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**Non-Abelian gravitational solitons
and black holes**

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Plan

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Gravity

The Einstein equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu} + \Lambda g_{\mu\nu}$$

Can be obtained from the action principle

$$S = S_g + S_M .$$

Some important solutions:

1. Flat space-time $g_{\mu\nu} = (+1, -1, -1, -1)$.
2. Schwarzschild:

$$ds^2 = \left(1 - \frac{2m}{r}\right)dt^2 - \frac{dr^2}{\left(1 - \frac{2m}{r}\right)} - r^2 d\Omega_2^2$$

3. Reissner-Nordstrom:

$$ds^2 = \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)dt^2 - \frac{dr^2}{\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)} - r^2 d\Omega_2^2$$

The EYM theory

Is defined by the action

$$S_{EYM} = \frac{1}{4\pi} \int \left(-\frac{c^3}{4G} R - \frac{1}{4g^2 c} F_{\mu\nu}^a F^{a\mu\nu} \right) \sqrt{-g} d^4x ,$$

The spherically symmetric metric can be parameterized as

$$ds^2 = A^2 B dt^2 - \frac{dr^2}{B} - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) .$$

It is common to express $B(r)$ through the “mass function” $m(r)$ defined by

$$B(r) = 1 - 2m(r)/r.$$

For the $SU(2)$ gauge field we take ‘monopole’ ansatz

$$W_0^a = 0, \quad W_i^a = \epsilon_{aij} \frac{n^j}{r} (1 - W(r)) .$$

From two coupling constants

$$[G] = L^3 T^{-2} M^{-1}, \quad [g] = L^{-1} T^{1/2} M^{-1/2},$$

one can form

$$l_{EYM} = \sqrt{G/cg^2}, \quad m_{EYM} = \sqrt{c/Gg^2}.$$

In dimensionless variables the reduced action is

$$S_{EYM}^{\text{red}} = \sqrt{\frac{c^5}{Gg^2}} \int A \left[\frac{1}{2}(B + rB' - 1) \right. \\ \left. + \left(BW'^2 + \frac{(1 - W^2)^2}{2r^2} \right) \right] dr.$$

Resulting field equations for $W, U \equiv W', B$ and A are

$$(BW')' = \frac{W(W^2 - 1)}{r^2} - \frac{2BW'^3}{r},$$
$$B' = \frac{1}{r} \left(1 - B - BW'^2 - \frac{(1 - W^2)^2}{r^2} \right),$$
$$\frac{A'}{A} = \frac{2W'^2}{r}.$$

These Eqs have **regular singular points** at $r = 0, \infty$ and at some r_h , where $B(r_h) = 0$.

At $r \rightarrow 0$ **one-parameter family** b :

$$W(r) = 1 - br^2 + O(r^4) ,$$

$$B(r) = 1 - 4b^2r^2 + O(r^4) .$$

At $r \rightarrow \infty$ **two-parameter family** c, m :

$$W(r) = \pm \left(1 - \frac{c}{r} + O\left(\frac{1}{r^2}\right)\right) ,$$

$$B(r) = 1 - \frac{2m}{r} + O\left(\frac{1}{r^4}\right) .$$

At $r \rightarrow r_h$ **two-parameter family** r_h, W_h :

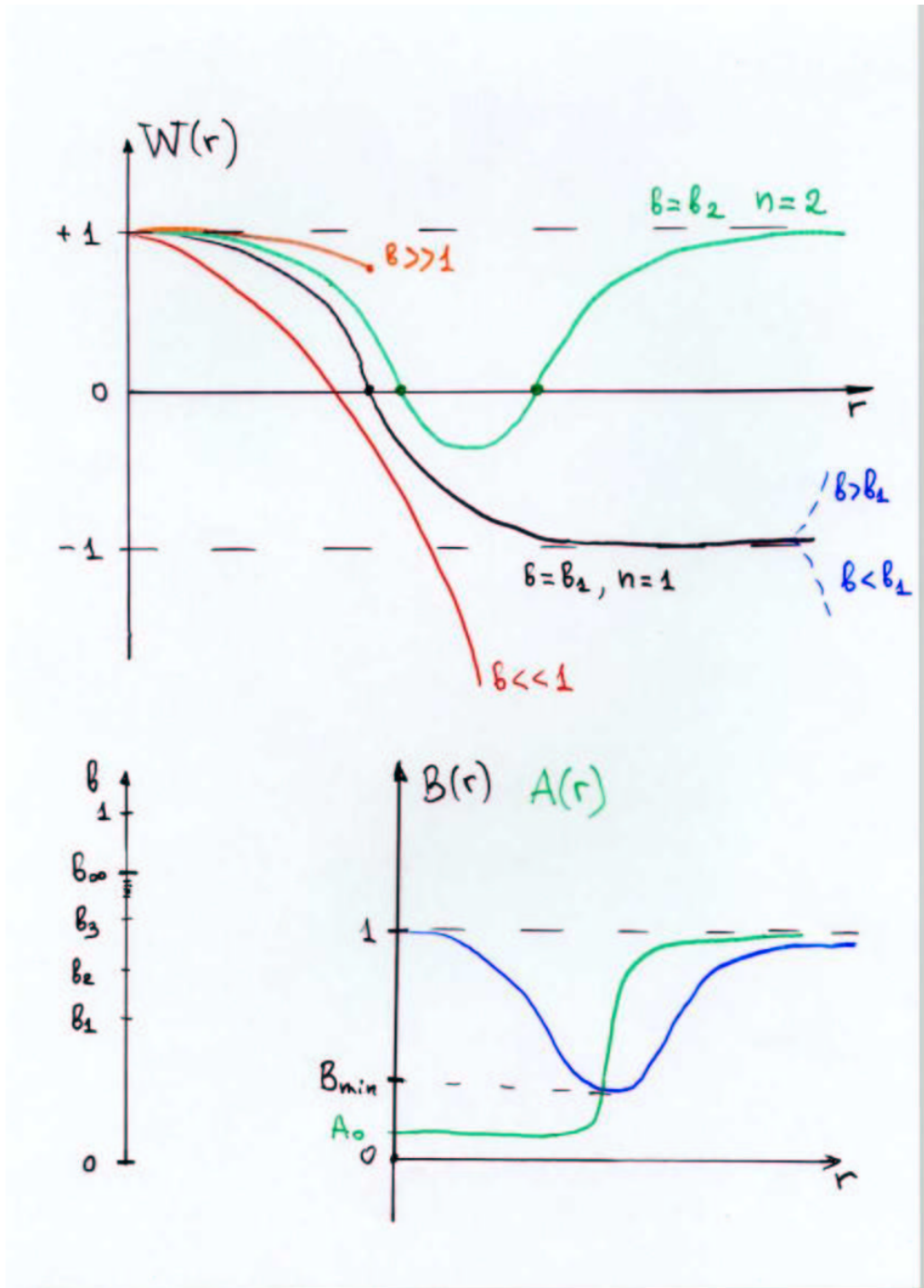
$$W(r_h + \rho) = W_h + W'_h \rho + O(\rho^2) ,$$

