Stability conditions and D-branes

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Some background from string theory

An $N = (2, 2)$ SCFT gives rise to two topological CFTs, the A-model and the B-model. Mathematically, these are A^{∞}-categories of a certain kind (Costello, Kontsevich).

The moduli space $\mathcal{M}$ of SCFTs thus has two foliations whose leaves correspond to SCFTs with fixed A- or B-models.

For example, at a point of $\mathcal{M}$ corresponding to the non-linear sigma model on a CY 3-fold $X_{I,\beta,\omega}$ the two TCFTs are

$$\mathcal{D}^b \operatorname{Fuk}(X_{\beta,\omega}) \text{ and } \mathcal{D}^b \operatorname{Coh}(X_I).$$

The corresponding leaves are the complex moduli space $\mathcal{M}_{\mathbb{C}}(X_{\beta,\omega})$ and the stringy Kähler moduli space $\mathcal{M}_{\mathcal{K}}(X_I)$.

Question : What do the points of the leaf $\mathcal{L} \subset \mathcal{M}$ corresponding to a fixed TCFT $\mathcal{D}$ correspond to in terms of $\mathcal{D}$?

Answer (Douglas) : Points of $\mathcal{L}$ determine an $\mathbb{R}$-graded subcategory

$$\mathcal{P} = \bigcup_{\phi \in \mathbb{R}} \mathcal{P}(\phi) \subset \mathcal{D}$$

of BPS branes, together with complex numbers (central charges)

$$Z(E) \in \mathbb{R}_{>0} \exp(i\pi\phi)$$

for all $E \in \mathcal{P}(\phi)$.

Example : Take the A-model

$$\mathcal{D} = \mathcal{D}^b \,\mathsf{Fuk}(X_{\beta,\omega})$$

above. Then points of $\mathcal{M}_{\mathbb{C}}(X_{\beta,\omega})$ determine the subcategory $\mathcal{P} \subset \mathcal{D}$ of special Lagrangians. The map Z is given by

$$Z(L) = \int_L \Omega$$

where $\Omega \in H^{3,0}(X)$ is a holomorphic 3-form.

Stability conditions

From now on $\mathcal{D}$ denotes a triangulated category. At some point we may wish to assume that $\mathcal{D}$ satisfies some extra conditions, to ensure that it is a topological twist of a SCFT. For example we could take $\mathcal{D} = \mathcal{D}^b \operatorname{Coh}(X)$ for X a smooth projective Calabi-Yau.

The aim is to axiomatise the properties of the subcategories $\mathcal{P} \subset \mathcal{D}$ of BPS branes and the map Z, and to obtain the corresponding leaf $\mathcal{L} \subset \mathcal{M}$ as the space of all possible choices of such data.

Definition 1 *A stability condition on $\mathcal{D}$ consists of a full additive subcategory $\mathcal{P}(\phi) \subset \mathcal{D}$ for each $\phi \in \mathbb{R}$, and a group homomorphism $Z\colon K(\mathcal{D}) \to \mathbb{C}$, such that*

(a) if $E \in \mathcal{P}(\phi)$ then $Z(E) \in \mathbb{R}_{>0}\,\mathrm{exp}(i\pi\phi)$,

(b) $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$ for all $\phi \in \mathbb{R}$,

(c) if $\phi_1 > \phi_2$ and $A_j \in \mathcal{P}(\phi_j)$ then

$$\mathrm{Hom}_{\mathcal{D}}(A_1, A_2) = 0,$$

(d) for each $0 \neq E \in \mathcal{D}$ there is a finite sequence of real numbers

$$\phi_1 > \phi_2 > \cdots > \phi_n$$

and a collection of triangles

$$\begin{tikzcd}
0 = E_0 \arrow[rr] & & E_1 \arrow[dl] \arrow[r] & \cdots \arrow[r] & E_{n-1} \arrow[rr] & & E_n = E \arrow[dl] \\
& A_1 \arrow[ul, dashed] & & & & A_n \arrow[ul, dashed] &
\end{tikzcd}$$

with $A_j \in \mathcal{P}(\phi_j)$ for all j.

