Constrained Willmore Tori in the 4–Sphere

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Constrained Willmore Surfaces

Definition
A conformal immersion \( f : M \rightarrow S^4 = \mathbb{R}^4 \cup \{\infty\} \) of a Riemann surface \( M \) is constrained Willmore if it is a critical point of the Willmore functional \( \mathcal{W} = \int |\hat{I}|^2 dA \) under \textit{conformal} variations.
(Willmore surfaces “=” critical pts. of \( \mathcal{W} \) under all variations.)

Functional and constraint are conformally invariant
\( \leadsto \) Möbius geometric treatment, e.g. in framework of quaternionic model of conformal 4–sphere \( S^4 = \mathbb{HP}^1 \)

Examples
- CMC in 3D space–forms \( \leadsto \) constrained Willmore
- Minimal in 4D space–forms \( \leadsto \) Willmore
- Hamiltonian Stationary Lagrangian in \( \mathbb{R}^4 \) \( \leadsto \) constr. Willmore
Prototype for Main Result: Harmonic Tori in $S^2$

Theorem

A harmonic map $f : T^2 \to S^2 = \mathbb{CP}^1$ is either

- holomorphic or
- of finite type.

More precisely:

If $\deg(f) \neq 0$, then $f$ is (anti–)holomorphic (Eells/Wood)
If $\deg(f) = 0$, then $f$ is of finite type (Pinkall/Sterling)

Finite type “=”

- attached to $f$ is a Riemann surface $\Sigma$ of finite genus called the spectral curve and
- the map $f$ is obtained by “algebraic geometric” or “finite gap” integration
The Main Result: Constrained Willmore Tori in $S^4$

**Theorem**

A constrained Willmore immersion $f : T^2 \to S^4 = \mathbb{HP}^1$ is either

- "holomorphic" (i.e., super–conformal or Euclidean minimal) or
- of finite type.

Where:

- super–conformal "=" $f$ is obtained by Twistor projection $\mathbb{CP}^3 \to \mathbb{HP}^1$ from holomorphic curve in $\mathbb{CP}^3$
- Euclidean minimal "=" there is a point $\infty \in S^4$ such that $f : T^2 \setminus \{p_1, ..., p_n\} \to \mathbb{R}^4 = S^4 \setminus \{\infty\}$ is an Euclidean minimal surface with planar ends $p_1, ..., p_n$. 

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Constrained Willmore Tori in the 4–Sphere
Holomorphic Case versus Finite Type Case

Theorem implies that all constrained Willmore tori admit explicit parametrization by methods of complex algebraic geometry.

Holomorphic case (e.g. twistor case):

\[
\begin{array}{cccc}
\mathbb{CP}^3 & \xrightarrow{\text{hol.}} & \mathbb{CP}^3 \\
\downarrow & & \downarrow^{\text{twistor}} \\
T^2 & \xrightarrow{f} & \mathbb{HP}^1
\end{array}
\]

Finite type case:

\[
\begin{array}{cccc}
\widehat{\text{Jac}}(\Sigma) & \xrightarrow{\text{hol.}} & \mathbb{CP}^3 \\
\uparrow^{\text{linear}} & & \downarrow^{\text{twistor}} \\
T^2 & \xrightarrow{f} & \mathbb{HP}^1
\end{array}
\]
Previous Results

- CMC tori are of finite type (Pinkall, Sterling; 1989)  
  \((\text{CMC} \iff \text{Gauss map } N: T^2 \to S^2 \text{ harmonic})\)

- 1.) Burstall, Ferus, Hitchin, Pedit, Pinkall, Sterling (≈ 90) 
  \(S^2\)-result generalizes to various symmetric target spaces

- 2.) Willmore \(\iff\) conformal Gauss map harmonic

- 1.) + 2.) \(\leadsto\) Conjecture: Willmore tori in \(S^3\) are of finite type

- Schmidt 2002: constrained Willmore in \(S^3\) are of finite type

- Willmore tori in \(S^4\) with non–trivial normal bundle are of “holomorphic” type (Leschke, Pedit, Pinkall; 2003)
Strategy: Adopt Hitchin Approach to Harmonic Tori in $S^2$

0.) Formulate as zero-curvature equation with spectral parameter $\nabla^\mu \mapsto$ associated family $\nabla^\mu$ of flat connections depending on spectral parameter $\mu \in \mathbb{C}^*$
   - Harmonic maps to $S^2 = \mathbb{CP}^1$: complex rank 2 bundle
   - constrained Willmore in $S^4 = \mathbb{HP}^1$: complex rank 4 bundle

1.) Which holonomy representations $H^\mu : \Gamma \to \text{SL}_2(\mathbb{C})$ or $\text{SL}_4(\mathbb{C})$ can occur for $\nabla^\mu$ if underlying surface is torus $T^2 = \mathbb{C}/\Gamma$?