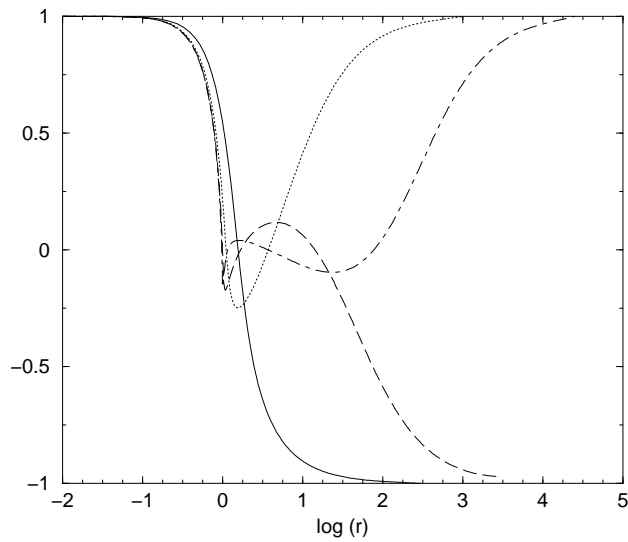
$$B(r_h + \rho) = B'_h \rho + O(\rho^2) ,$$

with $\rho = r - r_h$ and

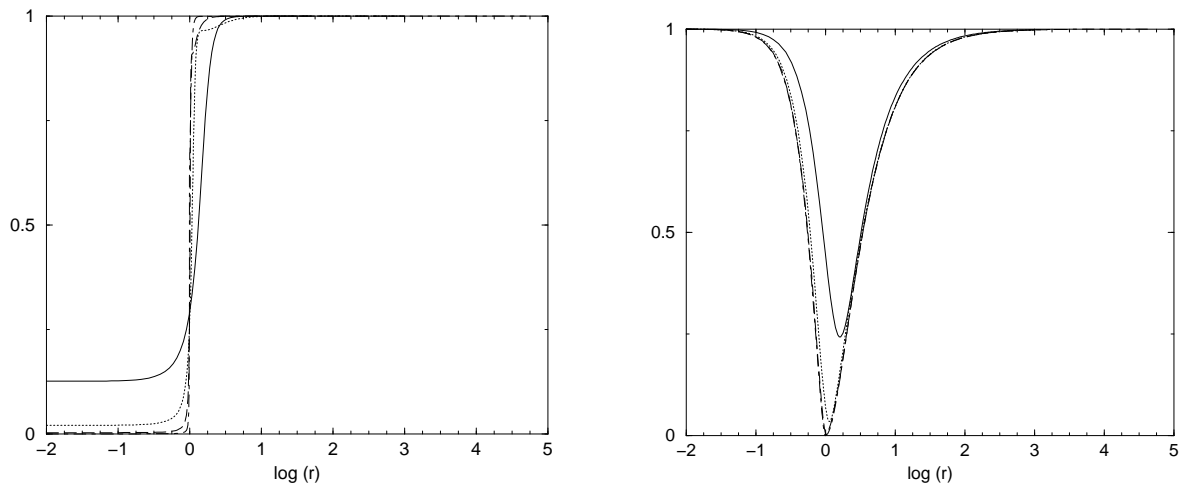
$$B'_h = \frac{1}{r_h} \left(1 - \frac{(W_h^2 - 1)^2}{r_h^2}\right) , \quad W'_h = \frac{W_h(W_h^2 - 1)}{B'_h r_h^2} .$$

Qualitative picture





The gauge field amplitude $W(r)$ for the first four globally regular solution of EYM theory.



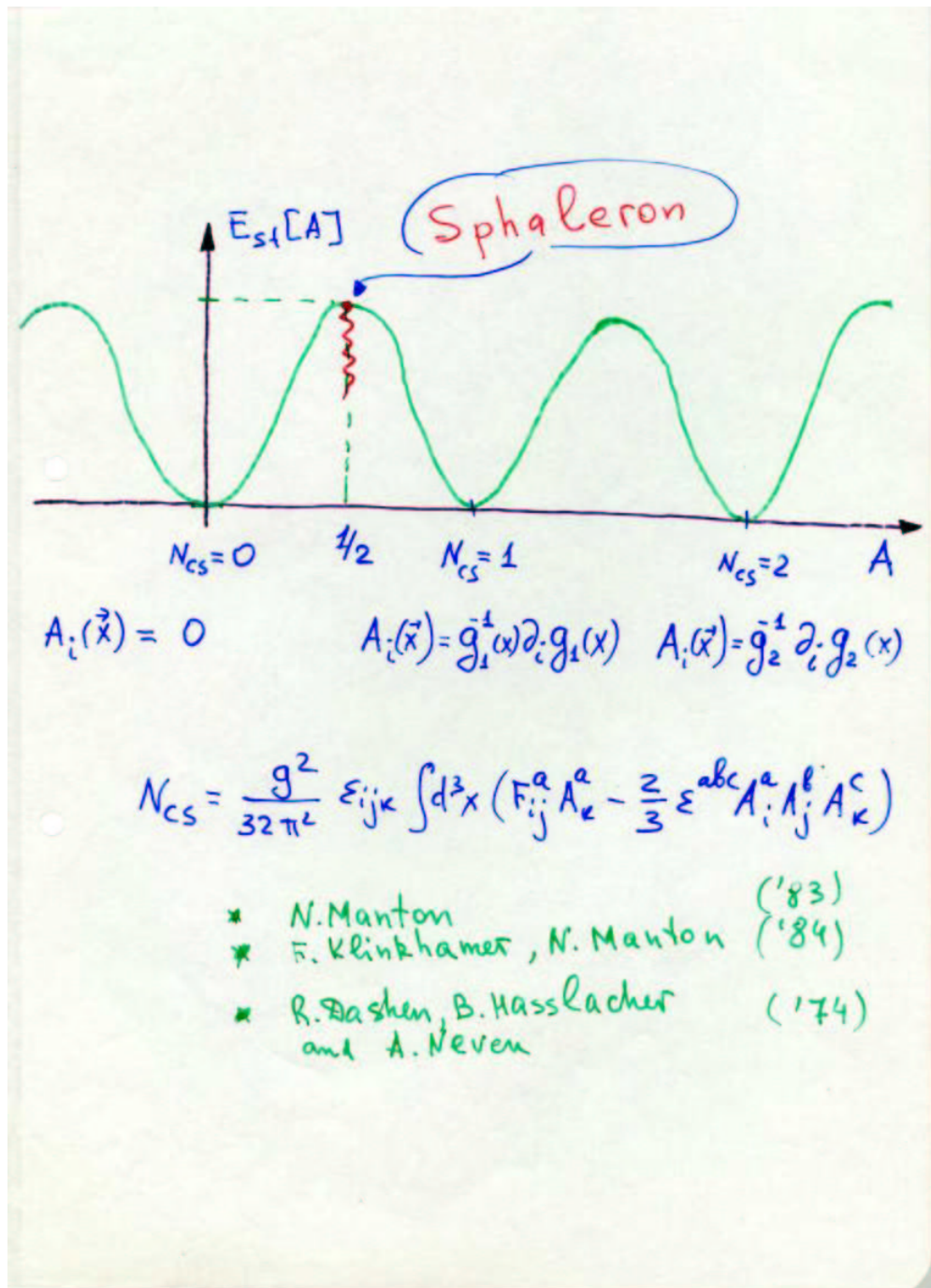
The metric functions $A(r)$ (left) and $B(r)$ (right) for the first four globally regular solution of EYM theory.

