London Mathematical Society
Durham Symposium

Geometry, Conformal Field Theory, and String Theory

22nd July – 1st August 2005

Separation of variables, Q operator, and Bäcklund transformation

E. K. Sklyanin (York) and V. B. Kuznetsov (Leeds)

Integrability: variety of approaches

Classical (Hamiltonian) mechanics

- Separation of variables in the Hamilton-Jacobi equation
- Hidden symmetries (Kepler problem etc.)
- Lax matrix method (inverse scattering etc.)

Quantum mechanics

- Separation of variables
- Hidden symmetries (quantum groups etc.)
- Bethe Ansatz (coordinate, algebraic, analytic etc.)
- Baxter's Q-operator

There are good reasons to believe that a **generalized Separation of Variables** can be a base for unification of all the above approaches.

1. Separation of variables in QM

1.1 Example: Cartesian coordinates

$$\Delta = H_1 + H_2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$$

$$[H_1, H_2] = 0, \qquad H_i \equiv \frac{\partial^2}{\partial x_i^2}$$

$$\Delta\Phi = h\Phi$$

$$\Phi(x_1, x_2) = \varphi_1(x_1)\varphi_2(x_2)$$

$$H_1\varphi_1 = h_1\varphi_1, \qquad H_2\varphi_2 = h_2\varphi_2,$$

$$h = h_1 + h_2$$

1.2 SoV example: parabolic coordinates

$$\begin{cases} x_1 &= \dfrac{z_1^2 - z_2^2}{2} \\ x_2 &= z_1 z_2 \end{cases}$$

$$\begin{aligned} H_1 &\equiv \Delta = \tfrac{\partial^2}{\partial x_1^2} + \tfrac{\partial^2}{\partial x_2^2} &&= \tfrac{1}{z_1^2+z_2^2}\left(\tfrac{\partial^2}{\partial z_1^2} + \tfrac{\partial^2}{\partial z_2^2}\right) \\ H_2 &\equiv -2x_1 \tfrac{\partial^2}{\partial x_2^2} + 2x_2 \tfrac{\partial^2}{\partial x_1 \partial x_2} + \tfrac{\partial}{\partial x_1} &&= \tfrac{1}{z_1^2+z_2^2}\left(z_2^2 \tfrac{\partial^2}{\partial z_1^2} - z_1^2 \tfrac{\partial^2}{\partial z_2^2}\right) \end{aligned}$$

$$[H_1, H_2] = 0 \qquad H_1\Phi = h_1\Phi, \qquad H_2\Phi = h_2\Phi$$

$$\Phi(z_1, z_2) = \varphi_1(z_1)\varphi_2(z_2)$$

$$\begin{cases} \frac{\partial^2}{\partial z_1^2} - z_1^2 H_1 - H_2 = 0 \\ \frac{\partial^2}{\partial z_2^2} - z_2^2 H_1 + H_2 = 0 \end{cases}$$

$$\begin{cases} \varphi_1'' - (h_1 z_1^2 + h_2)\varphi_1 = 0 \\ \varphi_2'' - (h_1 z_2^2 - h_2)\varphi_2 = 0 \end{cases}$$

1.3 SoV: general setting

$$\Psi = \Psi(\mathbf{x}) \equiv \Psi(x_1, x_2, \ldots, x_n).$$

Commuting Hamiltonians H_j, $j = 1, \ldots, n$

$$[H_j, H_k] = 0$$

$$H_i \Psi_\lambda = h_\lambda^{(i)} \Psi_\lambda, \qquad \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$$

Separating transformation $\mathcal{S}$

$$\mathcal{S} : \Psi_\lambda(\mathbf{x}) \mapsto \Phi_\lambda(\mathbf{z}) = \varphi_\lambda^{(1)}(z_1) \ldots \varphi_\lambda^{(n)}(z_n)$$

$$\mathcal{S} : \Psi(\mathbf{x}) \mapsto \Phi(\mathbf{z}) = \int d\mathbf{x}\, S(\mathbf{z}|\mathbf{x}) \Psi(\mathbf{x})$$

$$W_j \left(z_j, \frac{\partial}{\partial z_j}, H_1, \ldots, H_n \right) = 0$$

$$W_j \left(z_j, \frac{d}{dz_j}, h_\lambda^{(1)}, \ldots, h_\lambda^{(n)} \right) \varphi_\lambda^{(j)}(z_j) = 0$$