Given a stability condition $\sigma = (Z, \mathcal{P})$ on $\mathcal{D}$ and an object $0 \neq E \in \mathcal{D}$, the filtrations of axiom (d) are unique up to isomorphism. Thus we can define

$$\phi^+_\sigma(E) = \phi_1, \qquad \phi^-_\sigma(E) = \phi_n,$$

$$m_\sigma(E) = \sum_{i=1}^{n} |Z(A_i)| \in \mathbb{R}_{>0}.$$

The expression

$$\sup_{0 \neq E \in \mathcal{D}} \left\{ |\phi^\pm_\sigma(E) - \phi^\pm_\tau(E)|, |\log \frac{m_\sigma(E)}{m_\tau(E)}| \right\}$$

defines a metric $d(\sigma, \tau) \in [0, \infty]$ on the set of all stability conditions on $\mathcal{D}$.

Write $\text{Stab}(\mathcal{D})$ for the set of "locally-finite" stability conditions on $\mathcal{D}$ with the topology induced by this metric. There is a continuous map

$$\mathcal{Z} \colon \text{Stab}(\mathcal{D}) \longrightarrow \text{Hom}_{\mathbb{Z}}(K(\mathcal{D}), \mathbb{C})$$

sending a stability condition $\sigma = (Z, \mathcal{P})$ to its central charge Z.

Theorem 1 *For each connected component* $\Sigma \subset \mathsf{Stab}(\mathcal{D})$ *there is a linear subspace*

$$V(\Sigma) \subset \mathsf{Hom}_{\mathbb{Z}}(K(\mathcal{D}), \mathbb{C})$$

with a well-defined linear topology such that the map $\mathcal{Z}$ *induces a local homeomorphism* $\mathcal{Z} \colon \Sigma \to V(\Sigma)$ *onto an open subset.*

It follows that $\mathsf{Stab}(\mathcal{D})$ is a (possibly infinite-dimensional) complex manifold.

If X is a smooth projective complex variety, set

$$\mathcal{D} = \mathcal{D}^b \,\mathsf{Coh}(X).$$

Let $\mathsf{Stab}(X)$ be the subset of $\mathsf{Stab}(\mathcal{D})$ for which $Z \colon K(\mathcal{D}) \to \mathbb{C}$ factors via the Chern character

$$\mathsf{ch} \colon K(\mathcal{D}) \longrightarrow H^*(X, \mathbb{Q}).$$

Then $\mathsf{Stab}(X)$ is a finite-dimensional complex manifold.

Example 1 *Suppose X is an elliptic curve. Then*

$$\mathsf{Stab}(X) = \mathbb{C} \times \mathcal{H}$$

where $\mathcal{H}$ is the upper half-plane. Note that

$$\mathsf{Aut}(D) = \mathsf{Aut}(X) \times \mathsf{Pic}^0(X) \times \mathsf{SL}(2, \mathbb{Z}) \times \mathbb{Z}.$$

and hence the quotient

$$\mathsf{Stab}(X) / \mathsf{Aut}(\mathcal{D})$$

is a $\mathbb{C}^*$-bundle over the modular curve

$$\mathcal{H} \,/\, \mathsf{PSL}(2, \mathbb{Z}).$$

From categories to geometry

According to Kontsevich homological mirror symmetry for a pair $(X, \check{X})$ is an equivalence

$$\mathcal{D}^b\,\mathsf{Coh}(X) \cong \mathcal{D}^b\,\mathsf{Fuk}(\check{X}).$$

More traditionally mirror symmetry is supposed to identify an open subset of the stringy Kähler moduli space

$$\{\beta + i\omega \in H^2(X,\mathbb{C})/H^2(X,\mathbb{Z}) : \omega \gg 0 \text{ Kähler}\}$$

equipped with a VHS coming from Gromov-Witten invariants, with an open subset of the complex moduli space $\mathcal{M}_{\mathbb{C}}(\check{X})$ equipped with the VHS coming from Hodge theory.

To deduce the second statement from the first we need to know how to get from categories to spaces.

Barannikov/Kontsevich: Associate to $\mathcal{D}$ its moduli space of deformations as an A^∞ category.

$$\mathcal{D} \longrightarrow \mathsf{Def}_{A^\infty}(\mathcal{D}).$$

This as a formal germ of a manifold, naturally equipped with a semi-infinite VHS. In the case $\mathcal{D} = \mathcal{D}^b\,\mathsf{Coh}(X)$ this is an extended version of $\mathcal{M}_{\mathbb{C}}(X)$. The semi-infinite VHS allows one to pick out the submanifold

$$\mathcal{M}_{\mathbb{C}}(X) \subset \mathsf{Def}_{A^\infty}(\mathcal{D})$$

Alternatively, one can associate to $\mathcal{D}$ its space of stability conditions

$$\mathcal{D} \longrightarrow \mathsf{Stab}(\mathcal{D}).$$

This is a global complex manifold. Mirror symmetry leads one to expect that it should also have a semi-infinite VHS, which would allow one to pick out a submanifold

