DYNAMICAL SCALING IN TOPOLOGICAL DEFECT NETWORKS

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Some references

M.H., “Analytic scaling solutions for cosmic domain walls” *Phys. Rev. Lett.* **77**, 4495–4498 (1996).

G. Vincent, M.H. and M. Sakellariadou, “Correlations in Cosmic String Networks” *Phys. Rev.* **D55**, 573–581 (1997).

G. Vincent, M.H. and M. Sakellariadou, “Scaling and Small Scale Structure in Cosmic String Networks” *Phys. Rev.* **D56**, 637–646 (1997).

G. Vincent, N.D. Antunes and M.H., “Numerical simulations of string networks in the Abelian-Higgs model” *Phys. Rev. Lett.* **80** 2277–2280 (1998).

M.H., “Evolution of defect and brane networks,” Phys. Rev. D **68**, 043510 (2003)

T. Garagounis and M.H., “Scaling in numerical simulations of domain walls,” *Phys. Rev.* **D68** 103507 (2003).

Introduction

- Physics of elementary particles described by relativistic quantum field theories
- Field theories have extended “lump” solutions - topological defects
- Law of motion: acceleration $\propto$ mean curvature, extremal area (Nambu-Goto)
- Phase transitions in early Universe: phase ordering kinetics & dynamic scaling

  (Rajantie[a]: defect formation)
- String & M-theory: p-dimensional surfaces in D dimensions: branes

  Fundamental & “solitonic”.
- Brane gas Universe.[b]

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[a] A. Rajantie, Int. J. Mod. Phys. A 17, 1 (2002)

[b] R. H. Brandenberger and C. Vafa, Nucl. Phys. B **316**, 391 (1989);
S. Alexander, R. H. Brandenberger and D. Easson, Phys. Rev. D **62**, 103509 (2000)

This talk

- Model field theories & numerical solutions[a]
- Dynamic scaling for (domain walls & strings in 2& 3 dimensions)
- Nambu-Goto approximation [b]
- Level set method[c]

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[a] Vincent, Antunes, Hindmarsh, PRL 80, 227 (1998) [arXiv:hep-ph/9708427]; Garagounis & Hindmarsh *Phys. Rev.* **D68** 103507 (2003) [arXiv:hep-ph/0212359]

[b] Smith & Vilenkin PR D36, 990 (1987); Vincent, Hindmarsh, Sakellariadou PR D56, 637 (1997)

[c] Hindmarsh, PRL 77, 4495 (1996) [arXiv:hep-ph/9695332]; Phys. Rev. D **68**, 043510 (2003) [arXiv:hep-ph/0207267/]

A simple field theory

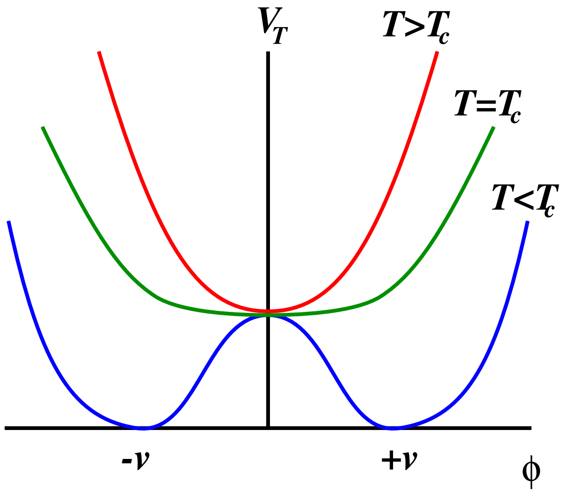
Scalar (real) field $\phi(\mathbf{x}, t)$ in D dimensions (“Higgs” field). Lagrangian density:

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - V(\phi), \qquad V(\phi) = V_0 - \frac{1}{2}\mu^2\phi^2 + \frac{1}{4!}\lambda\phi^4.$$

At high temperature T, can coarse-grain for wavenumbers $k < T$. $V \to V_T$, with

$$V_T(\phi) \simeq V_0 + (\frac{1}{24}\lambda T^2 - \frac{1}{2}\mu^2)\phi^2 + \frac{1}{4!}\lambda\phi^4$$

Phase transition at $T_c \simeq \mu\sqrt{24/\lambda}$.



The equation of motion of the (coarse-grained) field is (action principle)

$$\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi + (\frac{1}{12}\lambda T^2 - \mu^2)\phi + \frac{1}{3!}\lambda\phi^3 = 0$$

A phase transition in action

Early Universe: $T \propto t^{-1/2}$: departure from equilibrium at phase transition

- Model: $T \gg T_c \rightarrow T \ll T_c$ instantaneously,
- Dissipation: $\eta\dot{\phi}$ (models expansion of Universe: $\eta \rightarrow 0$ as $t \rightarrow \infty$)

Results:

- ϕ quickly reaches $\pm v$ almost everywhere
- Domain walls appear[a] around surfaces $\phi = 0$ ($\phi = v\tanh(\mu z/\sqrt{2})$)
- evolution of walls is self-similar: Area $\propto t^{-\alpha}$, with $\alpha \simeq 1$.[b]

Similar to phase ordering dynamics[c] in condensed matter.