1.4 SoV in Hamiltonian mechanics

$$\{p_j^{(z)}, z_k\} = \delta_{jk} \qquad \{H_j, H_k\} = 0$$

$$\Phi(z_1, \ldots, z_n) \sim \exp\left(\frac{i}{\hbar} F(z_1, \ldots, z_n)\right), \qquad \hbar \to 0$$

$$\varphi^{(j)}(z_j) \sim \exp\left(\frac{i}{\hbar} f^{(j)}(z_j)\right), \qquad \hbar \to 0$$

$$\Phi(\mathbf{z}) = \prod_{j=1}^{n} \varphi^{(j)}(z_j) \quad \Rightarrow \quad F(\mathbf{z}) = \sum_{j=1}^{n} f^{(j)}(z_j)$$

$$W_j(z_j, p_j, H_1, \ldots, H_n) = 0$$

$$W_j\left(z_j, \frac{df^{(j)}}{dz_j}, H_1, \ldots, H_n\right) = 0$$

Canonical transformation

$$S(\mathbf{z}|\mathbf{x}) \sim \exp\left(-\frac{i}{\hbar} s(\mathbf{z}|\mathbf{x})\right), \qquad \hbar \to 0$$

$$p_j^{(x)} = \frac{\partial s}{\partial x_j}, \qquad p_j^{(z)} = -\frac{\partial s}{\partial z_j}$$

1.5 Algebro-geometric approach

Lax matrix: $L(u, x, p^{(x)})$.

Characteristic polynomial:

$$w(v, u) \equiv \det\big(v - L(u)\big) \supset \{H_j\}$$

Recipe: choose z_j as the poles of the eigenvector $b(u)$ of $L(u)$

$$L(u)b(u) = v(u)b(u)$$

(in an appropriate normalization). Then choose $p_j^{(z)}$ as the corresponding eigenvalues

$$p_j^{(z)} = v(z_j)$$

Spectral curve $w(v, u) = 0$:

$$w(p_j^{(z)}, z_j) = \det\big(p_j^{(z)} - L(z_j)\big) = 0$$

2. Q-operator

Idea: *factorize one variable z after another.*

2.1 Definition of Q

$$\begin{aligned}
[Q_{z_1}, Q_{z_2}] &= 0 \qquad \forall z_1, z_2 \in \mathbb{C}, \\
[Q_z, H_i] &= 0 \qquad \forall z \in \mathbb{C}, \quad \forall i = 1, \dots, n,
\end{aligned}$$

$$[Q_z \Psi_\lambda](\mathbf{x}) = q_\lambda(z) \Psi_\lambda(\mathbf{x}).$$

$$H_j \Psi \lambda = h_\lambda^{(j)} \Psi_\lambda$$

$$W\left(z, \frac{d}{dz}; H_1, \dots, H_n\right) Q_z = 0,$$

$$W\left(z, \frac{d}{dz}; h_\lambda^{(1)}, \dots, h_\lambda^{(n)}\right) q_\lambda(z) = 0,$$

2.2 Separation of variables from Q

$$Q_{\mathbf{z}} = Q_{z_1} \cdots Q_{z_n}$$

$$Q_{\mathbf{z}}(\mathbf{y}|\mathbf{x}) = \int d\mathbf{t_1} \ldots \int d\mathbf{t_{n-1}}\, Q_{z_1}(\mathbf{y}|\mathbf{t_1}) Q_{z_2}(\mathbf{t_1}|\mathbf{t_2}) \ldots Q_{z_n}(\mathbf{t_{n-1}}|\mathbf{x}),$$

$$Q_{\mathbf{z}} \Psi_\lambda = q_\lambda(z_1) \ldots q_\lambda(z_n) \Psi_\lambda$$

$$\rho : \Psi(\mathbf{y}) \mapsto \int \rho(\mathbf{y}) \Psi(\mathbf{y})\, d\mathbf{y} \qquad \mathcal{S}_n^{(\rho)}(\mathbf{z}|\mathbf{x}) = \int d\mathbf{y}\, \rho(\mathbf{y}) Q_{\mathbf{z}}(\mathbf{y}|\mathbf{x}),$$

$$\mathcal{S}_n^{(\rho)} : \Psi_\lambda(x_1, \ldots, x_n) \mapsto \boldsymbol{c_\lambda} \prod_{i=1}^{n} \varphi_\lambda^{(i)}(z_i)$$

$$\varphi_\lambda^{(i)}(z_i) = q_\lambda(z_i) \qquad \text{and} \qquad c_\lambda = \int d\mathbf{y}\, \rho(\mathbf{y}) \Psi_\lambda(\mathbf{y}).$$

$$\mathcal{S}_n^{(\rho)} \Psi_\lambda = \rho Q_{z_1} \ldots Q_{z_n} \Psi_\lambda = (\rho \Psi_\lambda) q_\lambda(z_1) \ldots q_\lambda(z_n)$$

