A Pure Cotton Kink in a Funny Place

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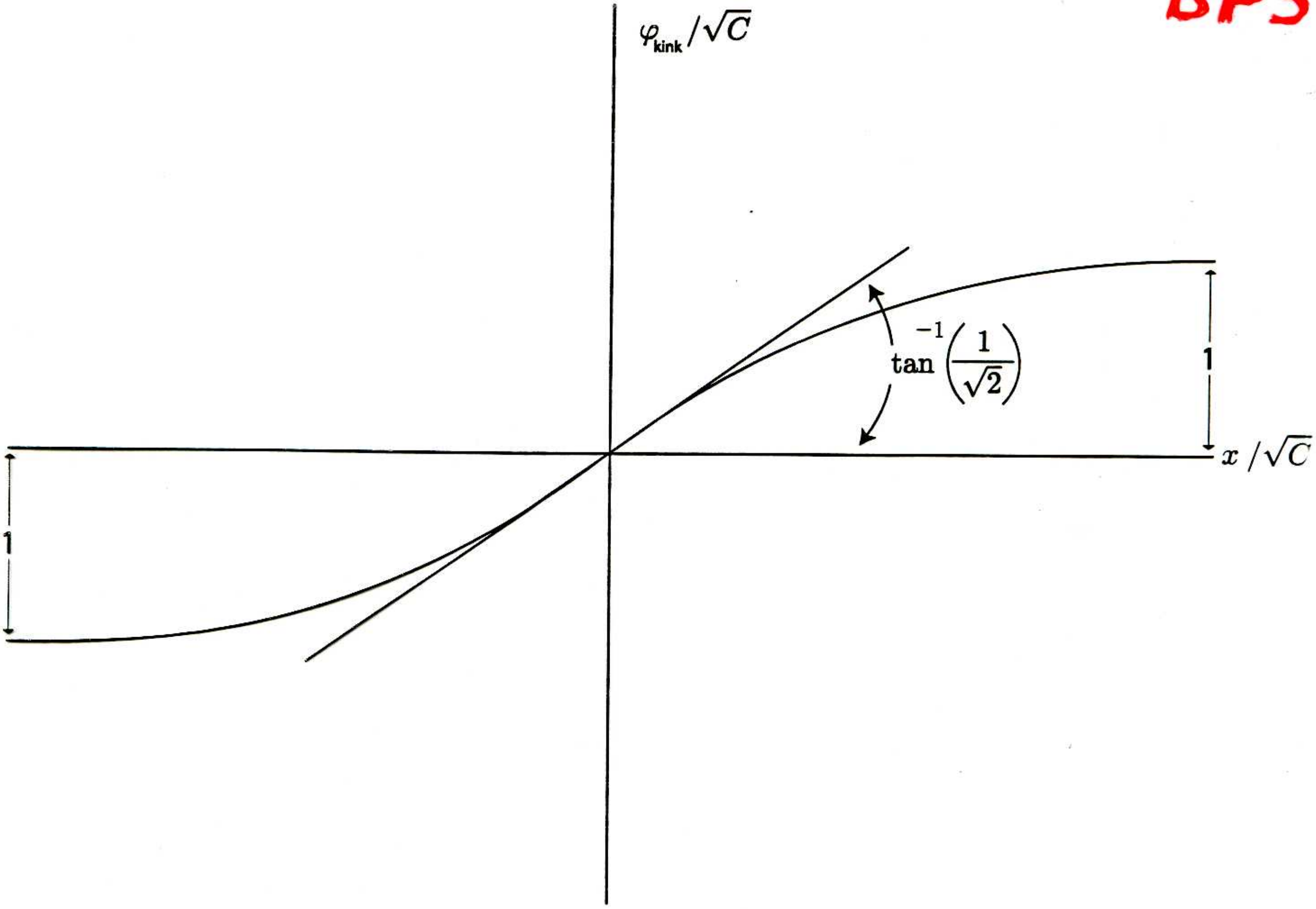
Background

Kink: $\Box f - Cf + f^3 = 0$

$d = 1$ static solution $-f'' - Cf + f^3 = 0$

$$\varphi_{\text{kink}} = \sqrt{C}\ \tanh\sqrt{\frac{C}{2}}\,x \quad \Rightarrow \quad f' = \frac{1}{\sqrt{2}}(C - f^2)$$

BPS



[physics: domain walls in polyacetylene, fractional fermions
mathematics: (other 1-d models) complete integrability]

3d Chern-Simons Term

[Deser, Templeton and RJ, Ann. Phys. **140**, 372 (1982)]

