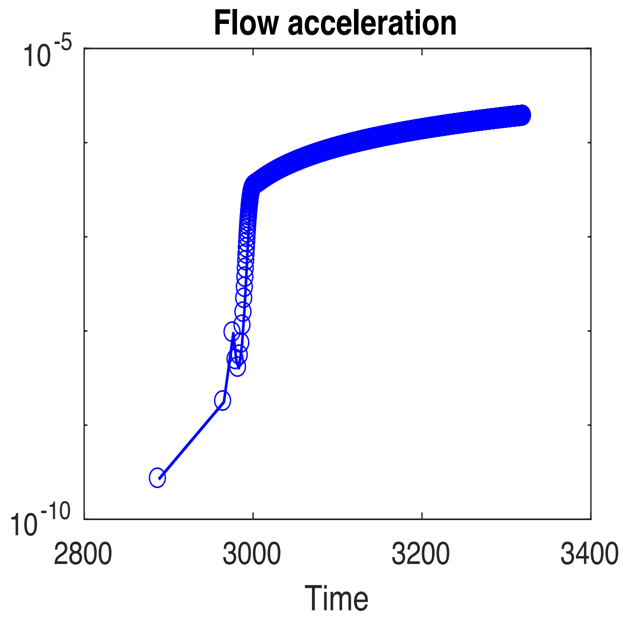
Impact of perturbation (lively)





Subcritical parameter
$\nu = 1/175$





Near-critical parameter
$\nu = 1/185.6$

Summarizing these results, for flow around obstacle:

- Transient iteration is again consistent with perturbation analysis
  – For sub-critical parameter, performance with perturbation is like that for no perturbation
  – For near-critical parameter, performance with perturbation is like that for super-critical regime
- Results affected by delicacy of stability analysis
  – Some instability is seen even for subcritical parameters
  Caused by truncation error in transient iteration

For both benchmark problems:

- New relatively cheap method for finding pseudospectra is predictive of behavior of simulation in time
- Refined understanding of simulation in time near stability limit