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[a] Related defect formation: Rajantie, Int. J. Mod. Phys. A **17**, 1 (2002) [arXiv:hep-ph/0108159].

[b] Press, Ryden, Spergel, ApJ (1990), Garagounis and Hindmarsh, arXiv:hep-ph/0212359 (2002)

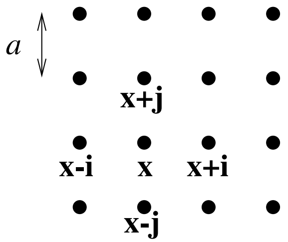
[c] Bray, Adv. Phys. (1994)

Simulations of scalar field theory

$$\frac{\partial^2\phi}{\partial t^2}+\eta(t)\frac{\partial\phi}{\partial t}-\nabla^2\phi+(\frac{1}{6}\lambda T^2-\mu^2)\phi+\frac{1}{6}\lambda\phi^3=0$$

Solve, in $R^D \times [t_{\rm i}, t_{\rm f}]$, with $\eta \longrightarrow 0$ when $t \gg t_{\rm i}$.

Numerical approx: periodic cubic lattice, spacing Δx, size L.



$$\begin{aligned} \text{Let} \qquad \Delta_i^{\pm}\phi_{\mathbf{x}} &= \pm\left(\phi_{\mathbf{x}\pm\mathbf{i}} - \phi_{\mathbf{x}}\right)/\Delta x, \\ \nabla^2\phi(\mathbf{x}) \to \triangle^2\phi_{\mathbf{x}} &= \sum_i \Delta_i^{+}\Delta_i^{-}\phi_{\mathbf{x}} \end{aligned}$$

Time evolution: Leapfrog (Verlet). Let $F_{\mathbf{x}}^n = \triangle^2\phi_{\mathbf{x}} - V'(\phi_{\mathbf{x}})$

$$\begin{aligned} \phi_{\mathbf{x}}^n &= \phi_{\mathbf{x}}^{n-1} + \pi_{\mathbf{x}}^{n-\frac{1}{2}} \cdot \Delta t, \\ \pi_{\mathbf{x}}^{n+\frac{1}{2}} &= [(1-\eta\Delta t/2)\pi_{\mathbf{x}}^{n-\frac{1}{2}} + F_{\mathbf{x}}^n \cdot \Delta t]/(1+\eta\Delta t/2), \end{aligned}$$

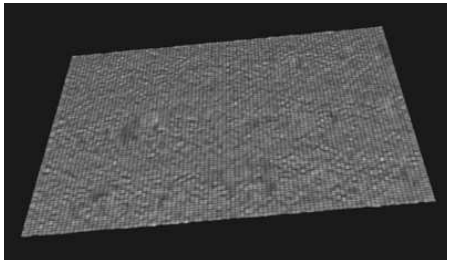
NB Simulation stops at $t_{\rm f} = L/2$ (periodic boundary conditions).

Simulations: initial conditions

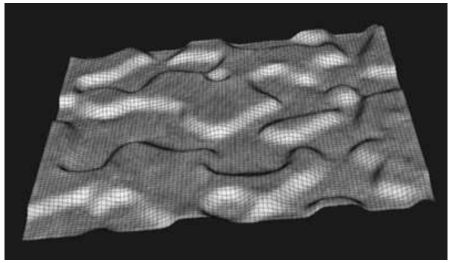
- $\phi_{\mathbf{x}}(t_{\mathrm{i}})$, $\dot{\phi}_{\mathbf{x}}(t_{\mathrm{i}})$ drawn from Gaussian Random Field.
- Simulates high T, if $\langle \phi_{\mathbf{x}}^2 \rangle = T^2/24$ and $\langle \dot{\phi}_{\mathbf{x}}^2 \rangle = \pi^2 T^4/180$
- Scaling hypothesis: Results as $t \longrightarrow \infty$ insensitive to initial conditions (?)

2D relativistic domain wall simulation

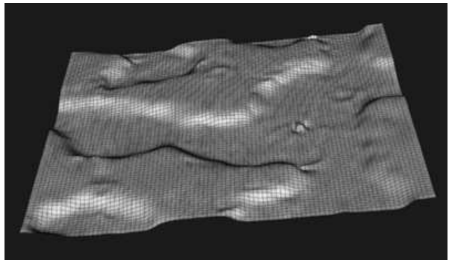
Simulation details:[a] $\Delta x = 0.7$, $\Delta t = 0.3$, $L = 400$