(i) gauge theory (non-Abelian, Abelian)

$$W(A) = \int d^3x\, \varepsilon^{\alpha\beta\gamma}\, tr\left(\frac{1}{2}A_\alpha\, \partial_\beta A_\gamma + \frac{1}{3}A_\alpha\, A_\beta A_\gamma\right)$$

$$\delta W(A) = \int d^3x \frac{1}{2}\, \varepsilon^{\alpha\beta\gamma} tr\left(F_{\alpha\beta}\delta A_\gamma\right)$$

[physics: Hall regime (mostly Abelian, some non-Abelian)

high T superconductivity (?)

mass for 3-d gauge fields, preserving gauge invariance

$$\mathcal{D}_\mu F^{\mu\nu} + \frac{m}{2}\, \varepsilon^{\nu\alpha\beta} F_{\alpha\beta} = 0$$

mathematics: Invariant against homotopically trivial
(≡deformable to identity) gauge transformations
∼ "small" gauge transformations
Non-invariant against homotopically non-trivial
(≡ not deformable to identity) gauge transformations
∼ "large" gauge transformations
⇒ in quantum theory m must be quantized for
consistency of non-Abelian theories.
knot theory: $W(A)$ is used in functional integral
formula for knot invariants]

N.B. Cannot avoid Chern-Simons term in presence of fermions
⇒ induced by radiative corrections.

(ii) gravity theory

$$W(\Gamma) = \int d^3x\, \varepsilon^{\alpha\beta\gamma} \left(\frac{1}{2} \Gamma^{\sigma}_{\alpha\,\tau} \partial_\beta \Gamma^{\tau}_{\gamma\,\sigma} + \frac{1}{3} \Gamma^{\rho}_{\alpha\,\sigma} \Gamma^{\sigma}_{\beta\,\tau} \Gamma^{\tau}_{\gamma\,\rho} \right)$$

$$\delta W(\Gamma) = \int d^3x\, \frac{1}{2}\, \varepsilon^{\alpha\beta\gamma} R^{\tau}{}_{\sigma\alpha\beta}\, \delta\, \Gamma^{\sigma}_{\gamma\,\tau}$$

$$\left(\text{as if } \Gamma^{\sigma}_{\gamma\tau} \sim (A_\gamma)^{\sigma}_{\tau},\; R^{\tau}{}_{\sigma\alpha\beta} \sim (F_{\alpha\beta})^{\tau}{}_{\sigma}\right)$$

but $g_{\mu\nu}$ (NOT $\Gamma^{\sigma}_{\gamma\,\tau}$) is fundamental variable

$$\delta\Gamma^{\sigma}_{\gamma\tau} = \frac{g^{\sigma\mu}}{2} (D_\gamma\, \delta g_{\mu\tau} + D_\tau\, \delta g_{\mu\gamma} - D_\mu \delta g_{\gamma\tau})$$

$$\Rightarrow \delta W(\Gamma) = -\int d^3x \sqrt{g}\, C^{\mu\nu} \delta g_{\mu\nu}$$

$$C^{\mu\nu} \equiv \frac{1}{2\sqrt{g}}\, \varepsilon^{\mu\alpha\beta} D_\alpha\, R^{\nu}_{\beta} + \frac{1}{2\sqrt{g}}\, \varepsilon^{\nu\alpha\beta} D_\alpha R^{\mu}_{\beta}$$

$\left(\text{can write in terms of Einstein tensor } G^{\mu}_{\nu} = R^{\mu}_{\nu} - \frac{1}{2}\, \delta^{\mu}_{\nu} R\right)$

$$C^{\mu\nu} = \frac{1}{2\sqrt{g}}\, \varepsilon^{\mu\alpha\beta} D_\alpha G^{\nu}_{\beta} + \frac{1}{2\sqrt{g}}\, \varepsilon^{\nu\alpha\beta} D_\alpha G^{\mu}{}_{\beta}$$

“Cotton Tensor”

NB : $C^{\mu\nu} = C^{\nu\mu}$; $C^{\mu}_{\mu} = 0$ [$W(\Gamma)$is conformally invariant]

$D_\mu C^{\mu\nu} = 0$ [$W(\Gamma)$is diffeomorphism invariant]

$C^{\mu\nu} = 0 \Leftrightarrow$ space-time is conformally flat

3-d geometry and gravity

in $d \geqslant 4$, Riemann tensor = Weyl tensor + Ricci scalar and tensor

Einsten equation in vacuum: Ricci terms vanish

curvature away from sources can be $\neq 0$

because Riemann tensor reduces to Weyl tensor

and remains non-vanishing.

in $d = 3$, Riemann tensor = Ricci scalar + tensor

Weyl tensor vanishes identically

Einstein theory: no curvature exterior to source,

vacuum is flat; no propagating modes.