2.3 Example: symmetric monomials

Let $\mathbf{x}^{\mathbf{a}} \equiv x_1^{a_1} \dots x_n^{a_n}$ for any $\mathbf{a} = (a_1, \dots, a_n)$.

Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$: $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$.

The *monomial symmetric functions* $m_\lambda(\mathbf{x})$:

$$m_\lambda(\mathbf{x}) = x_1^{\lambda_1} x_2^{\lambda_2} \dots x_n^{\lambda_n} + \text{permuted terms}$$

form a basis in $\mathbb{C}[\mathbf{x}]^{S_n}$

Let $\mathbf{1} = (1, \dots, 1)$. Normalization: $\bar{m}_\lambda(\mathbf{x}) \equiv m_\lambda(\mathbf{x})/m_\lambda(\mathbf{1})$

$$\bar{m}_\lambda(\mathbf{x}) = \frac{1}{n!} \sum_{\sigma \in S_n} \mathbf{x}^{\sigma(\lambda)}, \qquad \bar{m}_\lambda(\mathbf{1}) = 1,$$

Commuting Hamiltonians:

$$D_j = x_j \frac{\partial}{\partial x_j}, \qquad D_j \mathbf{x}^{\mathbf{a}} = a_j \mathbf{x}^{\mathbf{a}}, \qquad j = 1, \dots, n.$$

$$H_j = e_j(D_1, \dots, D_n), \qquad H_j \bar{m}_\lambda = e_j(\lambda) \bar{m}_\lambda,$$

$$\sum_{j=0}^{n} e_j(\mathbf{x}) t^j = \prod_{i=1}^{n} (1 + t x_j)$$

Define Q_z as

$$Q_z \bar{m}_\lambda = q_\lambda(z) \bar{m}_\lambda, \qquad q_\lambda(z) = \bar{m}_\lambda(z, 1, \ldots, 1) = \frac{1}{n} \sum_{j=1}^{n} z^{\lambda_j}.$$

or, equivalently,

$$Q_z = \frac{1}{n} \sum_{j=1}^{n} \ z^{D_j},$$

$$(Q_z f)(\mathbf{x}) = \frac{1}{n} \sum_{j=1}^{n} f(x_1, \ldots, x_{j-1}, z x_j, x_{j+1}, \ldots, x_n).$$

$$\rho : f(\mathbf{x}) \mapsto f(1, \ldots, 1) \equiv f(\mathbf{1})$$

$$\mathcal{S}_n^{(\rho)} = \rho Q_{z_1} \ldots Q_{z_n}$$

$$\mathcal{S}_n^{(\rho)} : \bar{m}_\lambda \mapsto \prod_{j=1}^{n} q_\lambda(z_j)$$

2.4 Bäcklund transformation

$$Q_z : \Psi(\mathbf{x}) \mapsto \int d\mathbf{x}\, \mathcal{Q}(\mathbf{y}|\mathbf{x})\Psi(\mathbf{x})$$

$$\mathcal{Q}_z(\mathbf{y}|\mathbf{x}) \sim \exp\left(-\frac{i}{\hbar}\mathcal{F}_z(\mathbf{y}|\mathbf{x})\right)$$

Canonical transformation $\mathbb{B}_z$:

$$p_j^{(x)} = \frac{\partial \mathcal{F}_z}{\partial x_j}, \qquad p_j^{(y)} = -\frac{\partial \mathcal{F}_z}{\partial y_j}.$$