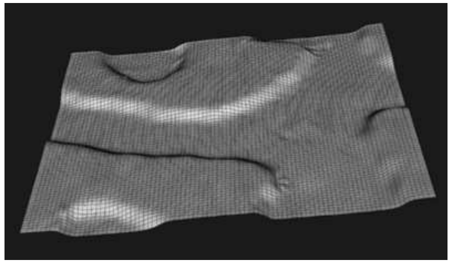
$t = 0\ \eta = 1$



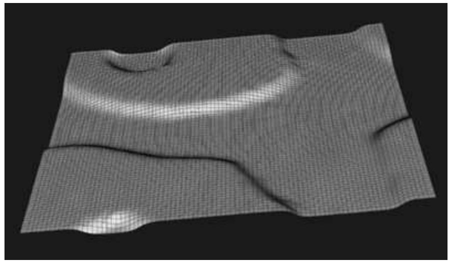
$t = 30\ \eta = 0$



$t = 60\ \eta = 0$



$t = 90\ \eta = 0$



$t = 126\ \eta = 0$

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[a] Garagounis and Hindmarsh, arXiv:hep-ph/0212359 (2002)

http://www.pact.cpes.susx.ac.uk/arXiv/hep-ph/0212359/

Domain wall scaling

2D simulations.[a] If length density[b] $L/A \propto t^{-1}$, then $\xi = (A/L) \propto t$

Linear fit to $\xi(t) = \xi_0 + At$

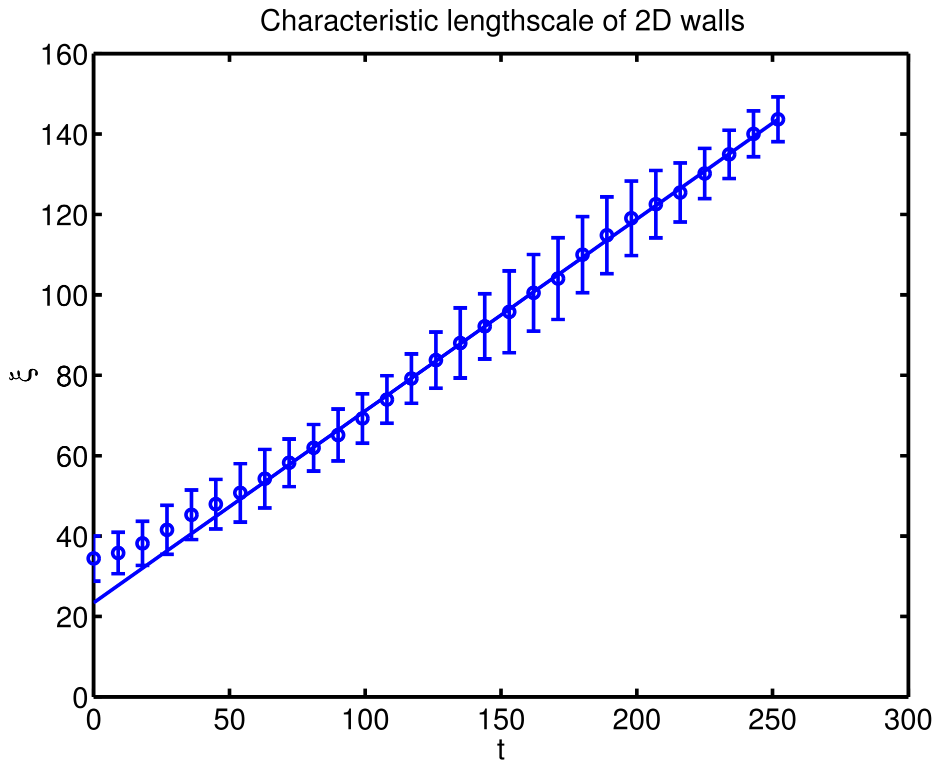
Average of 4 simulations.

Cooling:

$\eta = 0.2, n_{\mathrm{cool}} = 210,$

Lattice:

$\Delta x = 0.7, \Delta t = 0.3, N = 900$



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[a] Also: Press, Ryden, Spergel (1990); Coulson, Lalak, Ovrut (1996); Larsson, Sarkar, White (1997)

[b] 2D length density $L = \langle \delta(\phi) | \nabla \phi | \rangle$. Numerically: (weighted) sum of links $\mathbf{x}, i$ with $\phi_{\mathbf{x}} \phi_{\mathbf{x}+i} < 0$

Domain walls in the Universe

Q: Are there domain walls in the Universe?

A: No - unless very light

Dynamic scaling predicts that there will be O(1) domain wall stretching across observable Universe. Should see in Cosmic Microwave Background unless $\sigma \lesssim (1\mathrm{MeV})^3$.[a]

Many extensions of the Standard Model of particle physics have domain walls (e.g. spontaneous CP violation) – constrained by cosmology

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[a] Zel'dovich, Kobzarev, Okun' Sov. Phys. JETP (1975)

Abelian Higgs model phase transition

AHM has vortex “solitons”.

- Rapid quench makes vortices [a] - cosmic strings in 3D
- Evolution of strings is also self-similar
- Length density $(L/V) \propto t^{-\beta}$, with $\beta \simeq 2$

Q: Are there cosmic strings in the Universe?

A: Maybe ...

- string mass density $\rho_{\rm s} = \mu(L/V)$ constant small fraction of total $\rho \propto 1/Gt^2$
- GUT strings ($G\mu \sim 10^{-6}$) produce perturbations $\delta\rho/\rho$, $\delta T/T \simeq 10^{-5}$.

But ... CMB fluctuations mostly not from strings[b]

But ... particle production – too many cosmic rays ($G\mu < 10^{-9}$)?

