D-Branes, Superpotentials and A_∞ Algebras

Paul Aspinwall

Duke University

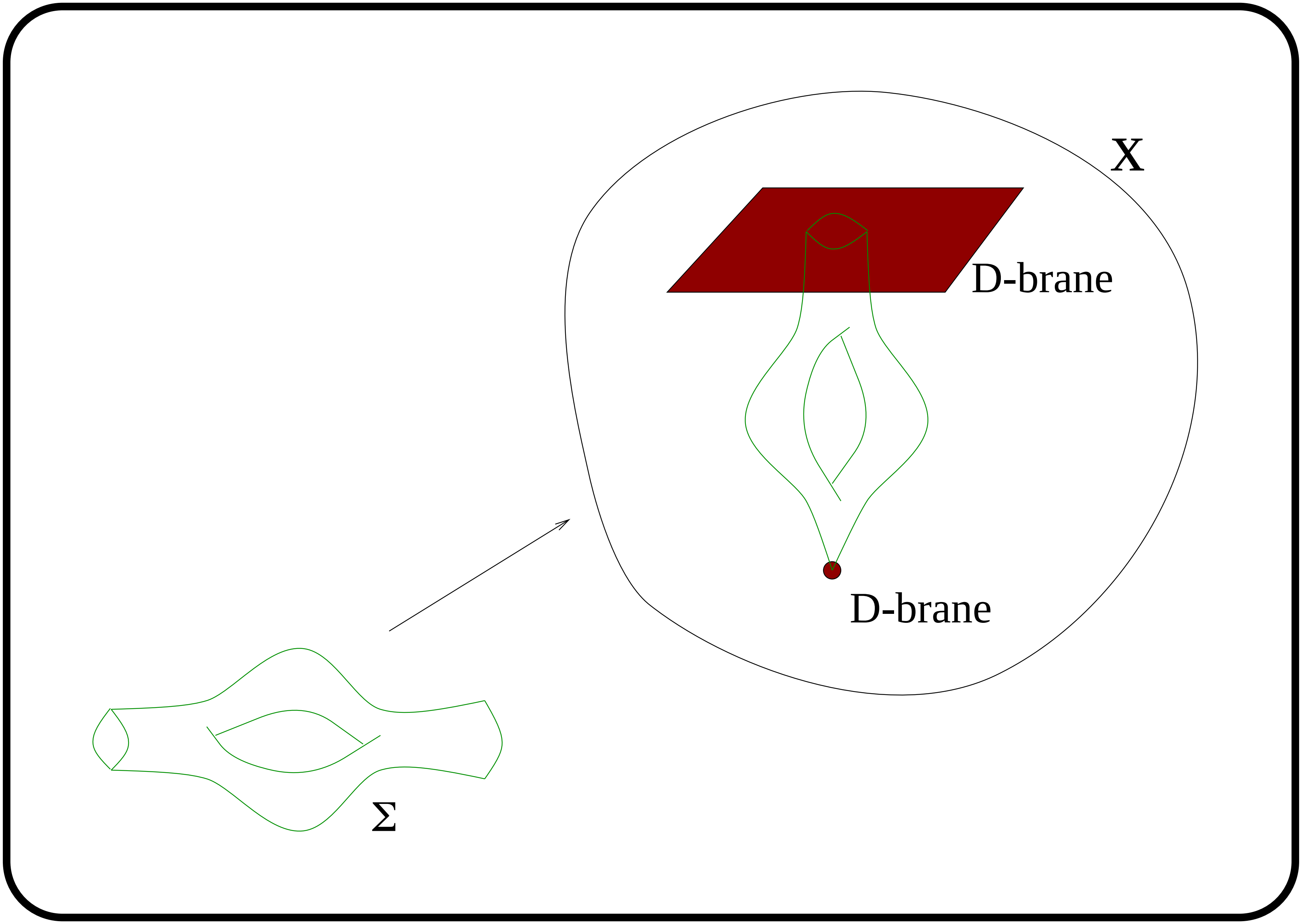
with Sheldon Katz hep-th/0412209.