Parameters of the globally regular solutions of EYM theory:

n	b_n	c_n	M_n	B_n^{min}
1	0.453716	0.89338	0.828647	0.2424
2	0.651726	8.86389	0.971345	0.03506
3	0.697040	$5.8932 \cdot 10^1$	0.995316	0.002974
4	0.704878	$3.6634 \cdot 10^2$	0.999236	0.000253
5	0.706169	$2.2519 \cdot 10^3$	0.999875	$5.609 \cdot 10^{-5}$
6	0.706379	$1.3817 \cdot 10^4$	0.999979	$3.325 \cdot 10^{-5}$
∞	0.7064204	1	-	0

P. Breitenlohner, P. Forgács, and D. Maison, CMP 1994

Sphaleron interpretation



Spherically symmetric $SU(2)$ field

The most general spherically symmetric ansatz for $SU(2)$ YM field (in Abelian gauge) can be written as

E.Witten 1977; P.Forgács, N.Manton 1980

$$\begin{aligned} W_t^a &= (0, 0, A_0), & W_\theta^a &= (\phi_1, \phi_2, 0), \\ W_r^a &= (0, 0, A_1), & W_\varphi^a &= (-\phi_2 \sin \theta, \phi_1 \sin \theta, \cos \theta). \end{aligned}$$

This ansatz is form-invariant under gauge transformations around the third isoaxis with A_α transforming as a $U(1)$ gauge field on the reduced space-time (t, r) , whereas $\phi = \phi_1 + i\phi_2$ is a scalar field of charge one w.r.t. the $U(1)$, $A_\alpha \rightarrow A_\alpha + \partial_\alpha \Omega$, $\phi \rightarrow e^{i\Omega} \phi$, W.r.t. this $U(1)$ one may define the ‘charge conjugation’ $A_\alpha \rightarrow -A_\alpha$, $\phi \rightarrow \bar{\phi}$. The even sector with respect to this charge conjugation is given by

$$A_0 = 0, \quad A_1 = 0, \quad \phi_1 \equiv W(r), \quad \phi_2 = 0.$$

In this sector the general ansatz is equivalent to the usual “monopole” ansatz.

Stability analysis

Straumann, Zhou, 1990

Gravitational (even) sector

$$\Phi_1 \rightarrow W(r) + \delta W(r)e^{i\omega t}$$

$$B \rightarrow B(r) + \delta B(r)e^{i\omega t}$$

$$A \rightarrow A(r) + \delta A(r)e^{i\omega t}$$

G.L., D. Maison, PLB 1995

Sphaleron (odd) sector

$$\Phi_2 \rightarrow 0 + \delta\xi(r)e^{i\omega t}$$

$$A_1 \rightarrow 0 + \delta a_1(r)e^{i\omega t}$$

$$A_0 \equiv 0$$

Conclusion: n th BK has $n + n = 2n$ unstable modes!

The numerical values for the energies $E = \omega^2$ of the negative modes of the first three BK solutions in the gravitational (even) and sphaleron (odd) sectors:

$n = 1$	$n = 2$	$n = 3$
$E_1 = -0.0525$	$E_1 = -0.0410$	$E_1 = -0.0339$
	$E_2 = -0.0078$	$E_2 = -0.0045$
		$E_3 = -0.0006$

$n = 1$	$n = 2$	$n = 3$
$E_1 = -0.0619$	$E_1 = -0.0360$	$E_1 = -0.0346$
	$E_2 = -0.0105$	$E_2 = -0.0037$
		$E_3 = -0.0009$

Volkov, Brodbeck, G.L., Straumann, PLB (1995)

The number of unstable modes in the sphaleron sector can be determined even analytically!

Main idea is that the zero energy wave function is proportional to the gauge field amplitude

$$\psi_0(r) \sim W(r) .$$

Black holes

P. Bizon 1990

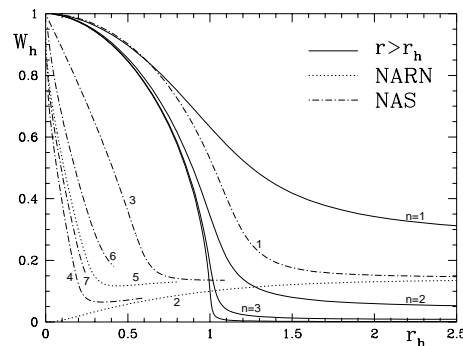
M.S. Volkov and D.V. Gal'tsov 1990

H.P. Künzle and A.K.M. Masood-ul-Alam 1990

Soon after Bartnik-McKinnon discovery corresponding black holes were found. Since these black holes have non-trivial YM field outside of horizon they are often called as 'colored' or 'hairy' black holes.

P.Breitenlohner, P.Forgács, D.Maison 1994

P.Breitenlohner, G.L., D.Maison 1998



Initial data for special solutions. The solid curves represent asymptotically flat solutions with n zeros of W . The other curves represent various NARN and NAS families.

EYMD theory

Consider theory defined by the action

$$S = \int [-R + 2(\nabla\phi)^2 - e^{2\gamma\phi} F^2] \sqrt{-g} d^4x ,$$

where ϕ is dilaton field, γ is dilatonic coupling constant, F is $SU(2)$ gauge field strength.

