Skyrmions & Harmonic Maps

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Rational Map Ansatz

- Classical effective theory used to describe nuclei; and the Skyrme field $U(\vec{x}) \in SU(N)$ describes pions
- Rational map ansatz: the angular dependence was determined by a rational map and the radial dependence was determined by a numerical solution of a nonlinear ordinary differential equation
- The ansatz has been a great step forward since it gave very good approximations to the solutions of the full equations which up to then could only be determined numerically (hours, days or weeks of CPU time)
- Energy a few % up on the true value and it was practically impossible to distinguish the energy density plots obtained with the use of the ansatz from the exact ones obtained numerically.
- Harmonic Maps $SU(N)$ non-embedded spherically symmetric solutions
- Improved Harmonic map ansatz: As long as the projectors represent **harmonic maps** - the profile functions can be more general, in addition to their dependence on r do depend, also, on z and $\bar{z}$ which allows for a more general angular dependence of the fields and density functions.
- $SU(2)$ Skyrme Model

$B = 1$	spherically symmetric energy density
$B = 2$	toroidal energy density
$B = 3$	the symmetry of tetrahedron
$B > 3$	Skyrmions are located on a spherical hollow shell

Skyrme Model

- Action

$$S = \int \left[\frac{\kappa^2}{4} \text{tr}\left(K_\mu K^\mu \right) + \frac{1}{32e^2} \text{tr}\left(\left[K_\mu, K_\nu \right] \left[K^\mu, K^\nu \right] \right) \right] d^4x$$

where $K_\mu = \partial_\mu U U^{-1}$ for $\mu = 0, 1, 2, 3$ U is the $SU(N)$ chiral field and κ, e are coupling constants

- Boundary Conditions

$$U \to I \qquad |x^\mu| \to \infty$$

This effectively compactifies the three-dimensional Euclidean space into S^3 and hence implies that the field configurations of the Skyrme model can be considered as maps from S^3 into $SU(N)$

- Baryon Number

$$B = \int B^0 \, d^3x$$

where

$$B^\mu = -\frac{1}{24\pi^2} \varepsilon^{\mu\nu\alpha\beta} \, \text{tr}\left(K_\nu K_\alpha K_\beta \right)$$

and $\varepsilon^{\mu\nu\alpha\beta}$ is the (constant) fully antisymmetric tensor.

Stereographic Projected Coordinates

- In terms of r, θ, ϕ the Riemann sphere variable z is defined by: $z = e^{i\phi}\tan(\theta/2)$.
- Action

$$S = \int dr dt\, dz d\bar{z}\ \mathrm{tr}\Big(\frac{\kappa^2 r^2}{2(1+|z|^2)^2} K_r^2 + \frac{\kappa^2}{2}|K_z|^2 + \frac{1}{8e^2}\Big|\,[K_r, K_z]\,\Big|^2 - \frac{(1+|z|^2)^2}{32e^2 r^2}\,[K_z, K_{\bar{z}}]^2\Big)$$

- Baryon

$$B = -\frac{1}{8\pi^2}\int \mathrm{tr}\left(K_r\,[K_z, K_{\bar{z}}]\right) dr\, dz\, d\bar{z}.$$

- Harmonic Map Ansatz

$$U = e^{2ih(P-I/N)} = e^{-2ih/N}\left[I + \left(e^{2ih} - 1\right) P\right]$$

where P is a $N \times N$ hermitian projector which depends only on the angular variables $(z, \bar{z})$ and h is the profile function which depends, at least, on r where $h(0) = \pi$ while the boundary value $U \to I$ at $r = \infty$ requires that $h(\infty) = 0$

- P: from S^2 into CP^{N-1} defined as

$$P(V) = \frac{V \otimes V^\dagger}{|V|^2} \qquad (1)$$

where V is a N-component complex vector (depending on z and $\bar{z}$).

- For $N = 2$ while

$$V = (1, f(z))$$

where $f(z)$ is a rational function we recover the rational map ansatz of Houghton et al

Harmonic Maps

- Action when $h = h(r, z, \bar{z})$ and $P = P(z, \bar{z})$

$$S = -\int dt\, dr\, dz\, d\bar{z} \Big(\kappa^2 A_N\, r^2\, h_r^2 + \kappa^2 B_N\, |h_z|^2 + \left[\mathcal{N}_1 \left(\kappa^2 + \frac{h_r^2}{e^2} \right) + \frac{\mathcal{N}_2}{e^2} \frac{|h_z|^2}{r^2} \right] \sin^2 h + \frac{\mathcal{I}}{e^2} \frac{\sin^4 h}{r^2} \Big)$$

where

$$A_N = 2i\, \frac{N-1}{N} \frac{1}{(1+|z|^2)^2}\,, \quad B_N = 2i\, \frac{N-1}{N},$$
$$\mathcal{N}_1 = i\, 2\, \frac{|P_+ V|^2}{|V|^2}\,,\ \mathcal{N}_2 = i\, \frac{|P_+ V|^2}{|V|^2}\, (1+|z|^2)^2,\ \mathcal{I} = i\, \frac{|P_+ V|^4}{|V|^4}\, (1+|z|^2)^2$$