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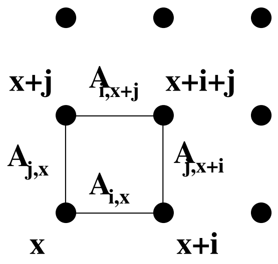
[a] M.H. and A. Rajantie, Phys. Rev. Lett. 85 (2000) 4660–63.

[b] COSMOS Consortium: Magueijo, Albrecht, Ferreira, Coulson 1996; Albrecht, Battye, Robinson 1997; Avelino, Shellard, Wu, Allen 1998; Contaldi, Magueijo, Hindmarsh 1999; Battye and Weller 1999.

Simulations of Abelian Higgs model



Numerical simulations: replace $\mathbf{A}(\mathbf{x},t)$ with $\theta_{i,\mathbf{x}} = -eaA_{i,\mathbf{x}}(t)$ defined on the links of a cubic lattice. Electric field $E_{i,\mathbf{x}}(t) = \dot{\theta}_{i,\mathbf{x}}$

Numerical approximations:

$$D_i\phi(\mathbf{x}) \to (D\phi)_{i,\mathbf{x}} = (e^{-i\theta_{i,\mathbf{x}}}\phi_{\mathbf{x}+\mathbf{i}} - \phi_{\mathbf{x}})/a$$

$$\frac{1}{2}\mathbf{B}^2 \to \frac{1}{2a^4e^2}\sum_{i,j}[1-\cos(\theta_{i,\mathbf{x}} + \theta_{j,\mathbf{x}+\mathbf{i}} - \theta_{i,\mathbf{x}+\mathbf{j}} - \theta_{j,\mathbf{x}})]$$

Time evolution: Leapfrog algorithm. Accurate to $O(h^2)$. Preserves discrete version of Gauss's Law $\nabla \cdot E = \rho$.

Initial conditions:

- Gaussian random field on ϕ, but $\theta_{i,\mathbf{x}} = 0$.
- Two dissipation periods: (1) $e = 0$ (2) e increasing to final value.

Cosmic string scaling

If length[a] density $L/V \propto t^{-2}$, then $\xi = \sqrt{(V/L)} \propto t$

Average of 5(10) simulations

Field: $e = 1, \lambda = 2,$

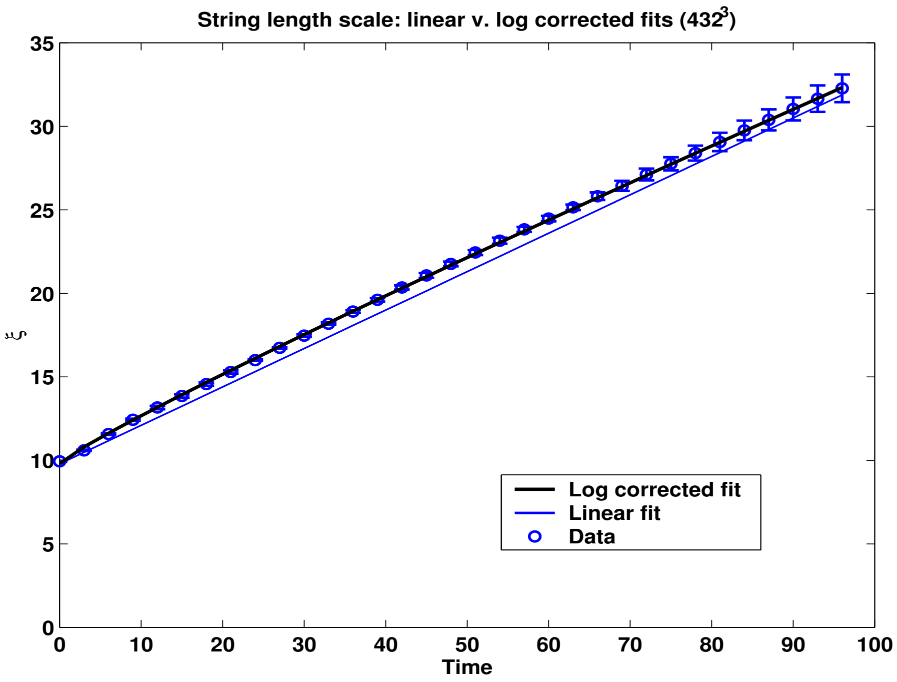
Lattice: $\Delta x = 0.5, \Delta t = 0.15,$

$N = 432(360)$

Fits to:

$$\xi(t) = \xi_0 + At,$$

$$\xi(t) = \xi_0 + At/\sqrt{\log(Bt)}$$



Log correction predicted by Level Set method[b] (see later)

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[a] Let n_s be number of plaquettes pierced by string (assigned by geodesic rule). Then $L = n_s \Delta x$.

[b] Hindmarsh (2003)

Ideal strings - Nambu-Goto approximation

Limit $w/R \to 0$? (w – width of defect, R – any curvature radius). If so,

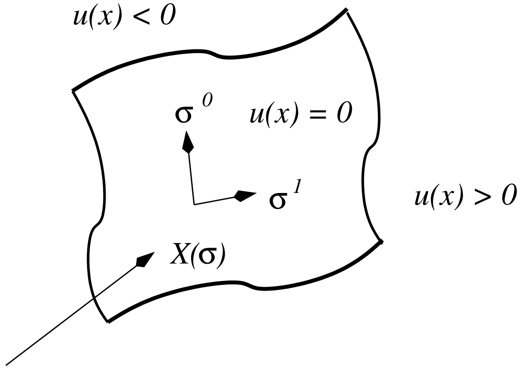
$$S_{\rm eff} \propto \text{Area} + \mathrm{O}(w^2/R^2)$$

Minkowski space-time: coordinates x^μ, with $\mu = 0, \ldots, D$.