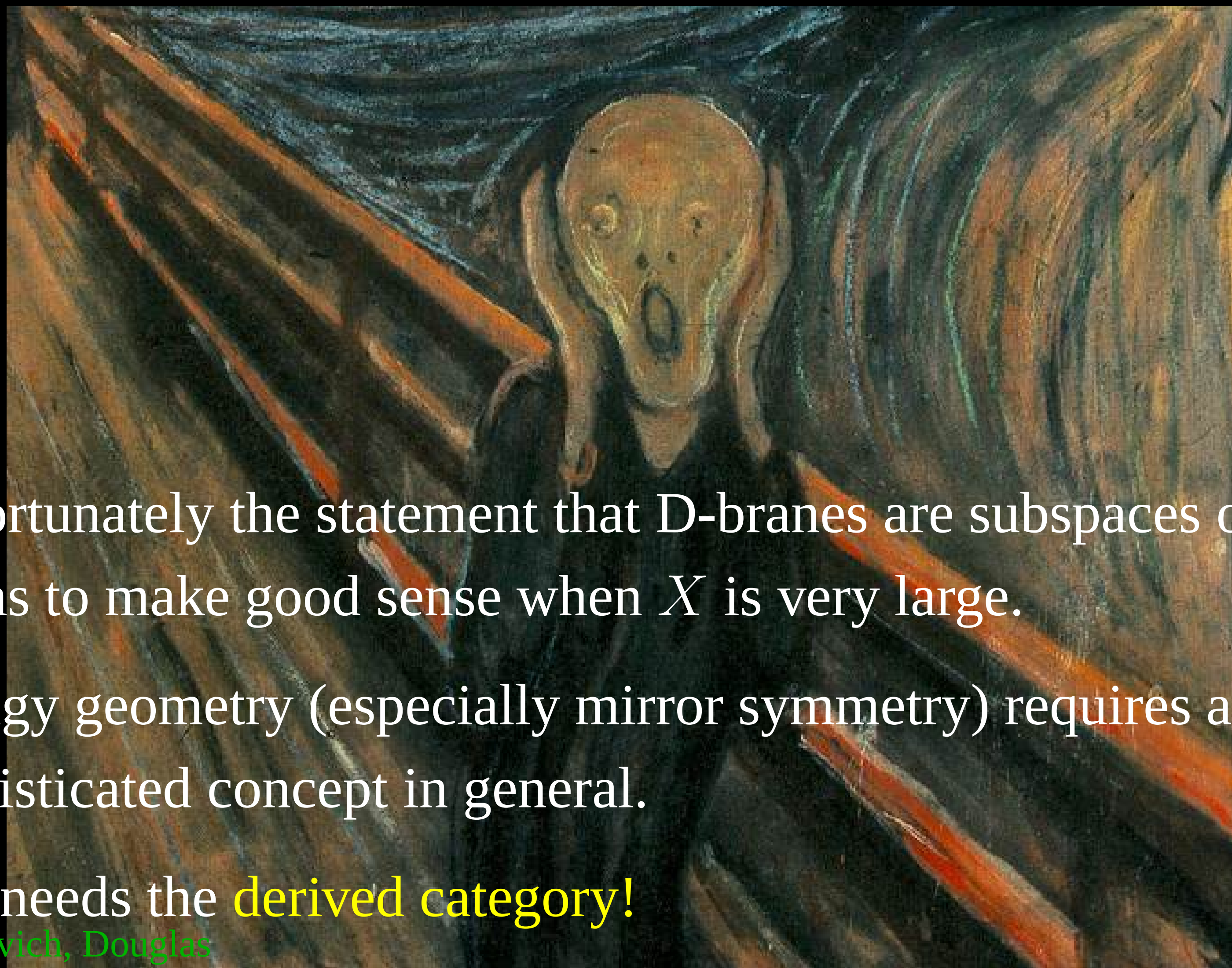
Consider a type IIB string propagating in the ten-dimensional spacetime $\mathbb{R}^{1,3} \times X$, where X is a smooth Calabi–Yau threefold.

The type IIB string contains open strings which must end on odd-dimensional D-Branes.

Suppose these D-branes fill the whole of $\mathbb{R}^{1,3}$ and have a remaining even number of dimensions in X.

A particularly easy class of D-branes to study are the BPS branes which, naïvely, correspond to holomorphic vector bundles over holomorphic submanifolds of X.


X
D-brane
D-brane
Σ



Unfortunately the statement that D-branes are subspaces only seems to make good sense when X is very large.

Stringy geometry (especially mirror symmetry) requires a more sophisticated concept in general.

One needs the derived category!
Kontsevich, Douglas

Actually the statement that we need the derived category is not surprising, nor is it bad news.

All our analysis may be done in terms of the topological B-model.

The study of topological field theories is a study of cohomology, and thus derived functors. . .

The derived category also provides a good conceptual framework for practical computations.

For our purposes a D-brane on X is a finite complex of coherent sheaves (which may ultimately be holomorphic vector bundles).

$$\ldots \longrightarrow A^0 \longrightarrow A^1 \longrightarrow A^2 \longrightarrow \ldots$$

A complex submanifold $W \subset X$ would be represented by the single coherent sheaf at position zero:

$$\ldots \longrightarrow \mathcal{O}_W \longrightarrow 0 \longrightarrow 0 \longrightarrow \ldots$$

A chain map between two complexes

$$\begin{tikzcd} \cdots \arrow[r] & A^0 \arrow[r] \arrow[d] & A^1 \arrow[r] \arrow[d] & A^2 \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & B^0 \arrow[r] & B^1 \arrow[r] & B^2 \arrow[r] & \cdots \end{tikzcd}$$

is a quasi-isomorphism if it induces the identity map on the cohomology of the two complexes.

Two D-branes are identical if they are represented by quasi-isomorphic complexes.

"Quotienting" by this "equivalence" yields the derived category.

A chain map between two complexes is simply an open string between the two D-branes.

More precisely, let $\mathrm{Hom}(A^\bullet, B^\bullet)$ represent the space of morphisms in the derived category between objects $A^\bullet$ and $B^\bullet$ and let $A^\bullet[q]$ represent the complex $A^\bullet$ shifted left by q places.

Then the Hilbert space of open strings in the topological B-model between the D-branes $A^\bullet$ and $B^\bullet$ is given by

$$\mathrm{Hom}(A^\bullet, B^\bullet[q]) = \mathrm{Ext}^q(A^\bullet, B^\bullet)$$

for all $q \in \mathbb{Z}$ (the ghost number).