- Baryon number: the topological charge of the CP^{N-1} sigma model

$$B = \frac{i}{\pi^2} \int \mathrm{tr}\, (P\, [P_z, P_{\bar{z}}])\ dz\, d\bar{z} \int_0^\infty \sin^2 h\, h_r\, dr = \frac{i}{2\pi} \int \frac{|P_+ V|^2}{|V|^2}\, dz\, d\bar{z}.$$

- Equation of Motion

$$\partial_x \left(\left[\frac{2(N-1)}{N} + \frac{2 \sin^2 h}{x^2} G \right] h_x\, x^2 \sin\theta \right) + \partial_\theta \left(\left[\frac{2(N-1)}{N} + \frac{\sin^2 h}{x^2} G \right] h_\theta\, \sin\theta \right) + \partial_\phi \left(\left[\frac{2(N-1)}{N} + \frac{\sin^2 h}{x^2} G \right] \frac{h_\phi}{\sin\theta} \right) - \left(1 + \frac{h_r^2}{2} + \frac{h_\theta^2}{2x^2} + \frac{h_\phi^2}{2x^2\, \sin^2\theta} + \frac{\sin^2 h}{x^2} G \right) \times \sin(2h) G \sin\theta = 0$$

where $G = \frac{|P_+ V|^2}{|V|^2} (1+|z|^2)^2$ is a function of θ and ϕ only and $x = e\kappa r$ the dimensionless coordinate

$SU(2)$ $B = 1, \ldots, 4$ Baryons

- To test our approach, we have calculated the profile function h for one skyrmion (i.e. for $B = 1$) and found no θ or ϕ dependence (as expected). The total energy is 1.232.

- Axially symmetric skyrmions with baryon number $B > 1$ can be obtained from vectors of the form

$$V = \left(z^B, 1\right)^t.$$

The action and the function h depends explicitly on θ but not on ϕ

- For $B = 2$ the energy is 2.382 which is closer to the value 2.342 of the "exact" solution (obtained numerically by solving the full equations) and lower than the value 2.416 obtained by the rational map ansatz

- For $B > 2$ the solutions (of this class) correspond to saddle points of the energy. Obviously, the improved harmonic map ansatz yields lower energies than the rational map ansatz

Platonic Symmetry

- For $B = 3$ and $B = 4$ we let the vector V to be

$$V = \left(\sqrt{3}iz^2 - 1, z(z^2 - i\sqrt{3})\right)^t$$
$$V = \left(z^4 + 2\sqrt{3}iz^2 + 1, z^4 - 2\sqrt{3}iz^2 + 1\right)^t.$$

All these expressions come from Houghton et al who established their form by minimizing the projector part of the action This time, the corresponding action depends explicitly on θ and ϕ

- Assume h to depend (only) on θ, in addition to r. We have performed the integration of the action over the azimuthal angle ϕ which has given us an effective action, which still contains an explicit θ dependence. The variation of this effective action with respect to $h(r,\theta)$ led to a partial differential equation for it.

- The θ dependence of h is very small for $B = 3$ while for $B = 4$ it is more pronounced, but still small compared to the axially symmetric solution. The improvement of the energy is rather small, compared to the improvement of the energy for the axisymmetric solutions.

	axial symmetry			platonic symmetry			
B	rat. map	improv	exact	rat. map	improv(no ϕ)	(with ϕ)	exact
1	1.232	1.232	1.232	–	-	–	–
2	1.208	1.191	1.181	–	-	–	–
3	1.256	1.214	1.194	1.184	1.183	1.168	1.143
4	1.322	1.243	1.216	1.137	1.133	1.130	1.116