2.) Non-trivial holonomy
   $\implies$ existence of polynomial Killing field
   $\implies$ finite type

3.) Trivial holonomy $\cong$ “holomorphic” case

Implementation of these ideas in constrained Willmore Case needs results from quaternionic holomorphic geometry.
The Quaternionic Model of Surface Theory in $S^4$

Immersion $f : M \rightarrow S^4 = \mathbb{HP}^1$ $\iff$ line subbundle $L \subset \mathbb{H}^2$

*Mean curvature sphere congruence* $\iff$
complex structure $S \in \Gamma(\text{End}(\mathbb{H}^2))$ with $S^2 = -\text{Id}$

2–sphere at $p \in M$ $\iff$ eigenlines of $S_p$ in $\mathbb{HP}^1$

$S$ induced decomposition of trivial connection $d$

$$d = \underbrace{\partial + \bar{\partial}}_{S \text{ commuting}} + \underbrace{A + Q}_{S \text{ anti–comm.}}$$

$\partial$ and $A$ are of type $K$, i.e., complex str. on $M$ acts by $\ast \omega = S \omega$

$\bar{\partial}$ and $Q$ are of type $\bar{K}$, i.e., complex str. on $M$ acts by $\ast \omega = -S \omega$
The Hopf Fields of a Conformal Immersion

A and \( Q \) are tensor fields called the *Hopf fields* of \( f \).

- the Hopf fields measure the local change of \( S \) along immersion
- Willmore functional measures “global change of \( S \)”

\[
\mathcal{W} = \int_M A \wedge \ast A = \int_M Q \wedge \ast Q
\]

- Euler–Lagrange Equation of constrained Willmore surfaces (for compact \( M \)) is

\[
d(2\ast A + \eta) = 0 \quad \text{for} \quad \eta \in \Omega^1(\text{End}(\mathbb{H}^2)), \quad \ker(\eta) = \text{im}(\eta) = L
\]

Lagrange–multiplier \( \eta \) “is” holomorphic quadratic differential
Willmore surface \( \longleftrightarrow \eta = 0 \)
The Associated Family of Constrained Willmore Surfaces

The associated family of a constrained Willmore immersion is the family of flat complex connections on the trivial complex rank 4 bundle $\mathbb{C}^4 = (\mathbb{H}^2, i)$

$$\nabla^\mu = d + (\mu - 1)A_{\circ}^{(1,0)} + (\mu^{-1} - 1)A_{\circ}^{(0,1)} \quad \mu \in \mathbb{C}^*$$

where

- $A_{\circ}$ is defined by $2 \ast A_{\circ} = 2 \ast A + \eta$ and where
- $(1, 0)$ and $(0, 1)$ denote the decomposition into forms satisfying $\ast \omega = \omega i$ and $\ast \omega = -\omega i$. 
Eigenlines of the Holonomy of $\nabla^\mu$ on Torus

Flat connections on torus $\leadsto$ study holonomy and its eigenlines

AIM: if possible, define eigenline spectral curve $\Sigma_{hol}$

“$\leadsto$” unique Riemann surface $\Sigma_{hol} \xrightarrow{\mu} \mathbb{C}^*$ parametrizing non–trivial eigenlines of $H^\mu(\gamma)$

Eigenvalue of holonomy $H^\mu(\gamma)$ for one $\gamma \in \Gamma$

$\Gamma$abelian $\leadsto$ simultaneous eigenline of $H^\mu(\gamma)$ for all $\gamma \in \Gamma$

$\Rightarrow$ section $\psi \in \Gamma(\tilde{\mathbb{H}}^2)$ on universal cover $\mathbb{C}$ of torus with