G.L. Maison 1993; Donets, Galtsov, 1993; Bizon 1993; Torii, Maeda 1993

The EYMD field equations are

$$(Ae^{2\gamma\phi} BW')' = Ae^{2\gamma\phi} \frac{W(W^2 - 1)}{r^2} ,$$

$$(ABr^2\phi')' = 2\gamma Ae^{2\gamma\phi} \left(BW'^2 + \frac{(1 - W^2)^2}{2r^2} \right) ,$$

$$B' = \frac{1}{r} \left(1 - B - r^2 B\phi'^2 - 2e^{2\gamma\phi} \left(BW'^2 + \frac{(1 - W^2)^2}{2r^2} \right) \right) ,$$

$$\frac{A'}{A} = \frac{2e^{2\gamma\phi} W'^2}{r} + r\phi'^2$$

This field equations have singular points at $r = 0$ and $r = \infty$ as well as at points where $B(r)$ vanishes. In the vicinity of $r = 0$

$$\begin{aligned} W(r) &= 1 - br^2 + O(r^4) , \\ B(r) &= 1 - 4b^2e^{2\gamma\varphi_0}r^2 + O(r^4) , \\ \varphi(r) &= \varphi_0 + 2\gamma e^{2\gamma\varphi_0}b^2r^2 + O(r^4) , \\ A(r) &= 1 + 4b^2e^{2\gamma\varphi_0}r^2 + O(r^4) . \end{aligned}$$

where b and φ_0 are arbitrary parameters. Similarly at $r = \infty$ one finds

$$\begin{aligned} W(r) &= \pm\left(1 - \frac{c}{r} + O\left(\frac{1}{r^2}\right)\right) , \\ B(r) &= 1 - \frac{2M}{r} + O\left(\frac{1}{r^2}\right) , \\ \varphi(r) &= \varphi_\infty - \frac{d}{r} + O\left(\frac{1}{r^2}\right) , \\ A(r) &= A_\infty\left(1 - \frac{d^2}{2r^2} + O\left(\frac{1}{r^4}\right)\right) . \end{aligned}$$

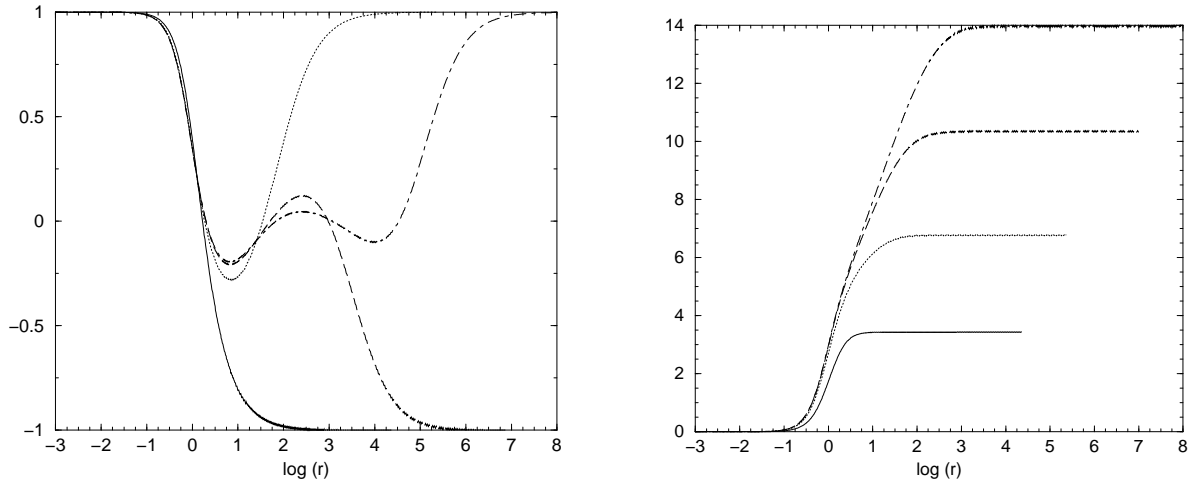
where again c, d, M, φ_∞ and A_∞ are arbitrary parameters.

The limits:

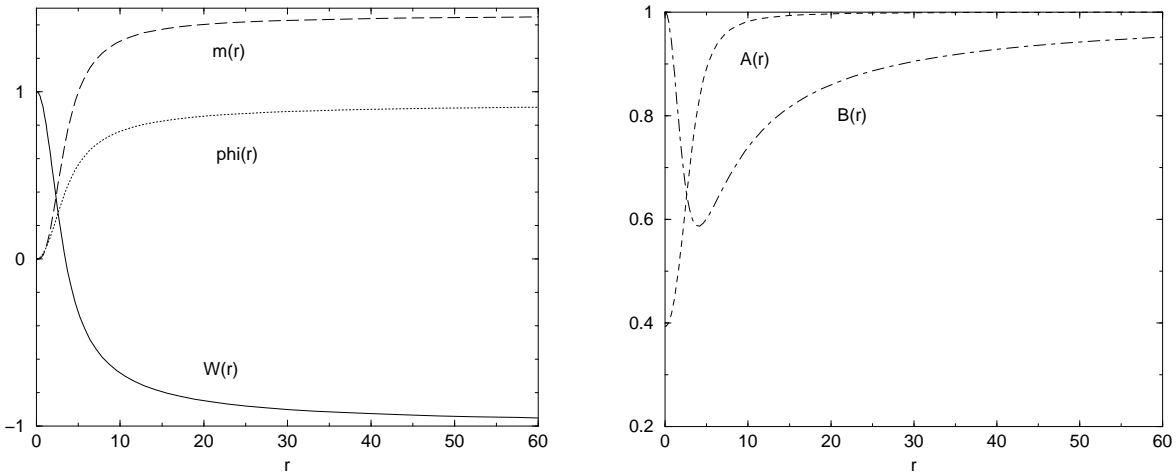
$\gamma = 0$ EYM theory Bartnik, McKinnon 1988

$\gamma \rightarrow \infty$ YM-dilaton theory in a flat space
G.L. Maison 1992, Bizon 1993

$\gamma = 1$ corresponds to a model obtained from string theory



The gauge field amplitude $W(r)$ (left) and the dilaton $\varphi(r)$ (right) for the first four globally regular solution of YMD theory.



The lowest $n = 1$ globally regular solution of EYMD theory. The gauge field amplitude $W(r)$ is shown by solid, the dilaton $\varphi(r)$ by dotted, mass function $m(r)$ by long dashed, metric functions $A(r)$ and $B(r)$ by dashed and dot-dashed lines respectively.

n	b	φ_∞	M	B_{min}
1	0.166666	0.932284	0.57695	0.5864
2	0.231800	1.792793	0.68481	0.3705
3	0.246862	2.692205	0.70344	0.2868
4	0.249484	3.597983	0.70651	0.2637
5	0.249916	4.504705	0.70702	0.2587
6	0.249986	5.411575	0.70709	0.2534

Parameters of the $n = 1..6$ globally regular solutions of EYMD theory for $\gamma = 1.0$.