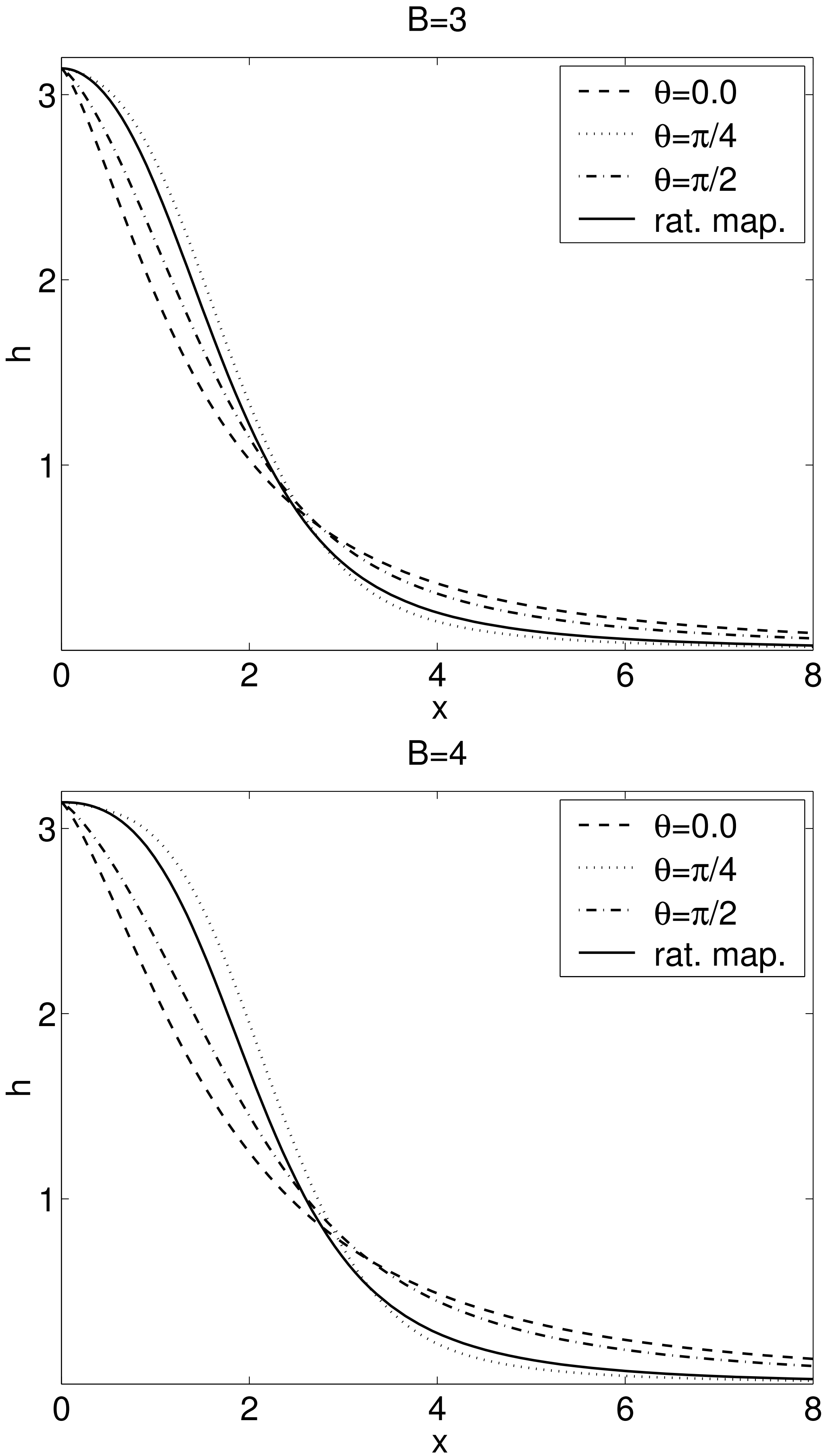

Figure 1: The skyrmion profile function $h(r, \theta)$ with axial symmetry obtained from the improved harmonic and rational map ansatze for $B = 3$ (left) and $B = 4$ (right).