An open string $f : A^\bullet \to B^\bullet$ may bind two D-branes (anti-A and B) together to form the mapping cone of f:

$$\cdots \longrightarrow \begin{matrix} A^0 \\ \oplus \\ B^{-1} \end{matrix} \xrightarrow{\begin{pmatrix} d^A & 0 \\ f & d^B \end{pmatrix}} \begin{matrix} A^1 \\ \oplus \\ B^0 \end{matrix} \xrightarrow{\begin{pmatrix} d^A & 0 \\ f & d^B \end{pmatrix}} \begin{matrix} A^2 \\ \oplus \\ B^1 \end{matrix} \longrightarrow$$

Any complex may be built from single-term complexes by iteratively using the cone construction.

Thus we may always view a complex in terms of a bound state of many branes corresponding to single-term complexes (so long as we carefully keep note of anti-branes in the right way).

Thus, every D-brane is some collection of more naïve branes.

Actually every D-brane may be viewed as a collection of vector bundles, i.e., branes that wrap all of X.

Whether or not an open string really does bind two D-branes together or not depends on whether that open string is tachyonic or not.

This depends on the value of the complexified Kähler form $B + iJ \in H^2(X, \mathbb{C}^*)$.

Thus, only certain objects in $\mathbf{D}(X)$ are stable in the sense that the complex will not decay via mapping cones. These objects are said to be Π-stable.
Douglas, Fiol, Römelsberger
PSA, Douglas
Bridgeland

So far we have focused on the Calabi–Yau part of spacetime.

Recall that our D-brane also filled flat spacetime $\mathbb{R}^{1,3}$.

This gives rise to a four-dimensional quantum field theory for the D-brane world-volume. Since the D-brane is BPS, we have $\mathcal{N} = 1$ supersymmetry.

The identity map in $\mathrm{Hom}(A^\bullet, A^\bullet)$ gives rise to an open string corresponding to a photon.

Thus we have an $\mathcal{N} = 1$ supersymmetric (at least) $\mathrm{U}(1)$-gauge theory in four dimensions.

An object $A^\bullet$ is simple if $\mathrm{Hom}(A^\bullet, A^\bullet) \cong \mathbb{C}$.

Suppose, instead, that

$$A^\bullet = \bigoplus_{j=1}^{N} C^\bullet,$$

where $C^\bullet$ is simple.

Then $\mathrm{Hom}(A^\bullet, A^\bullet) \cong \mathbb{C}^{N^2}$, and we have a $\mathrm{U}(N)$ gauge theory.

Open strings in $\operatorname{Ext}^1(A^\bullet, A^\bullet)$ give rise to massless chiral supermultiplets in our field theory (transforming in the adjoint of $\mathrm{U}(N)$).

Giving the scalar field a vacuum expectation value in such a field theory corresponds to deforming the theory, i.e., deforming or moving the D-brane.

An $N = 1$ field theory contains a superpotential W which is a polynomial of the chiral superfields x_j. The vacuum must satisfy

$$\frac{\partial W}{\partial x_j} = 0.$$

It is well-known that $\mathrm{Ext}^1(A^\bullet, A^\bullet)$ gives the space of first-order deformations of $A^\bullet$.

Whether or not such a first-order deformation can be integrated to a true deformation requires some knowledge of obstructions.

Since the superpotential also “knows” about these deformations, it must be that the superpotential encodes the information of obstruction theory.

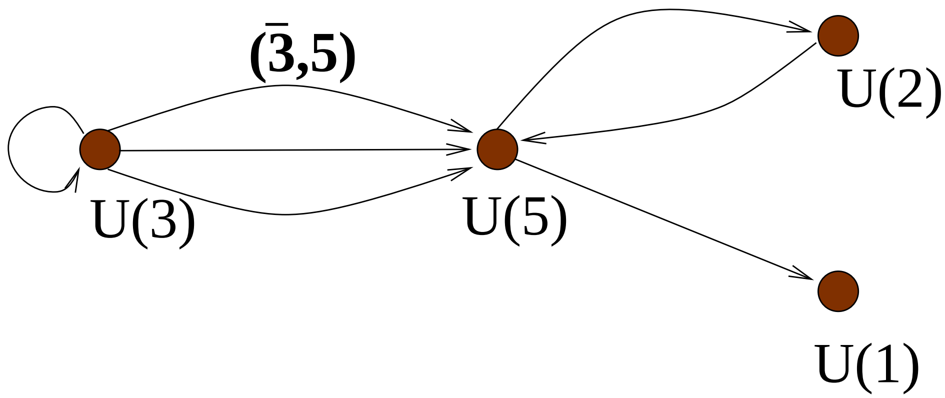
For a suitable value of $B + iJ$, a D-brane might be marginally stable with respect to a decay into N_1 copies of some D-brane, N_2 copies of some other D-brane, etc.

This gives an $\mathcal{N} = 1$ gauge theory with gauge group $\mathrm{U}(N_1) \times \mathrm{U}(N_2) \times \ldots$.