Metric tensor $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, \ldots, +1)$.

Two ways of specifying location of wall:

- Surface coordinates $X^\mu(\sigma^\alpha)$,
- Equation $u(x) = 0$.



New coordinates $\xi^\mu(x) = \{\sigma^\alpha(x), u(x)\}$, with $\alpha = 0, 1, \ldots, D-1$.

Motion of thin defects

Embedding metric $\gamma_{\alpha\beta} = \partial_\alpha X \cdot \partial_\beta X$. Hence $S_{\mathrm{eff}} \simeq -\sigma \int d^D\sigma \sqrt{-\det\gamma}$

Relativistic walls in $\mathbf{R}^2$

Can choose coordinates s.t. $\gamma_{\alpha\beta} \propto \mathrm{diag}(-1,1)$ and $\sigma^0 = X^0 \equiv \eta$.

$S_{\mathrm{eff}} \simeq -\sigma \int d\eta d\sigma \sqrt{(1-\dot{\mathbf{X}}^2)(\mathbf{X}')^2}$ where $\dot{\mathbf{X}} = \partial\mathbf{X}/\partial\eta$, $\mathbf{X}' = \partial\mathbf{X}/\partial\sigma$

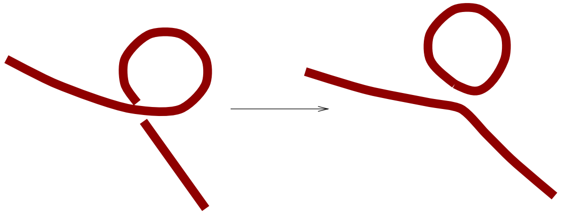
Constraints: $\dot{\mathbf{X}}^2 + \mathbf{X}'^2 = 1, \qquad \dot{\mathbf{X}} \cdot \mathbf{X}' = 0.$

Equations of motion: $\ddot{\mathbf{X}} - \mathbf{X}'' = 0$

Acceleration = Curvature

Worldsheet is 2d extermal surface embedded in 3D Minkowski space.

Non-linearity at self-intersections (from field theory)

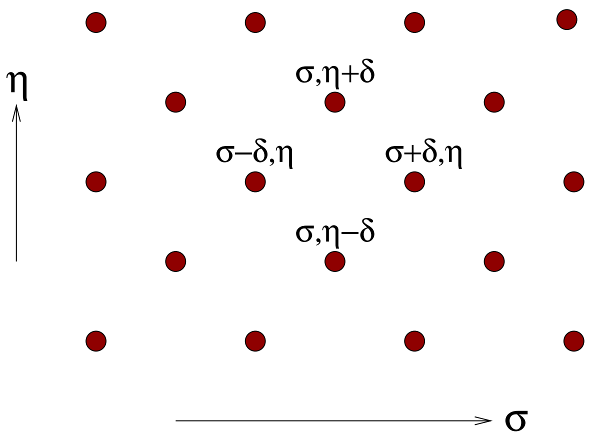


Numerical solutions for thin line defects (strings)

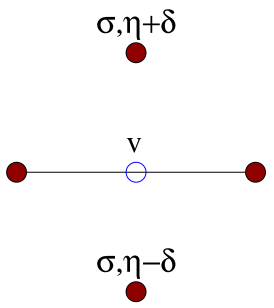
General solution: $\mathbf{X}(\sigma,\eta)=\frac{1}{2}(\mathbf{a}(\sigma-\eta)+\mathbf{b}(\sigma+\eta))$, with $\mathbf{a}'^2=1=\mathbf{b}'^2$.

Note $\mathbf{X}(\sigma,\eta+\delta)=\mathbf{X}(\sigma+\delta,\eta)+\mathbf{X}(\sigma-\delta,\eta)-\mathbf{X}(\sigma,\eta-\delta)$

(a)

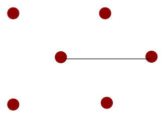
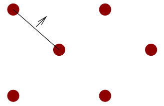
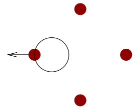


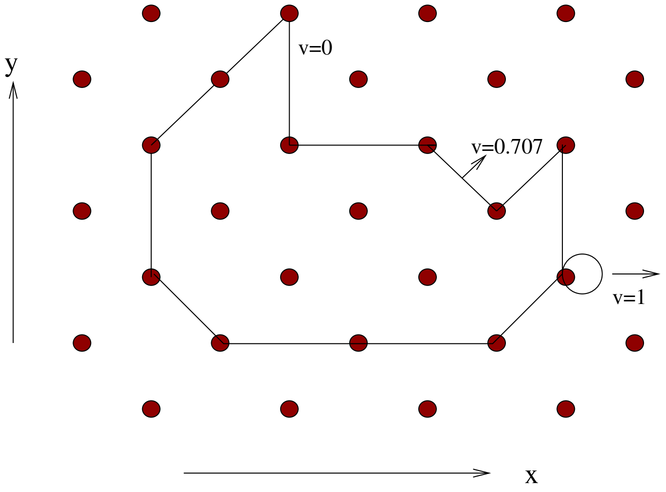
(b)