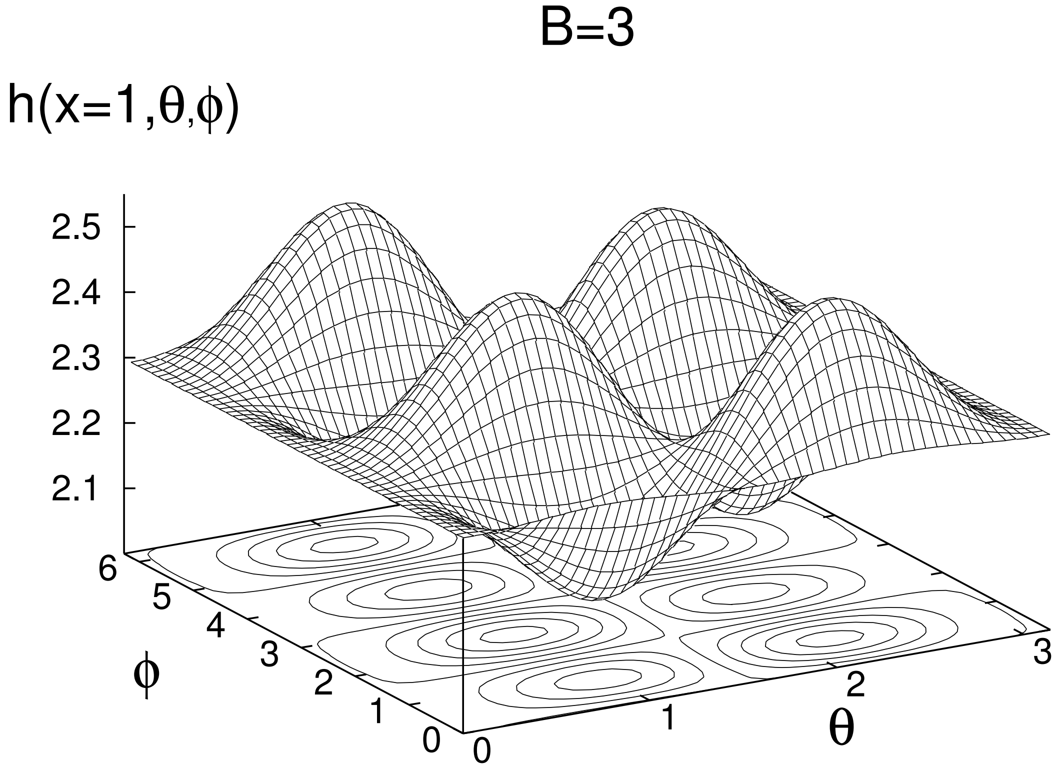

Figure 2: The $B = 3$ skyrmion profile function $h(r = 1, \theta, \phi)$ with tetrahedral symmetry obtained from the improved harmonic map ansatze.

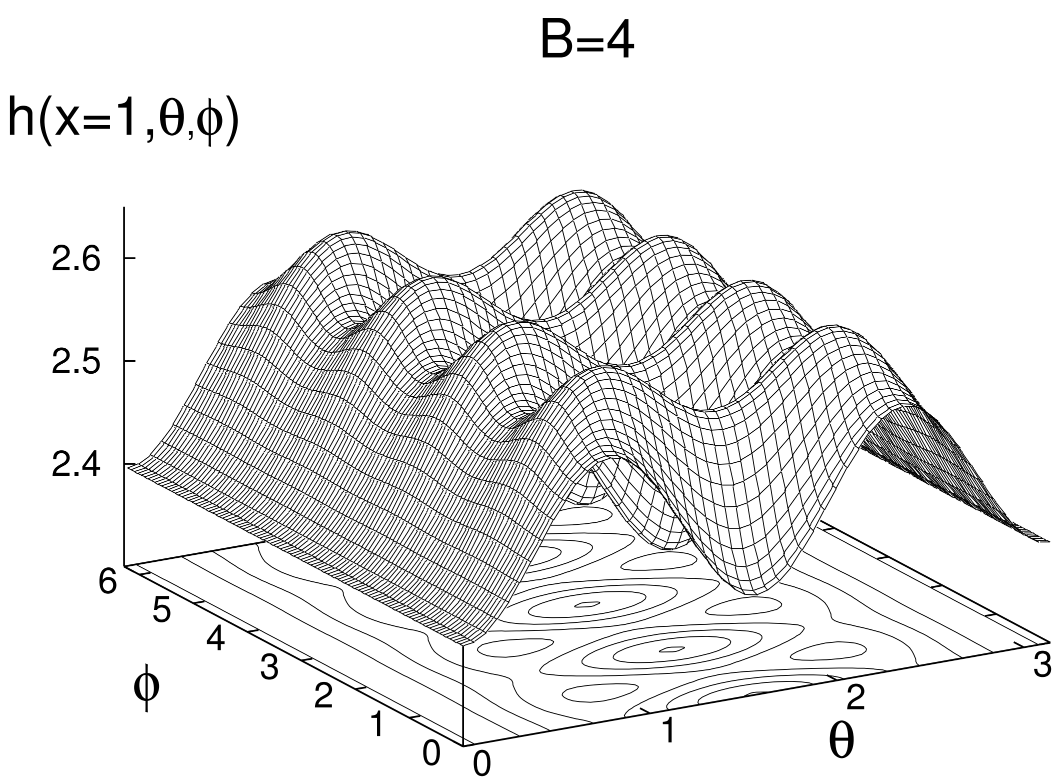

Figure 3: Same as Fig. 2 for $B = 4$ with octahedral symmetry.