The fact that the given D-brane is marginally stable means that there will be massless open strings between the decay products. These strings give rise to massless chiral supermultiplets in bifundamental $(\overline{\mathbf{N}}_1, \mathbf{N_2})$ representations etc.

A “quiver” gauge theory – also with a superpotential.

“Ext Quiver”



The arrows are the chiral superfields.

Giving nonzero values to these arrows yields a quiver representation.

Matrices

The superpotential is a function of all these chiral superfields.

$$W = \mathrm{Tr}(ABC + \ldots).$$

This superpotential can impose relations on quiver and higher order obstructions.

We want a general method for computing the superpotential!

Various techniques have been given in the literature for various examples:

- Douglas, Moore
- Brunner, Douglas, Lawrence, Römelsberger
- Klebanov, Witten
- Morrison, Plesser
- Kachru, Katz, Lawrence, McGreevy
- Cachazo, Katz, Vafa
- Douglas, Govindarayan, Jayaraman, Tomasiello
- Herbst, Lazaroiu, Lerche
- Aganagic, Vafa
- Ashok, Dell'Aquila, Diaconescu, Florea

Here we give a solution to the problem for $g_s \rightarrow 0$.

The key structure to the superpotential is an A_∞-algebra (for one type of D-brane), or its generalization to an A_∞-category (in the case of the quiver gauge theory).

Let A be a graded vector space over $\mathbb{C}$. We equip A with "higher products":

$$m_k : A^{\otimes k} \to A, \tag{1}$$

The map m_k has degree $2-k$ with respect to the grading.

We demand

$$\sum_{r+s+t=n} (-1)^{r+st} m_u(\mathbf{1}^{\otimes r} \otimes m_s \otimes \mathbf{1}^{\otimes t}) = 0, \tag{2}$$

for any $n > 0$, where $u = n + 1 - s$.

These may be viewed as generalized associativity conditions.

$$\begin{aligned} m_1 m_1 &= 0 \\ m_2(m_1 \otimes \mathbf{1} + \mathbf{1} \otimes m_1) &= m_1 m_2 \\ m_2(\mathbf{1} \otimes m_2 - m_2 \otimes \mathbf{1}) &= \ldots \end{aligned} \tag{3}$$

The A_∞ category may be defined similarly by giving the morphisms a higher product structure.

Now consider the topological B-model for open strings between B-type D-branes.

For simplicity, for now, assume we have only one D-brane $C^\bullet$.

The Hilbert space of open strings is given by

$$A = \bigoplus_q \mathrm{Ext}^q(C^\bullet, C^\bullet).$$

The grading is, of course, given by q (the ghost number).

The open strings are associated to local vertex operators ψ_i in the topological field theory.

These ψ_i's may be viewed as a basis for A.

To each such vertex operator, one may construct a 1-form operator

$$\psi_i^{(1)} = \frac{1}{\sqrt{2}} \left\{ G^-_{-\frac{1}{2}} + \overline{G}^-_{-\frac{1}{2}}, \psi_i \right\}, \tag{4}$$

These 1-form operators may be used to deform the topological field theory (at least to first order):

$$S \to S + \sum_i Z_i \oint \psi_i^{(1)}. \tag{5}$$

The Z_i are complex numbers as far as the topological field theory is concerned.

The Z_i (for $q = 1$) are (the scalar components of) chiral superfields in the effective world-volume theory.

The above deformations correspond to giving vacuum expectations values to these fields.

The chiral superfields are naturally dual to the vertex operators of the topological quantum field theory.

Define (up to suppressed signs) on a disk

$$B_{i_0,i_1,\ldots,i_k} = \langle \psi_{i_0}\, \psi_{i_1}\, P \int \psi^{(1)}_{i_2} \int \psi^{(1)}_{i_3} \ldots \int \psi^{(1)}_{i_{k-1}}\, \psi_{i_k} \rangle, \qquad (6)$$

In the case of N copies of a simple D-brane, the fields Z_i naturally form $N \times N$ matrices. We may now write the tree-level superpotential Brunner, Douglas, Lawrence, Römelsberger

$$\boxed{\mathbf{W} = \mathrm{Tr}\left(\sum_{k=2}^{\infty} \sum_{i_0,i_1,\ldots,i_k} \frac{B_{i_0,i_1,\ldots,i_k}}{k+1} Z_{i_0} Z_{i_1} \ldots Z_{i_k}\right).}$$

If X is a Calabi–Yau threefold, there is also a "trace map" of degree -3 (Serre duality: $\mathrm{Ext}^3(C^\bullet, C^\bullet) \cong \mathbb{C}$)

$$\gamma : A \to \mathbb{C}. \tag{7}$$

The correlation functions may be written in the form

$$B_{i_0,i_1,\ldots,i_k} = \gamma\left(m_2\left(m_k(\psi_{i_0}, \psi_{i_1}, \ldots, \psi_{i_{k-1}}), \psi_{i_k}\right)\right), \tag{8}$$

for maps of degree $2-k$

$$m_k : A^{\otimes k} \to A. \tag{9}$$

It can be shown that these products do indeed obey the conditions (2) and thus give A the structure of an A_∞ algebra.
Herbst, Lazaroiu, Lerche

There are two extreme versions of an A_∞-algebra.