Weyl tensor has another role:

template for conformal invariance in $d \geqslant 4$

$\Rightarrow$ vanishes if and only if space-time is conformally flat.

This role is taken by Cotton tensor in $d = 3$.

Adding Cotton to Einstein tensor $G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$

$$G^{\mu\nu} + \frac{1}{m} C^{\mu\nu} = \text{sources}$$

no quantization of coupling strength,

graviton propagates with mass m,

diffeomorphism invariance preserved!

[Einstein limit $m \rightarrow \infty$, massive graviton decouples

leaving no popagating modes]

This Research: $C^{\mu\nu} = 0$

[Guralnik, Iorio, Pi, and RJ, Ann. Phys. **308**, 222 (2003)]

Consider 2-d reduction:

3-d coordinates (t, x, y), demand no y dependence

Kaluza-Klein Ansatz for 3-*d* metric tensor

$$\text{3-}d \text{ metric tensor} = \varphi \begin{pmatrix} g_{\alpha\beta} - A_\alpha A_\beta & -A_\alpha \\ -A_\beta & -1 \end{pmatrix}$$

$g_{\alpha\beta}$: 2-*d* metric tensor, A_α: 2-component vector, φ:scalar

all functions of (t, x)

$$W(\Gamma) = -\frac{1}{2}\int d^2x\sqrt{-g}\left(F\,{}^{(2)}\!R + F^3\right)$$

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \equiv \sqrt{-g}\,\varepsilon_{\alpha\beta}\,F$$

$$CS \propto \int d\,A\,\left({}^{(2)}\!R + F^2\right);\int d^2x\,\Theta\,\varepsilon^{\alpha\beta}F_{\alpha\beta}$$

$$\Theta = {}^{(2)}\!R + F^2$$

Equations: $C^{\mu\nu} = 0$

$$\Rightarrow \qquad {}^{(2)}R + 3F^2 = \text{constant} \equiv C$$

$$\begin{aligned} 0 &= D^2F - CF + F^3 \\ 0 &= D_\alpha D_\beta F - \frac{1}{2} g_{\alpha\beta} D^2 F \end{aligned}$$

Note $F \rightarrow -F$ "symmetry" (action changes sign)

Homogenous Solutions

"symmetric" solution $F = 0,\ {}^{(2)}R = C$

"symmetry breaking" solution $F = \pm\sqrt{C},\ {}^{(2)}R = -2C < 0$

deSitter or anti-deSitter

Kink Solution

Above equations also support kink which interpolates between $F = \pm\sqrt{C}$ solutions

$$\begin{aligned} F(x) &= \sqrt{C} \tanh \frac{\sqrt{C}}{2} x \quad C > 0 \\ {}^{(2)}R &= -2C + \frac{3C}{\cosh^2 \frac{\sqrt{C}}{2} x} \end{aligned}$$

[dependence on one variable (x) rather than two (t, x) is a general result]

[More general solutions depend on 2 integration constants:

(i) origin of x coordinate

(ii) selection of kink in first integral

but local geometry is the same

[Grumiller & Kummer, Ann. Phys. **309**, 211 (2003)]

3-d Viewpoint

Returing to 3-d

$$R = {}^{(2)}R + \frac{1}{2}F^2 \qquad (\varphi = 1)$$

3-dimensional space-time is conformally flat $(C^{\mu\nu} = 0)$

"symmetric" solution

$$F = 0,\ {}^{(2)}R = C,\ \Rightarrow R = C$$

(a) $C > 0 \quad (ds)^2 = \frac{2}{C}\left[\left(\frac{dt}{t}\right)^2 - \left(\frac{dx}{t}\right)^2\right] - (dy)^2$

$$T = t\cosh\sqrt{\tfrac{C}{2}}y,\ Y = t\sinh\sqrt{\tfrac{C}{2}}y,\ X = x$$

$$\Rightarrow (ds)^2 = \frac{2}{C(T^2 - Y^2)}\left[(dT)^2 - (dX)^2 - (dY)^2\right]$$

(b) $C < 0 \quad (ds)^2 = \frac{2}{|C|}\left[\left(\frac{dt}{x}\right)^2 - \left(\frac{dx}{x}\right)^2\right] - (dy)^2$

$$X = x\cos\sqrt{\tfrac{|C|}{2}}y,\ Y = x\sin\sqrt{\tfrac{|C|}{2}}y,\ T = t$$

$$\Rightarrow (ds)^2 = \frac{2}{|C|(X^2 + Y^2)}\left[(dT)^2 - (dX)^2 - (dY)^2\right]$$