$$\mathcal{M}_{\mathcal{K}}(X) \subset \mathsf{Stab}(\mathcal{D}).$$

Furthermore, for a mirror pair, $(X, \check{X})$ one would expect the corresponding categories

$$\mathcal{D} = \mathcal{D}^b\,\mathsf{Coh}(X) \text{ and } \check{D} = \mathcal{D}^b\,\mathsf{Coh}(\check{X})$$

to satisfy

$$\mathsf{Stab}(\mathcal{D}) \cong \mathsf{Def}_{A^\infty}(\check{D}).$$

Example 1 : A surface singularity

Consider the family of hypersurfaces

$$f_s(x,y,z) = x^2 + y^2 + \prod_{i=0}^{n} (z - \alpha_i) = 0$$

in $\mathbb{C}^3$ parameterized by the points of

$$S = \{(\alpha_0, \cdots, \alpha_n) \in \mathbb{C}^{n+1} : \sum_{i=0}^{n} \alpha_i = 0\} / \, \mathrm{Sym}_{n+1} \, .$$

This family of surfaces is the universal unfolding of the A_n singularity $x^2 + y^2 + z^{n+1} = 0$.

The surface X_s is smooth unless s lies on the discriminant

$$\Delta = \{(\alpha_0, \cdots, \alpha_n) \in S : \alpha_i \text{ not all distinct}\}.$$

Each smooth surface X_s has a natural Kähler form restricted from $\mathbb{C}^3$ and a non-vanishing holomorphic two-form Ω_s obtained by taking the Poincaré residue of the form

$$\frac{dx \wedge dy \wedge dz}{f_s(x,y,z)}.$$

Take $\mathcal{D}$ to be the subcategory

$$\mathcal{D} \subset \mathcal{D}^b \,\mathsf{Fuk}(X_s)$$

generated by the vanishing cycles. This is independent of $s \in S \setminus \Delta$ since all the smooth surfaces X_s are isomorphic as symplectic manifolds.

Theorem 2 (R.P. Thomas) *There is a connected component of* $\mathsf{Stab}(\mathcal{D})$ *which is isomorphic to the universal cover of* $S \setminus \Delta$*.*

The tangent space to $\mathrm{Stab}(\mathcal{D})$ at a given point $\sigma = (Z, \mathcal{P})$ is just

$$V = \mathrm{Hom}_{\mathbb{Z}}(K(\mathcal{D}), \mathbb{C}).$$

The Euler form on $K(\mathcal{D})$ induces a form $(-,-)$ on V. This is given by

$$(\theta_1, \theta_2) = \sum_{L \in K(\mathcal{D})} \theta_1(L)\theta_2(L)$$

Here the sum is taken over the classes $L \in K(\mathcal{D})$ such that $\chi(L, L) = 2$, i.e. those which are represented by spheres.

We can also define triple-point functions on V by

$$(\theta_1, \theta_2, \theta_3) = \sum_{L \in K(\mathcal{D})} \frac{\theta_1(L)\theta_2(L)\theta_3(L)}{Z(L)}$$

and hence a product $V \otimes V \to V$ satisfying

$$(\theta_1 \circ \theta_2, \theta_3) = (\theta_1, \theta_2, \theta_3) = (\theta_1, \theta_2 \circ \theta_3).$$

This makes V into a Frobenius algebra with identity Z. The resulting structure on Stab($\mathcal{D}$) is not quite a Frobenius manifold (the identity is not flat). It is the "almost-dual" Frobenius manifold (in the sense of Dubrovin) to the Frobenius structure on the unfolding space written down by Kyoji Saito.

Saito's construction works for any isolated hypersurface singularity. Does the above picture generalise? Of course, for non-simple singularities there will be infinitely many roots, so convergence becomes a problem.

Example 2 : A non-compact CY threefold

Let $X = \mathcal{O}_{\mathbb{P}^2}(-3)$ be the total space of the canonical bundle of $\mathbb{P}^2$.

The McKay correspondence shows that

$$\mathcal{D}^b \operatorname{Coh}(X) \cong \mathcal{D}^b \operatorname{Coh}_{\mathbb{Z}_3}(\mathbb{C}^3)$$

where $\mathbb{Z}_3$ acts on $\mathbb{C}^3$ with weights $(1,1,1)$.