- $\nabla^\mu \psi = 0$ and
- $\gamma^* \psi = \psi h_\gamma$ for all $\gamma \in \Gamma$ and some $h \in \text{Hom}(\Gamma, \mathbb{C}^*)$

Be definition, such solution to $\nabla^\mu \psi = 0$ satisfies

$$d\psi = (1 - \mu) A_{\omega}^{(1,0)} + (1 - \mu^{-1}) A_{\omega}^{(0,1)} \in \Omega^1(\tilde{\mathbb{L}})$$
Link to Quaternionic Holomorphic Geometry

Immersion $f \rightsquigarrow$ quaternionic holomorphic structure on $\mathbb{H}^2/L$ (operator $D$ whose kernel contains projections of all $v \in \mathbb{H}^2$)

**Lemma**

*For every* $h \in \text{Hom}(\Gamma, \mathbb{C}^*)$, *there is 1–1-correspondence between*

- *holomorphic sections* $\varphi$ of $\mathbb{H}^2/L$ *with monodromy* $h$ *and*
- *sections* $\psi \in \Gamma(\tilde{\mathbb{H}}^2)$ *with*

$$
\begin{align*}
    d\psi &\in \Omega^1(\tilde{L}) & \text{and} & \gamma^*\psi = \psi h_\gamma & \text{for all} & \gamma \in \Gamma.
\end{align*}
$$

*The correspondence is given by* $\psi \mapsto \varphi = [\psi]$.

**Definition**

The section $\psi$ is called *prolongation* of the holomorphic section $\varphi$.

The map $L^\# := \psi\mathbb{H}$ is called a *Darboux transform* of $f$. 
Taimanov–Schmidt Spectral Curve of Degree 0 Tori

Definition

Taimanov–Schmidt spectral curve $\Sigma_{\text{mult}}$ of a conformally immersed torus $f$ in $\mathbb{S}^4 = \mathbb{HP}^1$ with trivial normal bundle is normalization of

$$\{ h \in \text{Hom}(\Gamma, \mathbb{C}^*) \mid \text{monodromy of holomorphic section of } \mathbb{H}^2/L \}$$

Theorem

*The set* $\{ h \in \ldots \}$ *is a 1–dimensional complex analytic subset of* $\text{Hom}(\Gamma, \mathbb{C}^*) \cong \mathbb{C}^* \times \mathbb{C}^*$. *Moreover, for generic* $h \in \Sigma_{\text{mult}}$, *the space of holomorphic sections is complex 1–dimensional.*

This implies that generic holonomies $H^\mu(\gamma)$ have

- an even number of simple eigenvalues that are non–constant as functions of $\mu$ (called *non–trivial eigenvalues*) and
- $\lambda = 1$ as an eigenvalue of even multiplicity (*trivial eigenvalue*).
The non–trivial Normal Bundle Case (Degree \(\neq 0\))

In the case of non–trivial normal bundle, the quaternionic Plücker formula implies that the only possible eigenvalue of the holonomies \(H^\mu(\gamma)\) is \(\lambda = 1\).
List of possible Types of Holonomy Representations

Lemma

For constrained Willmore tori in $S^4$, the holonomy $H^\mu(\gamma)$ of the associated family $\nabla^\mu$ belongs to one of the following cases:

I. generically $H^\mu(\gamma)$ has 4 different eigenvalues,

II. generically $H^\mu(\gamma)$ has $\lambda = 1$ as an eigenvalue of multiplicity 2 and 2 non–trivial eigenvalues,

IIIa. all holonomies $H^\mu(\gamma)$ are trivial, or

IIIb. all holonomies $H^\mu(\gamma)$ are of Jordan type with eigenvalue 1 (and have $2 \times 2$ Jordan blocks).