4 Killing vectors spanning S0(2.1) x S0(2)
(2-d deSitter or anti-deSitter space embedded in 3-d)

"symmetry breaking" solution

$$F = \pm\sqrt{C},\ {}^{(2)}\!R = -2C < 0, \Rightarrow R = -\frac{3}{2}C < 0$$

$$(ds)^2 = -\frac{2}{\sqrt{C}x}dtdy - \left(\frac{dx}{\sqrt{C}x}\right)^2 - (dy)^2$$

6 Killing vectors spanning S0(2.1) x S0(2.1)
$R^\mu_\nu = \frac{1}{3}\delta^\mu_\nu R = -\frac{1}{2}\delta^\mu_\nu C$, maximally symmetric space
⇒ anti-deSitter space, conformally flat! ($C^{\mu\nu}$ vanishes)

"kink" solution

$$F(x) = \sqrt{C}\tanh\frac{\sqrt{C}}{2}x,\ {}^{(2)}\!R = -2C + \frac{3C}{\cosh^2\frac{\sqrt{C}}{2}x},$$

$$\Rightarrow R = -\frac{3C}{2} + \frac{5C}{2\cosh^2\frac{\sqrt{C}}{2}x}$$

$$(ds)^2 = -\frac{2}{\cosh^2\frac{\sqrt{C}}{2}x}dtdy - (dx)^2 - (dy)^2$$

⇒ conformally flat ($C^{\mu\nu}$ vanishes)
asymplotically anti-deSitter space

Conformally flat coordinates for "symmetry breaking" and "kink" solutions

(M. Brigante, R. Sharma)

In kink set

$$\bar{t} = t + y/2, \qquad \bar{x} = \frac{1}{\sqrt{C}} \sinh^2 \frac{\sqrt{C}}{2} x \qquad , \bar{y} = y$$

$$\begin{aligned} (ds)^2_{kink} &= -\frac{2}{\cosh^2 \frac{\sqrt{C}}{2} x} dt dy - (dx)^2 - (dy)^2 \\ &= \frac{\bar{x}}{\bar{x} + 1/\sqrt{C}} \underbrace{\left(-\frac{2}{\sqrt{C}\,\bar{x}} d\bar{t} d\bar{y} - \left(\frac{d\bar{x}}{\sqrt{C}\,\bar{x}} \right)^2 - (d\bar{y})^2 \right)}_{\text{“symmetry breaking”} = AdS} \end{aligned}$$

transforming "symmetry breaking" = AdS:

$$\begin{aligned} T + Y &= \bar{t} + \bar{x} \tanh \tfrac{\sqrt{C}}{2} \bar{y} \\ T - Y &= -\tfrac{1}{\sqrt{C}} \tanh \tfrac{\sqrt{C}}{2} \bar{y} \\ X &= \sqrt{\frac{\bar{x}}{\sqrt{C}}} \frac{1}{\cosh \frac{\sqrt{C}}{2} \bar{x}} \end{aligned}$$

$$\begin{aligned} (ds)^2_{AdS} &= \frac{4}{CX^2} \left((dT)^2 - (dX)^2 - (dY)^2 \right) \\ (ds)^2_{kink} &= \frac{4}{1 - C(T-Y)^2 + CX^2} \left((dT)^2 - (dX)^2 - (dY)^2 \right) \end{aligned}$$

NB: $(ds)^2_{kink} \longrightarrow (ds)^2_{AdS}$ when $X \rightarrow large$

Supersymmetric extension

[L. Bergamin, D. Grumiller, A. Iorio and C. Nuñez (in preparation)]

Known for 3-d Chern-Simons term and Cotton tensor

Kaluza-Klein reduction $\Rightarrow$
supersymmetric generalization of our 2-d gravity

Previous solutions continue to hold when
supplemented by vanishing Fermion fields

Supersymmetry properties:
"symmetry preserving" solutions
are not preserved by supersymmetry
transformations

"symmetry breaking" and "kink" solutions
are preserved by $1/2$ of the
supersymmetry transformations

Also: our model controls some BPS solutions to $N = 2$, $D = 4$ gauged supergravity. [S. Cacciatori, M. Caldarelli, D. Klemm and D. Mansi, hep-th/0406238.]