The abelian category $\operatorname{Coh}_{\mathbb{Z}_3}(\mathbb{C}^3)$ is equivalent to the category of representations of a quiver with relations of the form

$$\begin{tikzcd} \bullet \arrow[rr, "3"] & & \bullet \arrow[dl, "3"] \\ & \bullet \arrow[ul, "3"] & \end{tikzcd}$$

Let $\mathcal{D} \subset \mathcal{D}^b \operatorname{Coh}(X)$ be the full subcategory of objects supported on the zero-section $\mathbb{P}^2 \subset X$. Under the above equivalence these objects correspond to equivariant sheaves supported at the origin and hence to nilpotent representations of the quiver.

Theorem 3 *There is a connected open subset* $\mathsf{Stab}_0(X) \subset \mathsf{Stab}(X)$ *which as a set is a disjoint union of regions*

$$\mathsf{Stab}_0(X) = \bigsqcup_{g \in G} D(g),$$

where G *is the affine braid group with presentation*

$$G = \left\langle \tau_0, \tau_1, \tau_2 \mid \tau_i \tau_j \tau_i = \tau_j \tau_i \tau_j \text{ for all } i, j \right\rangle.$$

The stability conditions in a given region $D(g)$ *all have the same heart* $\mathcal{A}(g) \subset \mathcal{D}$.

Each region $D(g)$ *is mapped isomorphically by* $\mathcal{Z}$ *onto a locally-closed subset of*

$$\mathsf{Hom}_{\mathbb{Z}}(K(\mathcal{D}), \mathbb{C}) \cong \mathbb{C}^3,$$

and the closures of two regions $D(g_1)$ *and* $D(g_2)$ *intersect in* $\mathsf{Stab}_0(X)$ *precisely if* $g_1 g_2^{-1} = \tau_i^{\pm 1}$ *for some* $i \in \{0, 1, 2\}$.

The abelian subcategories $\mathcal{A}(g) \subset \mathcal{D}$ are all distinct, and each is equivalent to a category of nilpotent representations of a quiver with relations of the form

$$\begin{tikzcd} \bullet \arrow[rr, "a"] & & \bullet \arrow[dl, "b"] \\ & \bullet \arrow[ul, "c"] & \end{tikzcd}$$

with $a^2 + b^2 + c^2 = abc$.

For each $g \in G$ let $S_0(g), S_1(g), S_2(g)$ be the three simple objects of $\mathcal{A}(g)$. These are spherical objects. The associated Seidel-Thomas twist functors induce pseudo-reflections

$$\phi_{S_0(g)}, \phi_{S_1(g)}, \phi_{S_2(g)} \in \operatorname{Aut} K(\mathcal{D}),$$

which with respect to the fixed basis of $K(\mathcal{D})$ defined by the classes of the objects $S_i = S_i(e)$ are given by a triple of matrices

$$M_0(g), M_1(g), M_2(g) \in \mathrm{SL}(3, \mathbb{Z}).$$

The same system of matrices come up in the study of the quantum cohomology of $\mathbb{P}^2$.

Dubrovin showed how the quantum cohomology of $\mathbb{P}^2$ can be analytically continued to a semisimple Frobenius structure on a dense open subset M of the universal cover of

$$\{(u_0, u_1, u_2) \in \mathbb{C} : i \neq j \implies u_i \neq u_j\}.$$

The Frobenius structure defines a flat connection ∇ (the second structure connection) on $p^*\mathcal{T}_M$ where

$$W = \{(m, z) \in M \times \mathbb{C} : z \neq u_i(m)\}$$

and $p\colon W \to M$ is the projection.

At each point $m \in M$ the restriction ∇_m is a meromorphic connection on a rank 3 trivial bundle on $\mathbb{P}^1$ with simple poles at $u_i(m)$ and at ∞. These connections vary isomonodromically.

Fix a point $m \in M$ such that $u_i(m)$ are the three cube roots of unity. Choose also a basis of flat sections of ∇_m near $0 \in \mathbb{P}^1$.

Note that G is a subgroup of the fundamental group of

$$\{(u_0, u_2, u_2) \in \mathbb{C}^* : i \neq j \implies u_i \neq u_j\} / \,\mathsf{Sym}_3\,.$$

So for each $g \in G$ we get a point $g(m) \in M$ such that the $u_i(g(m))$ are the cube roots of unity, and a basis of flat sections of $\nabla_{g(m)}$ near $0 \in \mathbb{P}^1$.

Taking monodromy about loops encircling the roots of unity gives the same matrices

$$M_0(g), M_1(g), M_2(g) \in \mathsf{SL}(3, \mathbb{Z}).$$