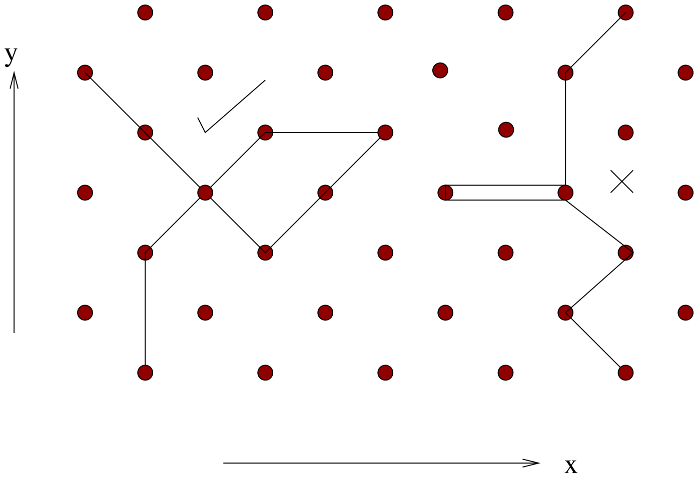


$$\left.\begin{array}{l}\mathbf{X}(m\delta,0)\text{ is on even elements of square lattice}\\ \mathbf{X}(m\delta,\delta)\text{ is on odd elements of square lattice}\end{array}\right\}\mathbf{X}(m\delta,n\delta)\text{ always on square lattice.}$$

Numerical solutions: allowed configurations

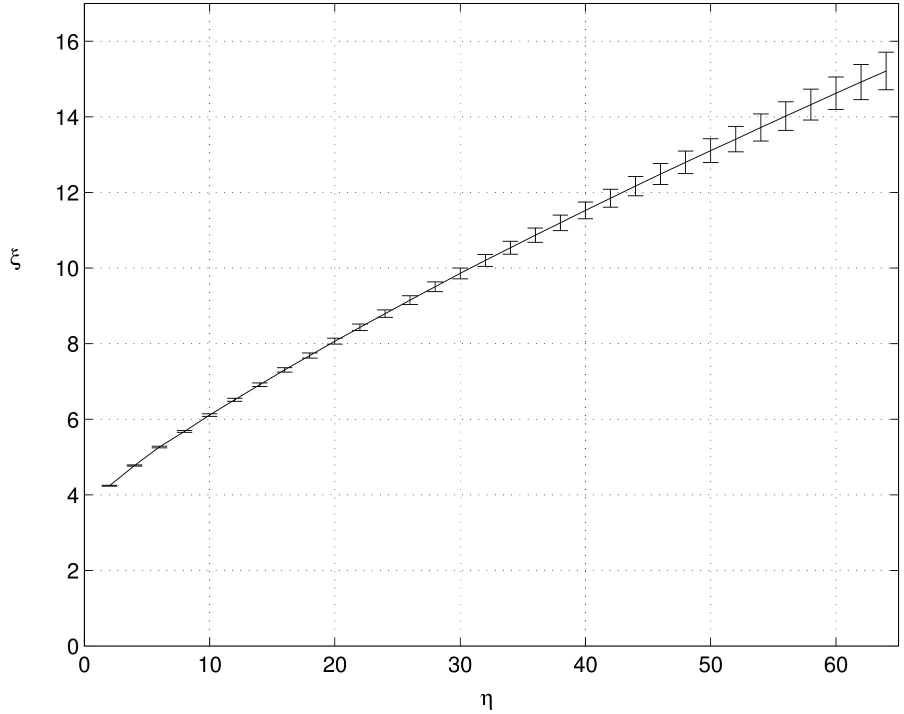
Link type	velocity **v** example	tangent **u** example	picture of example on 2-d lattice
Stationary	0	1 (1,0,0)	
Diagonal	1/sqrt(2) (1,1,0)	1/sqrt(2) (1,-1,0)	
Cusp	1 (-1,0,0)	0	





3D simulations of thin strings on the lattice

Approximate scaling ($\xi = \sqrt{(V/L_{\mathrm{string}})}$), where L_{string} excludes shortest loops.



At end of simulation most of string in the form of shortest loops (entropy).

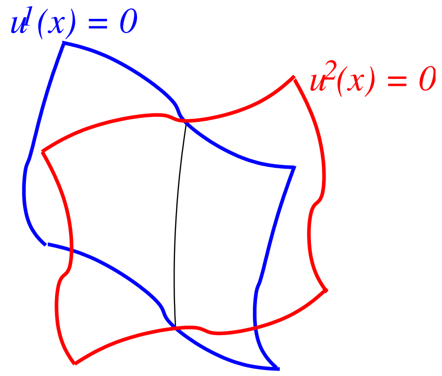
Field theory: energy of defects goes to wave modes of field.