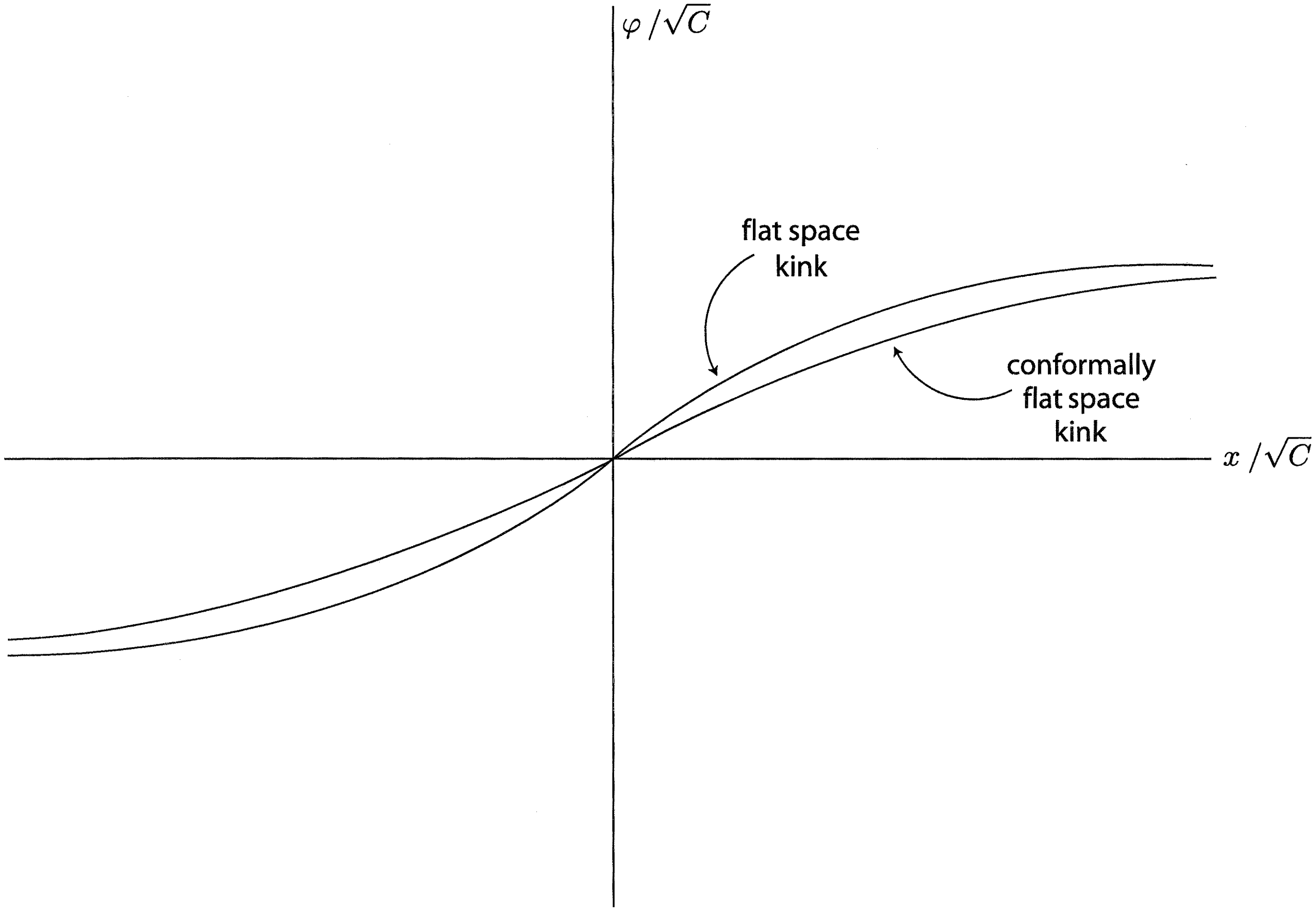
Connection to flat 2-d kink

$$\Box f - Cf + f^3 = 0$$

$$\text{flat kink}: f = \sqrt{C}\tanh\sqrt{\frac{C}{2}}x$$

differs from conformally flat kink by scale of x

$$F(x) = \sqrt{C}\tanh\frac{\sqrt{C}}{2}x$$



General result: if

$$\Box f + \mathrm{V}'(f) = 0$$

has kink solution $k(x)$, then

$$D^2 F + \mathrm{V}'(F) = 0$$
$$D_\alpha D_\beta F - \frac{1}{2} g_{\alpha\beta} D^2 F = 0$$

has solution

$$F(x) = k(x/\sqrt{2})$$

and

$$(ds)^2 = \mathrm{V}(F)(dt)^2 - (dx)^2$$
$${}^{(2)}R = -\mathrm{V}''(F)$$

Question: Why is there a kink in 3-d conformally flat space?