A very interesting situation occurs for the special value of the dilatonic coupling constant $\gamma = 1$, corresponding to a model obtained from superstring theory.

In addition to

$$g_{00} \equiv A^2 B = e^{2\phi} \quad (\rightarrow M = d)$$

for the $n = 1$ solution one finds that

- $b = \frac{1}{6}$ rational number!
- $c = 2M$

Based on these observations it was suggested that the $n = 1$ solution can be obtained in a closed form.

G.L., Maison 1993

Bogomol'nyi equations

It was noticed that the EYMD Lagrangian almost coincides with the bosonic part of Friedman-Schwarz supergravity model.

$$U(\phi) = -\frac{1}{8}(g_1^2 + g_2^2)e^{-2\phi} \quad N = 4 \text{ gauged} \\ SU(2) \times SU(2)$$

⇓

0

⇓

$$N=4 \text{ gauged} \\ SU(2) \times SU(1, 1)$$

With the 'imaginary trick' $g_2 = ig_1$ potential term in the Friedman-Schwarz model disappears and the resulting theory exactly coincides with the EYMD (for $\gamma = 1$). Applying the supersymmetry machinery it was possible to find corresponding Bogomol'nyi equations.

Choosing the metric in the form

$$ds^2 = N^2 dt^2 - e^{2\tau} \left(\frac{d\tau^2}{\nu^2} + d\Omega^2 \right) ,$$

where ν is defined through equation

$$\nu^2 = \frac{1 - e^{2\psi} (W^2 - 1)^2}{2 - \psi'^2 - 2e^{2\psi} W'^2} ,$$

with $\psi = \varphi - \tau$ and prime denoting differentiation w.r.t. τ , the Bogomol'nyi equations were found to be:

$$\nu W' = -2\nu^2(\psi' + 2) + W^2 + 1 ,$$

$$\nu\psi' = -\nu + (X + 1) \frac{W(X - 1) - \nu}{X - 1 - 8W\nu e^{2\psi}} ,$$

$$2\nu^2 = W^2 + 1 + e^{2\psi} (W^2 - 1)^2 ,$$

$$N = e^{\varphi - \varphi_\infty} ,$$

with $X = 2e^{2\psi} (W^2 - 1)$ and it is confirmed that the $n = 1$ solution of EYMD theory studied previously numerically is indeed satisfies these Bogomol'nyi equations.

EYM_Λ theory with Λ > 0

The action of EYM theory with the cosmological constant Λ has the form:

$$S_{EYM_\Lambda} = \frac{1}{4\pi} \int \left(-\frac{1}{4G}(R+2\Lambda) - \frac{1}{4g^2} F_{\mu\nu}^a F^{a\mu\nu} \right) \sqrt{-g} d^4x .$$

It turns out that the situation is rather different in case of positive and negative values of cosmological constant.

$$(BW')' = \frac{W(W^2 - 1)}{r^2} - \frac{2BW'^3}{r} ,$$

$$m' = \left(BW'^2 + \frac{(W^2 - 1)^2}{2r^2} \right) ,$$

$$\frac{A'}{A} = \frac{2W'^2}{r}$$

with $B(r) = 1 - \frac{2m(r)}{r} - \frac{\Lambda}{3}r^2$.

Note that equation for A decouples as in pure EYM case.

At $r \rightarrow 0$ one-parameter family b (for given Λ):

$$W(r) = 1 - br^2 + O(r^4) ,$$

$$B(r) = 1 - \left(4b^2 + \frac{\Lambda}{3}\right)r^2 + O(r^4) ,$$

In $\Lambda > 0$ case the solutions have a cosmological horizon, where $B(r)$ vanishes. Close to it one finds

$$W(r) = W_h + W'_h x + O(x^2), \quad B(r) = B'_h x + O(x^2) ,$$

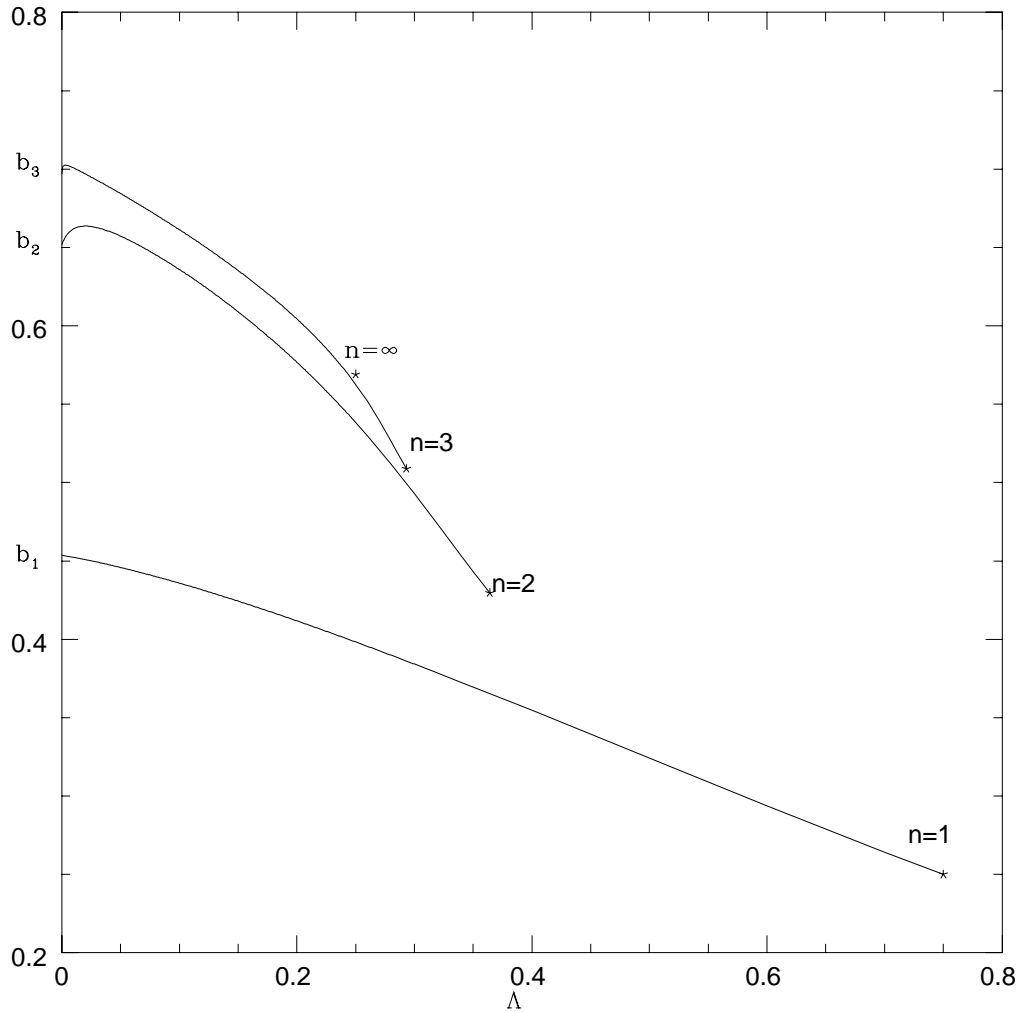
where $x = r - r_h$ and W'_h and $B'_h < 0$ are some functions.