Consistency of Nambu-Goto approximation?

Level set method

Consider p-dimensional surface in $D+1$-dimensional spacetime (Brane).

Metric $g_{\mu\nu} = a^2(\tau)\text{diag}(-1,+1,\ldots,+1)$ (cosmology: $a(\tau) = \tau + \tau^2$).



Two ways of specifying location of brane:

- Surface coordinates $X^\mu(\sigma^\alpha)$,
- $N = D - p$ equations $u^A(x) = 0$.

New coordinates $\xi^\mu(x) = \{\sigma^\alpha(x), u^A(x)\}$, with $\alpha = 0, 1, \ldots, p$,
$A = 1, \ldots, N$ (codimension of surface).

New metric at $u^A = 0$:

$$G_{\mu\nu} = \begin{pmatrix} \gamma_{\alpha\beta} & 0 \\ 0 & (h^{-1})_{AB} \end{pmatrix} \quad \text{with} \quad \begin{aligned} \gamma_{\alpha\beta} &= \partial_\alpha x \cdot \partial_\beta x \\ h^{AB} &= \partial u^A \cdot \partial u^B \end{aligned}$$

Extremal branes

Invariant $p+1$-dim area ($\gamma = \det\gamma_{\alpha\beta}$, $g = \det g_{\mu\nu}$, $h = \det h^{AB}$):

$$\mathcal{A}[X] = \int d^{p+1}\sigma\sqrt{-\gamma(X)}, \quad \mathcal{A}[u^A] = \int d^d x\sqrt{-g}\sqrt{h}\,\delta^N(u)$$

Equation of extremal area brane:

$$\frac{\delta A[X]}{\delta X_\mu(\sigma)} = {}^{(p+1)}\Box X^\mu(\sigma) + \Gamma^\mu_{\nu\rho}\gamma^{\alpha\beta}\partial_\alpha X^\nu \partial_\beta X^\rho = 0$$

Equation for N fields $u^A(x)$ whose zeros are extremal branes:

$$\left[g^{\mu\nu} - h_{AB}\partial^\mu u^A \partial^\nu u^B\right]\left(\partial_\mu\partial_\nu u^C - \Gamma^\rho_{\mu\nu}\partial_\rho u^C\right)\delta^N(u) = 0.$$

NB1 Projector: $P_\parallel^{\mu\nu} = \gamma^{\alpha\beta}\partial_\alpha X^\nu \partial_\beta X^\rho = \left[g^{\mu\nu} - h_{AB}\partial^\mu u^A \partial^\nu u^B\right]$

NB2 Extrinsic curvatures $K^A = P_\parallel^{\mu\nu}\nabla_\mu\partial_\nu u^A = 0.$

Random brane networks: Mean field equation for $u^A(x)$

Let $u^A(x)$ be a Gaussian random field (GRF)

Average the field equation for u^A with respect to probability distribution

$$dP[u^A] = \prod_C \mathcal{D}u^C \exp\left(-\frac{1}{2}\int (dx)(dy)\, u^A(x) C^{-1}_{AB}(x,y) u^B(y)\right).$$

2-point correlator $C^{AB}(x,y) = \langle u^A(x) u^B(y)\rangle = C(\tau_x, \tau_y, |\mathbf{x}-\mathbf{y}|)\delta^{AB}$.

Write $C(\tau,\tau,0) = C(\tau)$

Obtain linearised equation of the form $\ddot{u}^C + \frac{\mu(\tau)}{\tau}\dot{u}^C - v^2\nabla^2 u^C = 0.$

μ, v^2 depend on $T(\tau) = \langle \dot{u}^2\rangle\big|_{x=y}$, $S(\tau) = \langle (\boldsymbol{\nabla} u)^2\rangle\big|_{x=y}$ and $\alpha = \tau\dot{a}/a$.

Self-consistent solution for μ, v^2 (with $\nu = \pm(1-\mu)/2$)

$$u^C_{\mathbf{k}}(\tau) = A^C_{\mathbf{k}}\,(\tau/\tau_{\mathrm{i}})^{(1-\mu)/2+\nu}\, J_\nu(kv\tau)/(kv\tau)^\nu,$$

Area densities

Proper area density of p-brane in d space-time dimensions:

$$\mathcal{A}^p_D(x) = \int d^{p+1}\sigma' \,\sqrt{-\gamma}\delta^d(x - X(\sigma'))/\sqrt{-g}.$$

In terms of fields $u^A(x)$: $\boxed{\mathcal{A}^p_D = \delta^N(u^A)\sqrt{h}.}$

Recall $h = \det h^{AB}$, $h^{AB} = \partial_\mu u^A \partial_\nu u^B g^{\mu\nu}$.

Projected area density of p-brane in D space dimensions:

$$A^p_D = \int d^p\sigma' \,\sqrt{\gamma_D}\delta^D(\mathbf{x} - \mathbf{X}(\sigma'))/\sqrt{g_D},$$

g_{Dij} is spatial part of metric, $\gamma_{Dab} = \partial_a X^i \partial_b X^j g_{Dij}$, ($a, b = 1, \ldots, p$).