Commutativity: $\mathbb{B}_{z_1} \circ \mathbb{B}_{z_2} = \mathbb{B}_{z_2} \circ \mathbb{B}_{z_2}$
Preserving Hamiltonians: $\mathbb{B}_z : H^{(x)} \mapsto H^{(y)}$.
Spectrality: let $p^{(z)} = -\partial\mathcal{F}_z/\partial z$. Then

$$W(p^{(z)}, z; H_1, \ldots, H_n) = 0.$$

2.5 Example: Calogero-Moser system

Classical case:

$$L(u) = \begin{pmatrix} p_1 & igf_{12} & \dots & igf_{1n} \\ igf_{21} & p_2 & \dots & igf_{2n} \\ \dots & \dots & \dots & \dots \\ igf_{n1} & igf_{n2} & \dots & p_n \end{pmatrix} \qquad f_{jk} \equiv \cot u + \cot(x_j - x_k)$$

Hamiltonians:

$$\det\bigl(v - L(u)\bigr) \supset \{H_j\}$$

$$H_1 = p_1 + \ldots + p_n$$

$$H_2 = \sum_{j=1}^{n} \frac{p_j^2}{2} + \sum_{j<k} \frac{g^2}{\sin^2(x_j - x_k)}$$

Quantum case:

$$H_1 = -i\frac{\partial}{\partial x_1} - \ldots - i\frac{\partial}{\partial x_n}$$

$$H_2 = \sum_{j=1}^{n} -\frac{\partial^2}{\partial x_j^2} + \sum_{j<k} \frac{g(g\mathbf{-1})}{\sin^2(x_j - x_k)}$$

Eigenfunctions are parametrized by partitions $\lambda = (\lambda_1, \ldots, \lambda_n)$:

$$\Psi_\lambda(x_1, \ldots, x_n) = \Omega(\mathbf{x}) P_\lambda^{(1/g)}(e^{2ix_1}, \ldots, e^{2ix_n})$$

$$\Omega(\mathbf{x}) = \left| \prod_{j<k} \sin(x_j - x_k) \right|^g$$

and expressed in terms of Jack polynomials $P_\lambda^{(1/g)}$ (symmetric polynomials of n variables).

Case $g = 0$ corresponds to monomial symmetric functions m_λ.

Bäcklund transformation

$$\mathcal{F}_z(\mathbf{y}|\mathbf{x}) = z\sum_{j=1}^{n}(x_j - y_j) + \sum_{j,k} ig \ln\sin|x_j - y_k| - \sum_{j<k} ig \ln\sin|x_j - x_k| - \sum_{j<k} ig \ln\sin|y_j - y_k|$$

Quantization:

$$\mathcal{Q}_z(\mathbf{y}|\mathbf{x}) \sim \exp\left(-\frac{i}{\hbar}\mathcal{F}_z(\mathbf{y}|\mathbf{x})\right)$$

$$Q_z(\mathbf{y}|\mathbf{x}) = \prod_{j=1}^{n} e^{-iz(x_j-y_j)} \prod_{j,k} \sin^{g\mathbf{-1}}|x_j - y_k| \times \left(\prod_{j<k} \sin|x_j - x_k| \sin|y_j - y_k|\right)^{-g\mathbf{+1}}$$

$$q_\lambda(z) = \frac{P_\lambda(z,1,\ldots,1)}{P_\lambda(1,\ldots,1)}$$

References

[1] V.B. Kuznetsov and E.S., *Factorization of symmetric polynomials;* ArXiv: http://arxiv.org/abs/math.CA/0501257

[2] V.B. Kuznetsov, V.V. Mangazeev and E.S., *Q-operator and factorized separation chain for Jack polynomials.* Indag. Mathem. *14*(3,4) (2003), pp. 451–482. ArXiv: http://arxiv.org/abs/math.CA/0306242

[3] E.S., *Bäcklund transformations and Baxter's Q-operator.* In: "Integrable systems: from classical to quantum" (Montreal, QC, 1999), 227–250, CRM Proc. Lecture Notes 26, Amer. Math. Soc., Providence, RI, 2000; ArXiv: http://arxiv.org/abs/nlin.SI/0009009

[4] V.B. Kuznetsov and E.S., *On Bäcklund transformations for many-body systems,* J. Phys. A: Math. Gen. **31** (1998) 2241–2251; ArXiv: http://arxiv.org/abs/solv-int/9711010

[5] E.S., *Separation of variables. New trends.* Progress in Theoretical Physics Supplement **118** (1995) 35–60; ArXiv: http://arxiv.org/abs/solv-int/9504001