At $r = \infty$ for asymptotically de Sitter solutions

$$W(r) = W_\infty + \frac{p}{r} - \frac{3W_\infty(W_\infty^2 - 1)}{4\Lambda r^2} + O\left(\frac{1}{r^3}\right) ,$$

$$B(r) = 1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2 + \frac{Q_{eff}^2}{r^2} + O\left(\frac{1}{r^3}\right) ,$$

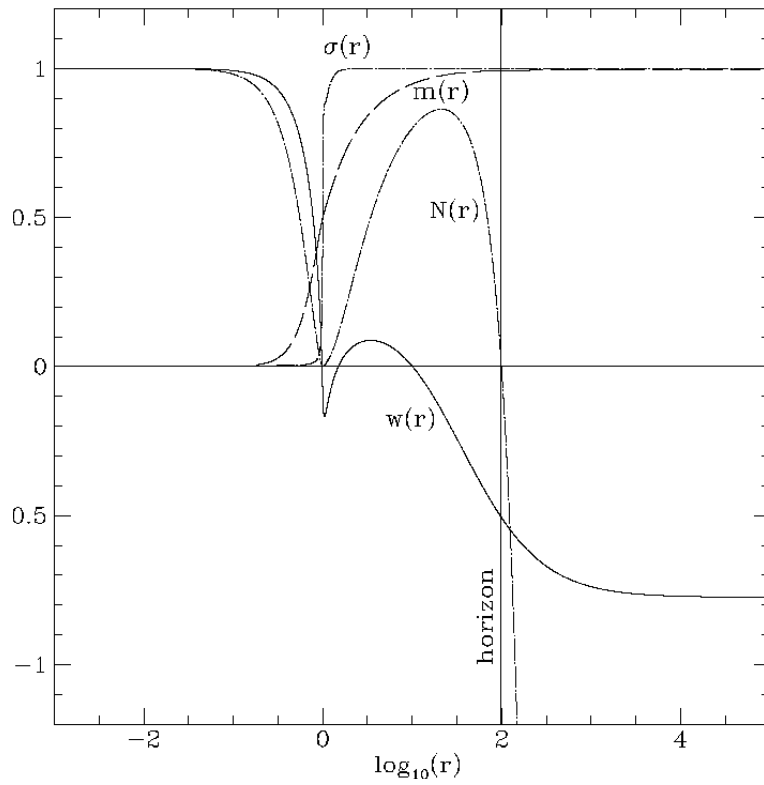
with $Q_{eff}^2 = (W_\infty^2 - 1)^2 - \frac{2}{3}\Lambda p^2$. Here b, r_h, W_h, W_∞, p and M are six “shooting” parameters.



Shooting parameter b versus cosmological constant (for $\Lambda > 0$) for different solutions of the EYM_Λ theory.

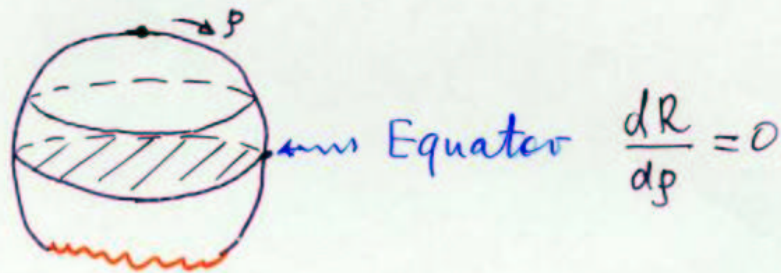
n	$\Lambda_{crit}(n)$	$\Lambda_\diamond(n)$	$\Lambda_\star(n)$
1	0.330	0.334	0.75
2	0.239	0.250	0.364
3	0.237	0.247	0.293

The special values of Λ for the lowest n 's.



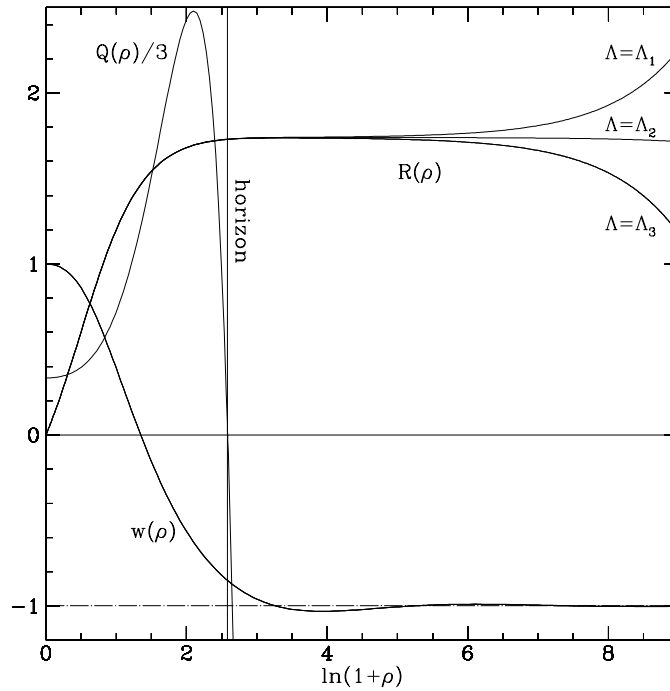
Asymptotically deSitter solution with $\Lambda = 3 \times 10^{-4}$ and $n = 3$. For this solution one finds $b = 0.6998$, $r_h = 98.99$, $w_h = -0.505$, $w_\infty = -0.774$, $M = m(\infty) = 0.994$, $a = -37$, $\sigma(0) = 2 \times 10^{-3}$, $\sigma(r_h) = 0.99999$ and $N_{min} = 1.7 \times 10^{-3}$. Here $N \equiv B$, $\sigma \equiv A$.

Solutions with 'equator' cannot be obtained in Schwarzschild-like coordinates!