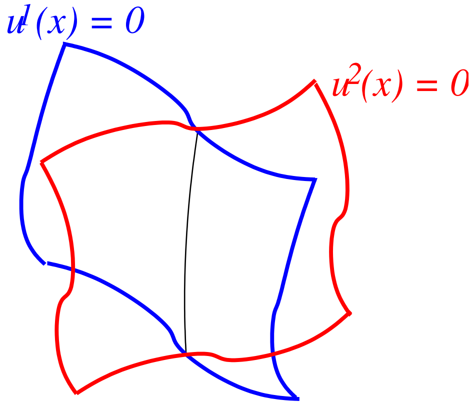
In terms of fields $u^A(x)$: $\boxed{\mathcal{A}^p_D = \delta^N(u^A)\sqrt{h_D},}$ where $h^{AB} = \partial_i u^A \partial_j u^B g^{ij}_D$.

Average projected area density

$N = 1$ Domain walls[a]

$$\langle A_D^{D-1} \rangle = \sqrt{\frac{S}{\pi C}} \frac{\Gamma[(D+1)/2]}{\Gamma(D/2)},$$

Higher N:



Scherrer & Vilenkin (strings in $D = 3$): string located at the intersection of two surfaces $u^1 = 0$ and $u^2 = 0$. Hence $A_3^2 = A_2^1 A_3^2$,

General formula:

$$A_D^p = \prod_{n=1}^{D-p} A_{D-n+1}^{D-n} = \left(\frac{S}{\pi C}\right)^{N/2} \frac{\Gamma[(D+1)/2]}{\Gamma[(D-N+1)/2]}.$$

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[a] agrees with Ohta, Jasnow, Kawasaki PRL 49, 1223 (1982)

Area density: quantitative results for $N = 1$

D=3 $A_3^2 = \frac{2}{\pi}\sqrt{\frac{S}{C}}$			D=2 $A_2^1 = \frac{1}{2}\sqrt{\frac{S}{C}}$	
α	Theory	Simulation	Theory	Simulation
0	$1.91\tau^{-1}$	$0.88(0.14).\tau^{-1.00(0.03)}$	$1.11\tau^{-1}$	$0.77(0.23).\tau^{-0.99(0.03)}$
1	$2.02\tau^{-1}$	$0.93(0.13).\tau^{-0.99(0.01)}$	$1.18\tau^{-1}$	$0.93(0.17).\tau^{-1.00(0.02)}$
2	$2.16\tau^{-1}$	$0.96(0.12).\tau^{-1.00(0.01)}$	$1.24\tau^{-1}$	$1.15(0.23).t^{-0.99(0.01)}$

NB Simulations: Garagounis & Hindmarsh (2003)

Area density: quantitative results for $N = 2$

D=3 $A_3^1 = \frac{S}{\pi C}$		
α	Theory	Simulation
0	$3.6\tau^{-2}\log(\tau\Lambda)$	$(11 \pm 1)\tau^{-2}$
1	$6.8\tau^{-2}$	$(24 \pm 8)\tau^{-2}$
2	$7.1\tau^{-2}$	$(18 \pm 5)\tau^{-2}$

NB1 Simulations: G. Vincent, N. D. Antunes and M. Hindmarsh, Phys. Rev. Lett. **80**, 2277 (1998) [arXiv:hep-ph/9708427]; J. N. Moore, E. P. Shellard and C. J. Martins, Phys. Rev. D **65**, 023503 (2002) [arXiv:hep-ph/0107171].

NB2 Log fits not attempted to length density in Minkowski space.

Area densities: quantitative results (Minkowski space)

N	D	Theory	Field simulation
1	2	$1.91\tau^{-1}$	$0.88(0.14).\tau^{-1.00(0.03)}$
1	3	$1.11\tau^{-1}$	$0.77(0.23).\tau^{-0.99(0.03)}$
2	3	$3.6\tau^{-2}\log(\tau\Lambda)$	$(11 \pm 1)\tau^{-2}$

$N = 1$: Simulations from Garagounis & Hindmarsh (2003)

$N = 2$: Simulations from G. Vincent, N. D. Antunes and M. Hindmarsh, Phys. Rev. Lett. **80**, 2277 (1998) [arXiv:hep-ph/9708427]; J. N. Moore, E. P. Shellard and C. J. Martins, Phys. Rev. D **65**, 023503 (2002) [arXiv:hep-ph/0107171].

Summary

Simulations of random networks of p-branes evolve (in cosmological spacetimes).

- Dynamic scaling hypothesis: network has a single length scale ξ, with $\xi \propto \tau^z$
- Field theories ($N = 1, 2$ in $D = 2, 3$) consistent with $\xi \propto \tau$ (corrections?)
- Nambu-Goto in $D = 3$: consistent with $\xi \propto \tau$ (log correction in Minkowski)
- Level sets: $\xi \propto \tau$. Scaling amplitudes for walls and strings in $D = 2, 3$

Future directions:

- Level sets: simulations with higher D, simulations & calculations with higher N.
- Nambu-Goto approx for extended defects? Asymptotic behaviour.
- Physics: observational consequences (CMB, matter density, cosmic rays).