The first is where all the higher products vanish:

$$m_k = 0, \qquad k > 2.$$

This is a $\boxed{\text{differential graded algebra}}$ (dga).

m_1 is the differential and m_2 is the product.

The second is a $\boxed{\text{minimal model}}$, where $m_1 = 0$ but all higher products can be nonzero.

An $\boxed{A_\infty \text{ morphism}}$ is a set of maps

$$f_k : A^{\otimes k} \to B. \tag{10}$$

such that (up to signs)

$$\sum_{r+s+t=n} f_u(\mathbf{1}^{\otimes r} \otimes m_s \otimes \mathbf{1}^{\otimes t}) = \sum_{\substack{1 \le r \le n \\ i_1+\ldots+i_r=n}} m_r(f_{i_1} \otimes f_{i_2} \otimes \cdots \otimes f_{i_r}), \tag{11}$$

for any $n > 0$ and $u = n + 1 - s$.

Given a dga A, one may construct the space $B = H^*(A)$.

Thanks to a theorem by Kadeishvili, we may define an A_∞ structure on B such that

1. There is an A_∞ morphism f from B to A with f_1 equal to an embedding $i : B \hookrightarrow A$.

2. $m_1 = 0$ (i.e. B is minimal).

This A_∞ structure on B is unique up to A_∞-isomorphisms.

The fact that the A_∞-algebra can only be identified up to some ambiguity is unfortunate but actually inevitable given that we are only using topological field theory methods.

In the language of topological field theory there are intrinsic ambiguities in the definition of the correlation functions arising from contact terms.

In the language of the effective $\mathcal{N} = 1$ supersymmetric field theory we do not know the kinetic terms and thus only know the superpotential up to nonlinear field redefinitions.

Actually we do better than knowing W up to any nonlinear field redefinition but we will not pursue this here.

The combinatorics of constructing B from A is identical to a tree-level ϕ^3 quantum field theory. Kontsevich and Soibelman

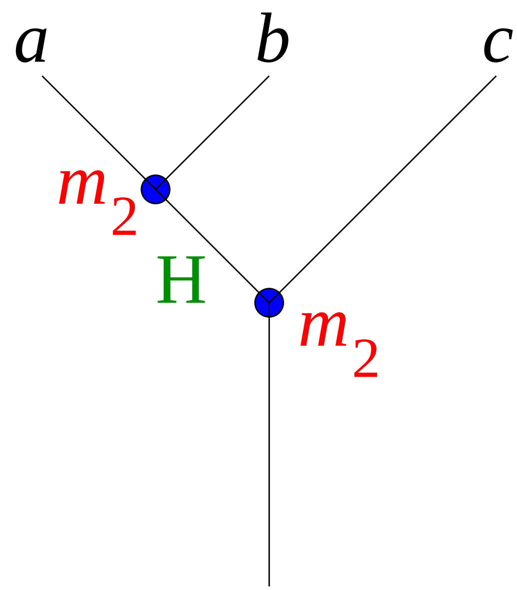
m_k for B is given by a sum over trees with k “in” legs and 1 “out” leg.

The vertex is given by m_2 of A.

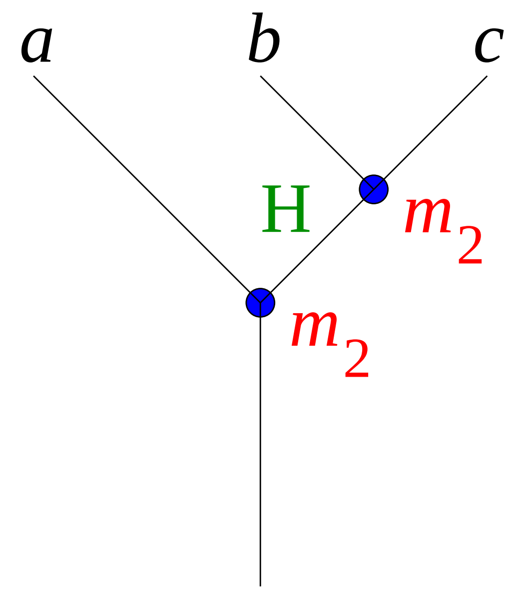
The propagator H is given by “the inverse” of m_1 of A. More precisely, if $p : A \to B$ is a map such that $p \circ i = 1$, then

$$1 - i \circ p = m_1 H + H m_1.$$

$m_3(a,b,c) \; =$



$+$



The A-infinity structure in the general case is highly non-trivial and includes interesting products between morphisms of many gradings.

E.g. Polishchuk has analyzed derived category of an elliptic curve.

In our case, we only consider collections of objects that make a sensible quantum field theory. (No Hom's between nodes in quiver.)

The effect of this is to make the A-infinity structure fairly boring except for the products between "$q = 1$" morphisms.

Thus we focus only on these products from now on.

Let us restrict attention to 9-branes, i.e., vector bundles E covering X.

Witten showed that the correlation functions required for our superpotential could be computed, at tree-level, by a cubic field theory:

Holomorphic Chern–Simons theory:

$$S = \int_X \mathrm{Tr}\left(\mathsf{A} \wedge \bar{\partial}\mathsf{A} + \tfrac{2}{3}\mathsf{A} \wedge \mathsf{A} \wedge \mathsf{A}\right) \wedge \Omega, \tag{12}$$

where A is a 1-form valued in $\mathrm{End}(E)$.