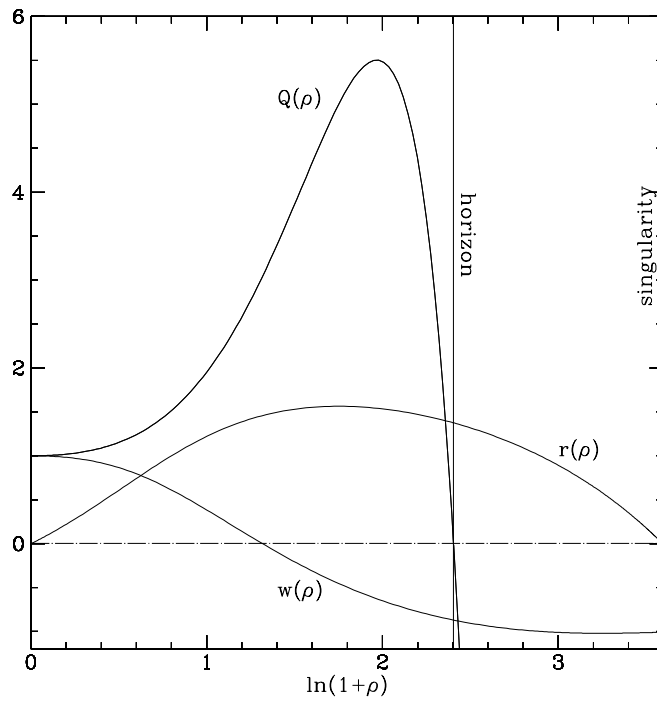


Way out is to choose another coordinate system, like

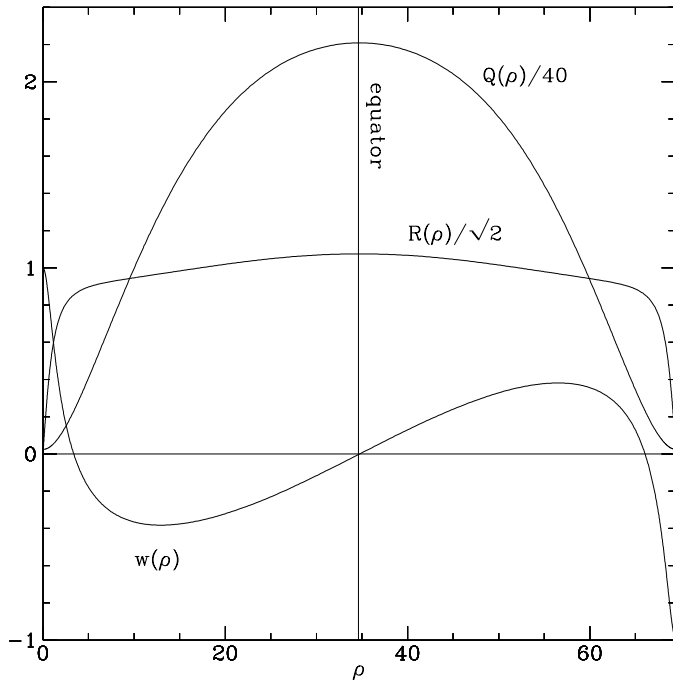
$$ds^2 = Q(p) dt^2 - \frac{dp^2}{Q(p)} - R^2(p) d\Omega_2^2$$



Change of the topology of the EYM solutions. The solution with $\Lambda = \Lambda_1 = 0.3304$ is asymptotically deSitter, whereas the one with $\Lambda = \Lambda_3 = 0.3306$ is of the bag of gold type. The value $\Lambda = \Lambda_2 = 0.3305$ is very close to Λ_{crit} . The functions $Q(\rho)$ and $w(\rho)$ for the three solutions are almost identical.



The bag of gold solution with $\Lambda = 0.4$ and $n = 1$.



The $n = 3$ compact solution.

EYM_Λ theory with $\Lambda < 0$

In this case no cosmological horizon anymore. Many conditions are relaxed.

- * Solutions can be nodeless in W (\rightarrow stability!)
- * Solutions with non-zero electric part are allowed
- * W_∞ is not necessary ± 1