That is,

$\boxed{\text{the } A_\infty\text{-algebra for the correlation functions is a minimal } A_\infty\text{-algebra computed from a dga given by differential forms with } m_1 = \bar{\partial} \text{ and } m_2 = \wedge.}$

Note these differential forms are valued in $\mathrm{End}(E)$ for a single 9-brane; or $\mathrm{Hom}(E, F)$ for the quiver version with several 9-branes.

One may show that differential forms with $m_1 = \bar{\partial}$ and $m_2 = \wedge$ my be replaced by:

Čech cochains with $m_1 = \delta$ and $m_2 = \cup$ or

a similar structure in $\mathbf{D}(X)$ given by maps between complexes.

This latter structure is an intrinsic A_∞-structure in $\mathbf{D}(X)$ (studied by Polishchuk for example).

It allows us to extend the computation of the superpotential over the whole of $\mathbf{D}(X)$ (not just 9-branes).

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{d} & \mathscr{E}^{n-1} & \xrightarrow{d} & \mathscr{E}^{n} & \xrightarrow{d} & \mathscr{E}^{n+1} & \xrightarrow{d} & \cdots \\
 & & \Big\downarrow{\scriptstyle g_{n-1}} & & \Big\downarrow{\scriptstyle g_{n}} & & \Big\downarrow{\scriptstyle g_{n+1}} & & \\
\cdots & \xrightarrow{d} & \mathscr{E}^{n-1} & \xrightarrow{d} & \mathscr{E}^{n} & \xrightarrow{d} & \mathscr{E}^{n+1} & \xrightarrow{d} & \cdots
\end{array}$$

and

$$\mathfrak{d} = d \circ g - (-1)^n g \circ d.$$

Thus, a $\mathfrak{d}$-closed map is a chain map and a $\mathfrak{d}$-exact map is a chain homotopy.

The product is simply a composition of chain maps.

Yet another presentation of the same mathematics is given by another, more practical picture.

Let $\mathscr{E}^\bullet$ be a complex of *locally-free* sheaves representing a given D-brane.

Now build a $\boxed{\text{double complex}}$ with entries

$$\bigoplus_{p+q=n} \check{C}^p \left(\mathfrak{U}, \mathscr{H}\!om(\mathscr{E}^\bullet, \mathscr{E}^\bullet[q])\right), \tag{13}$$

with m_1 and m_2 from above.

This can be generalized to several branes in the obvious way.

For example, let $C \cong \mathbb{P}^1$ and let X be the (noncompact) total space of the bundle $\mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$.

The 3-brane corresponds to a point in X. An example of a 5-brane is $\mathcal{O}_C$.

We may vary $B + iJ$ on this space and vary the area of C. At a special value of $B + iJ$, we have a conifold.

By Π-stability arguments, the 3-brane on the conifold point decays marginally into two 5-branes $\mathcal{O}_C$ and $\mathcal{O}_C(-1)[1]$. Thus, if we considered N coincident 3-branes at this conifold point, we would have a $\mathrm{U}(N) \times \mathrm{U}(N)$ quiver gauge theory.

To produce a local model for this case, let X be the total space of the normal bundle $\mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$. Thus we have bundle map $\pi : X \to C$. An affine open cover of X is then given by two patches: U_0, with coordinates (x, y_1, y_2); and U_1, with coordinates (w, z_1, z_2). The transition functions are obviously

$$\begin{aligned} w &= x^{-1} \\ z_1 &= xy_1 \\ z_2 &= xy_2 \end{aligned} \tag{14}$$

Now $\mathcal{O}_C$ is not a locally-free sheaf on X. Define $\mathcal{O}(1) = \pi^*\mathcal{O}_C(1)$. We then have an exact sequence

$$\mathcal{O}(2) \xrightarrow[\begin{pmatrix} -z_2 \\ z_1 \end{pmatrix}]{\begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}} \mathcal{O}(1) \oplus \mathcal{O}(1) \xrightarrow[\begin{pmatrix} z_1 & z_2 \end{pmatrix}]{\begin{pmatrix} y_1 & y_2 \end{pmatrix}} \mathcal{O} \longrightarrow \mathcal{O}_C, \tag{15}$$

where we have given the explicit sheaf maps in both patches. This provides the locally-free resolution of $\mathcal{O}_C$, and thus $\mathcal{O}_C(-1)[1]$ too by tensoring the resolution by $\mathcal{O}(-1)$ and shifting one place to the left.