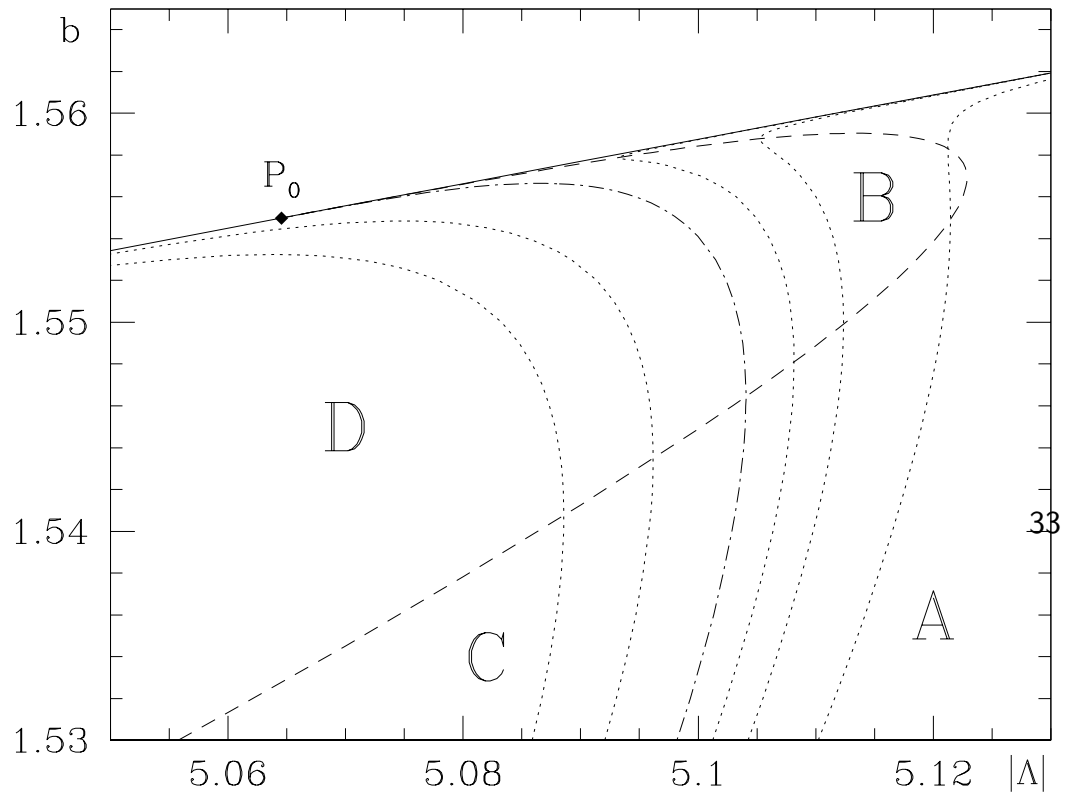
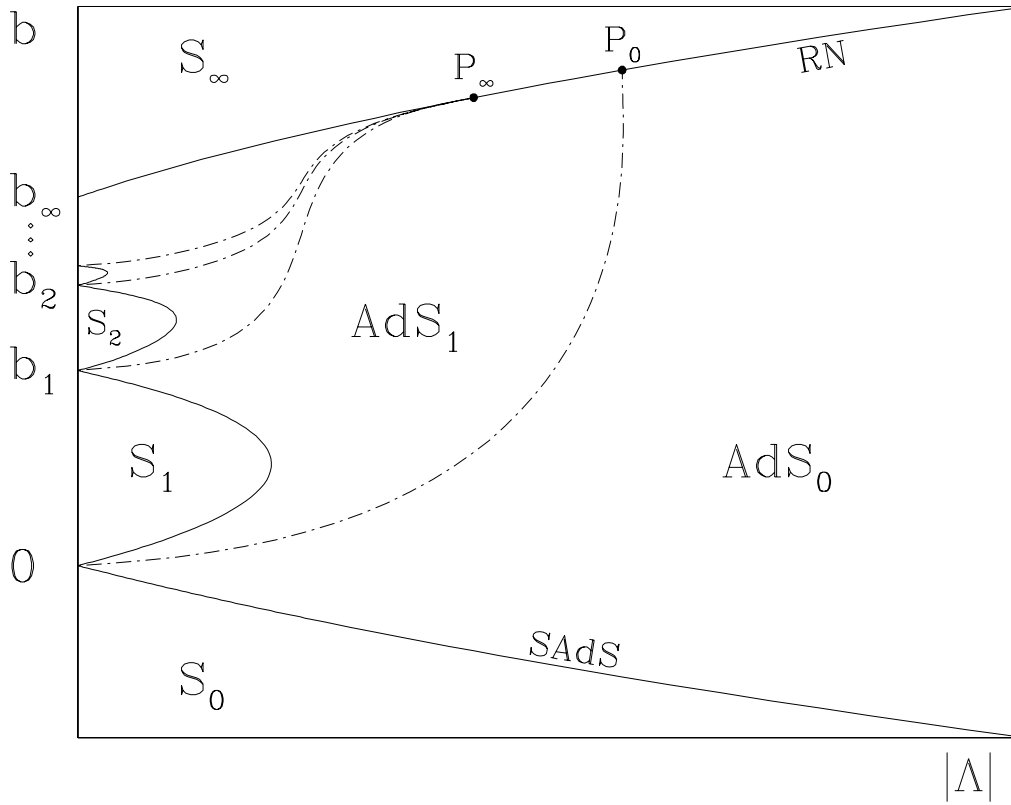
As a result it was found that there are

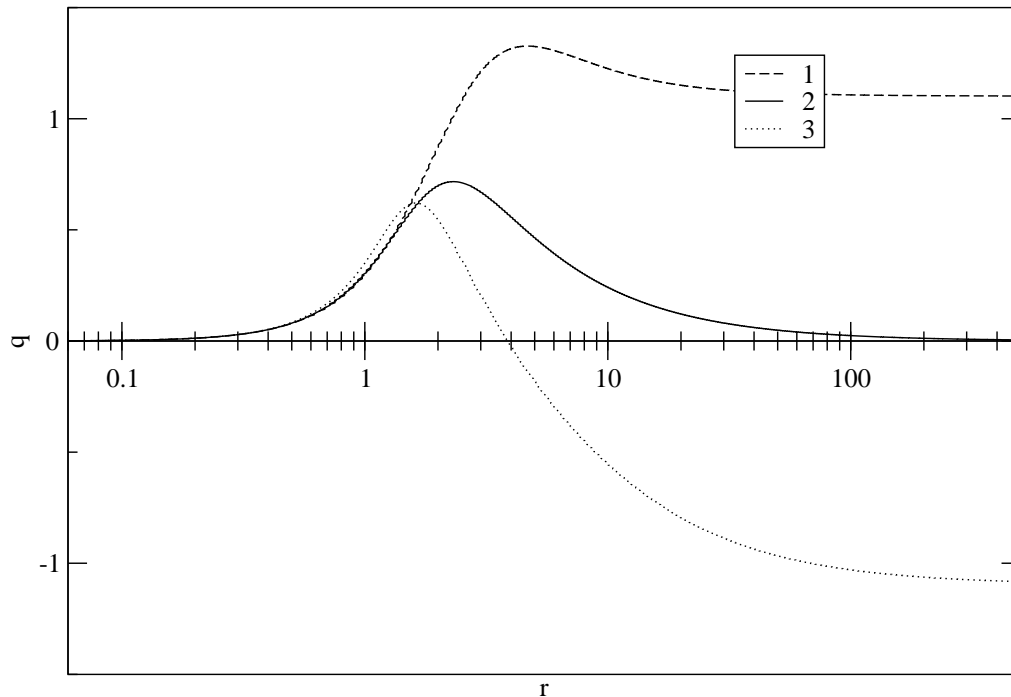
E.Winstanley 1999; J. Bjoraker, Y.Hosotani 2000

- * Neutral solutions
- * Magnetically charged
- * Electrically charged
- * Dyons

with any number of nodes in W , including **nodeless** and corresponding black holes.

Moduli space of the EYM_Λ solutions for $\Lambda < 0$.





Zero energy wave functions of Schrödinger equation of linear perturbations (gravitational sector) about background soliton for three different values of parameters Λ, b : 1-the wave function for under critical values (dashed line) has no nodes, 2-the wave function for critical values (solid line) is a normalizable zero mode, 3-the wave function for above critical values (dotted line) has one node.

Some non-Abelian lessons

- * the YMH sphaleron
+ gravitational (EYM), dilatonic (YMD) and cosmological (EYM_Λ) relatives.
- * $n = 1$ EYMD solution \rightarrow new SUGRA
 $N = 4$ gauged $SU(2) \times SU(1, 1)$
- * black holes exterior: counter examples to 'no-hair' conjecture
- * black hole interior: new type of singularity
- * asymptotically dS, 'bag of gold', compact
- * asymptotically AdS - neutral, 'monopoles', e-poles, dyons (and some of them are linearly stable!)

Other non-Abelian solutions

- * Gravitating sphalerons and sphaleronic black holes
- * Gravitating monopoles and corresponding black holes
- * Gravitating Skyrmions and Skyrme black holes
- * Rotating solutions
- * Axially symmetric generalizations
- * Black holes with non-spherically symmetric horizon
- * Solutions with higher gauge groups
- * Beyond 4 dimensions