$\mathrm{Ext}^1(\mathcal{O}_C(-1)[1], \mathcal{O}_C)$ and $\mathrm{Ext}^1(\mathcal{O}_C, \mathcal{O}_C(-1)[1])$ are both isomorphic to $\mathbb{C}^2$. Thus we have a quiver:

$$\begin{tikzcd}[column sep=huge] \mathcal{O}_C(-1)[1] \arrow[r, bend left=40, "a"] \arrow[r, bend left=12, "b"] & \mathcal{O}_C \arrow[l, bend left=12, "c"] \arrow[l, bend left=40, "d"] \end{tikzcd} \tag{16}$$

The classes in $\mathrm{Ext}^1(\mathcal{O}_C(-1)[1], \mathcal{O}_C)$ are represented by elements of $\check{C}^0(\mathfrak{U}, \mathscr{H}\!om(\mathcal{O}_C(-1), \mathcal{O}_C))$ as follows. Using the notation described above, let one generator of this group, denoted a, be represented by

$$\begin{tikzcd} \mathcal{O}(1) \arrow[r, "\begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}"] \arrow[d, "1"] & \mathcal{O} \oplus \mathcal{O} \arrow[r, "\begin{pmatrix} y_1 & y_2 \end{pmatrix}"] \arrow[d, "-\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}"] & \mathcal{O}(-1) \arrow[d, "1"] \\ \mathcal{O}(2) \arrow[r, "\begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}"] & \mathcal{O}(1) \oplus \mathcal{O}(1) \arrow[r, "\begin{pmatrix} y_1 & y_2 \end{pmatrix}"] & \mathcal{O} \end{tikzcd} \tag{17}$$

and b by the same thing with 1 replaced by x in the vertical maps.

Next, the two generators of $\mathrm{Ext}^1(\mathcal{O}_C, \mathcal{O}_C(-1)[1])$ can be represented by elements of $\check{C}^1(\mathfrak{U}, \mathscr{H}om(\mathcal{O}_C, \mathcal{O}_C(-1)[1])$. Let c be represented by

$$\begin{tikzcd} & \mathcal{O}(2) \arrow[r, "\begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 0 \\ -\frac{1}{x} \end{pmatrix}_{01}"] & \mathcal{O}(1) \oplus \mathcal{O}(1) \arrow[r, "\begin{pmatrix} y_1 & y_2 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} \frac{1}{x} & 0 \end{pmatrix}_{01}"] & \mathcal{O} \\ \mathcal{O}(1) \arrow[r, "\begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}"] & \mathcal{O} \oplus \mathcal{O} \arrow[r, "\begin{pmatrix} y_1 & y_2 \end{pmatrix}"] & \mathcal{O}(-1) & \end{tikzcd} \tag{18}$$

and d by something similar.

Finally, the generator of $\mathrm{Ext}^3(\mathcal{O}_C(-1)[1], \mathcal{O}_C(-1)[1])$ can be represented by a 1-cochain in $\check{C}^1(\mathfrak{U}, \mathscr{H}om(\mathcal{O}_C(-1), \mathcal{O}_C(-1)[2]))$:

$$
\begin{tikzcd}
& & \mathcal{O}(1) \arrow[r, "\begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}"] \arrow[d, "(\frac{1}{x})_{01}"] & \mathcal{O} \oplus \mathcal{O} \arrow[r, "\begin{pmatrix} y_1 & y_2 \end{pmatrix}"] & \mathcal{O}(-1) \\
\mathcal{O}(1) \arrow[r, "\begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}"] & \mathcal{O} \oplus \mathcal{O} \arrow[r, "\begin{pmatrix} y_1 & y_2 \end{pmatrix}"] & \mathcal{O}(-1) & &
\end{tikzcd}
\tag{19}
$$

The composition $\mathsf{c} \star \mathsf{a}$ gives a map

$$\begin{tikzcd}
 & \mathcal{O}(1) \arrow[r, "\begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 0 \\ -\frac{1}{x} \end{pmatrix}_{01}"] & \mathcal{O} \oplus \mathcal{O} \arrow[r, "\begin{pmatrix} y_1 & y_2 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} -\frac{1}{x} & 0 \end{pmatrix}_{01}"] & \mathcal{O}(-1) \\
\mathcal{O}(1) \arrow[r, "\begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}"] & \mathcal{O} \oplus \mathcal{O} \arrow[r, "\begin{pmatrix} y_1 & y_2 \end{pmatrix}"] & \mathcal{O}(-1) &
\end{tikzcd} \tag{20}$$

This is δ-exact. Thus $m_2(\mathsf{c}, \mathsf{a}) = 0$.

More precisely, $\mathsf{c} \star \mathsf{a}$ is a Čech coboundary of the map which is zero in patch 0 and in patch 1 given by the chain map

$$\begin{tikzcd} & \mathcal{O}(1) \arrow[r, "\begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 0 \\ -1 \end{pmatrix}_1"] & \mathcal{O} \oplus \mathcal{O} \arrow[r, "\begin{pmatrix} y_1 & y_2 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} -1 & 0 \end{pmatrix}_1"] & \mathcal{O}(-1) \\ \mathcal{O}(1) \arrow[r, "\begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}"] & \mathcal{O} \oplus \mathcal{O} \arrow[r, "\begin{pmatrix} y_1 & y_2 \end{pmatrix}"] & \mathcal{O}(-1) & \end{tikzcd} \quad (21)$$

The above represents $H(\mathsf{c} \star \mathsf{a})$.

Continuing this way, $\mathsf{b} \star H(\mathsf{c} \star \mathsf{a}) + H(\mathsf{b} \star \mathsf{c}) \star \mathsf{a}$ is

$$
\begin{tikzcd}
 & \mathcal{O}(1) \arrow[r, "\begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 0 \\ 1 \end{pmatrix}"] & \mathcal{O} \oplus \mathcal{O} \arrow[r, "\begin{pmatrix} y_1 & y_2 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} -1 & 0 \end{pmatrix}"] & \mathcal{O}(-1) \\
\mathcal{O}(2) \arrow[r, "\begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}"] & \mathcal{O}(1) \oplus \mathcal{O}(1) \arrow[r, "\begin{pmatrix} y_1 & y_2 \end{pmatrix}"] & \mathcal{O} &
\end{tikzcd}
\tag{22}
$$

When composed with d this gives the Ext^3 of (19) but when composed with c it gives zero. Thus $m_3(\mathsf{b}, \mathsf{c}, \mathsf{a})$ is Serre dual to d.

Denoting by A the $N = 1$ superfield dual to a etc., we thus have a term in the superpotential equal to $\mathrm{Tr}(BCAD)$.

Checking all combinations (and being careful with signs!) one obtains:

$$\boxed{W = \mathrm{Tr}(BCAD - ACBD).}$$

in agreement with Klebanov and Witten.

For another example, consider a 5-brane wrapping a $\mathbb{P}^1$ in X which has normal bundle $\mathcal{O}_C \oplus \mathcal{O}_C(-2)$ but whose deformations are obstructed.

An example of such a $\mathbb{P}^1$ can be given explicitly in patches using the transition functions

$$\begin{aligned} w &= x^{-1} \\ z_1 &= x^2 y_1 + x y_2^n \\ z_2 &= y_2 \end{aligned} \tag{23}$$

The Picard group is one-dimensional and we denote the corresponding twisted sheaf $\mathcal{O}(1)$.

By using a locally-free resolution, $\mathcal{O}_C$ is quasi-isomorphic to

$$\mathcal{O} \xrightarrow{\begin{pmatrix} y_2 \\ -1 \\ x \end{pmatrix}} \begin{matrix} \mathcal{O} \\ \oplus \\ \mathcal{O}(1) \\ \oplus \\ \mathcal{O}(1) \end{matrix} \xrightarrow{\begin{pmatrix} 1 & y_2 & 0 \\ -x & 0 & y_2 \\ -y_2^{n-1} & -s & -y_1 \end{pmatrix}} \begin{matrix} \mathcal{O}(1) \\ \oplus \\ \mathcal{O}(1) \\ \oplus \\ \mathcal{O} \end{matrix} \xrightarrow{\begin{pmatrix} s & y_1 & y_2 \end{pmatrix}} \mathcal{O}. \tag{24}$$

where $s = xy_1 + y_2^n$.

Define x to be the following generator of $\mathrm{Ext}^1(\mathcal{O}_C, \mathcal{O}_C) \cong \mathbb{C}$:

$$
\begin{tikzcd}
 & \mathcal{O} \arrow[r] \arrow[d, "\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}"] & \begin{matrix} \mathcal{O} \\ \oplus \\ \mathcal{O}(1) \\ \oplus \\ \mathcal{O}(1) \end{matrix} \arrow[r] \arrow[d, "\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ y_2^{n-2} & 0 & 0 \end{pmatrix}"] & \begin{matrix} \mathcal{O}(1) \\ \oplus \\ \mathcal{O}(1) \\ \oplus \\ \mathcal{O} \end{matrix} \arrow[r] \arrow[d, "(0 \; 0 \; 1)"] & \mathcal{O} \\
\mathcal{O} \arrow[r] & \begin{matrix} \mathcal{O} \\ \oplus \\ \mathcal{O}(1) \\ \oplus \\ \mathcal{O}(1) \end{matrix} \arrow[r] & \begin{matrix} \mathcal{O}(1) \\ \oplus \\ \mathcal{O}(1) \\ \oplus \\ \mathcal{O} \end{matrix} \arrow[r] & \mathcal{O} &
\end{tikzcd}
\tag{25}
$$

Now $\mathsf{x} \star \mathsf{x}$ is a chain homotopy, i.e., a boundary, if $n \geq 3$. It is the boundary of k:

$$
\begin{tikzcd}
 & \mathscr{F}_3 \arrow[r] \arrow[d, "\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}"] & \mathscr{F}_2 \arrow[r] \arrow[d, "\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ y_2^{n-3} & 0 & 0 \end{pmatrix}"] & \mathscr{F}_1 \arrow[r] \arrow[d, "\begin{pmatrix} 0 & 0 & 0 \end{pmatrix}"] & \mathscr{F}_0 \\
\mathscr{F}_3 \arrow[r] & \mathscr{F}_2 \arrow[r] & \mathscr{F}_1 \arrow[r] & \mathscr{F}_0 &
\end{tikzcd}
\tag{26}
$$

But $\mathsf{x} \star \mathsf{k} + \mathsf{k} \star \mathsf{x}$ yields the same map as $\mathsf{x} \star \mathsf{x}$ except that n is replaced by $n - 1$.

Thus the process continues iteratively.

Ultimately one recovers the superpotential

$$W = X^{n+1}$$

as has been derived in many other ways previously.

Conclusion

- This computation can be, and has been, extended to other examples.
- It is applicable to any set of B-type D-branes in any Calabi–Yau threefold.
- It is not clear how much stamina is required for complicated examples!
- The precise nature of the ambiguities is not fully understood.
- It is only valid to tree-level.