Non–trivial normal bundle $\leadsto$ holonomy belongs to Case III
Non–trivial Holonomy, Polynomial Killing Field (Case I)

Can define eigenline curve $\Sigma_{hol} \xrightarrow{\mu} \mathbb{C}^*$ of $\mu \mapsto H^\mu(\gamma)$

- $\Gamma$ abelian $\leadsto$ independent of choice of $\gamma \in \Gamma \setminus \{0\}$
- map $\Sigma_{hol} \to \Sigma_{mult}$ is (essentially) biholomorphomic

AIM: construct polynomial Killing field, i.e., a family of sections of $\text{End}_\mathbb{C}(\mathbb{H}^2, \mathbf{i})$ that is polynomial in $\mu$ and satisfies $\nabla^\mu \xi(\mu, .) = 0$ or, equivalently, a solution $\xi(\mu, p) = \sum_{j=0}^k \xi_j(p) \mu^j$ to Lax–equation

$$d\xi = [(1 - \mu)A_0^{(1,0)} + (1 - \mu^{-1})A_0^{(0,1)}], \xi].$$

Such $\xi$ commutes with all $H^\mu(\gamma)$

$\leadsto$ same eigenline curve

$\leadsto \Sigma_{hol}$ can be compactified by filling in points over $\mu = 0, \infty$
Hitchin Trick

The $(1,0)$ and $(0,1)$–parts of $\nabla^\mu$ extend to $\mathbb{C}$ and $\mathbb{C}^* \cup \{\infty\}$:

\[
\partial^\mu = (\nabla^\mu)^{(1,0)} = \partial + (\mu - 1)A_o^{(1,0)}
\]

\[
\bar{\partial}^\mu = (\nabla^\mu)^{(0,1)} = \bar{\partial} + (\mu^{-1} - 1)A_o^{(0,1)}
\]

Theorem

For a holomorphic family $F(\lambda), \lambda \in U \subset \mathbb{C}$ of Fredholm operators,

- the minimal kernel dimension of $F(\lambda), \lambda \in U$ is generic and
- the holomorphic bundle $\mathcal{V}_\lambda = \ker(F(\lambda))$ defined at generic points holomorphically extends through the isolated points of higher dimensional kernel.

Apply to $\partial^\mu$ and $\bar{\partial}^\mu$ on $\text{End}_\mathbb{C}(\mathbb{C}^4) = \text{End}_\mathbb{C}(\mathbb{H}^2, i) \rightsquigarrow \text{rank 4 bundle } \mathcal{V} \to \mathbb{CP}^1$ whose fiber $\mathcal{V}_\mu, \mu \in \mathbb{C}^*$ is $\{\nabla^\mu - \text{parallel sections}\}$ and whose meromorphic sections are polynomial Killing fields.
The Case of Trivial Holonomy

**Case IIIa:** apply Hitchin trick to $\partial^\mu$ and $\bar{\partial}^\mu$ on $\mathbb{C}^4 = (\mathbb{H}^2, \mathbf{i})$

$\leadsto$ rank 4 bundle $\mathcal{V} \rightarrow \mathbb{C}P^1$ whose fiber $\mathcal{V}_\mu$, $\mu \in \mathbb{C}^*$ is

$\{\nabla^\mu - \text{parallel sections of } \mathbb{C}^4\}$

Investigating the asymptotics of holomorphic sections of $\mathcal{V}$ at $\mu = 0$ or $\infty$ shows that Case IIIa is only possible if $f$ is super-conformal or Euclidean minimal.

**Case IIIb:** Hitchin trick $\leadsto$ existence of polynomial Killing field $\xi$ with $\xi^2 = 0$

Investigating the asymptotics of $\xi$ at $\mu = 0$ or $\infty$ shows that Case IIIb is only possible if $f$ is Euclidean minimal.
Theorem

Let \( f : T^2 \to S^4 \) be constrained Willmore. Then either

I. \( f \) is of finite type and \( \mu \) extends to covering \( \Sigma \xrightarrow{\mu} \mathbb{CP}^1 \) or

II. \( f \) is of finite type and \( \mu \) extends to covering \( \Sigma \xrightarrow{\mu} \mathbb{CP}^1 \) or

IIIa. all holonomies are trivial and \( f \) is super–conformal or an algebraic Euclidean minimal surface or

IIIb. all holonomies are of Jordan type and \( f \) is a non–algebraic Euclidean minimal surface.

Non–trivial normal bundle (\( \deg(\perp_f) \neq 0 \))

\( \leadsto \) “Holomorphic” Case IIIa or IIIb \( \leadsto \) Willmore

Trivial normal bundle (\( \deg(\perp_f) = 0 \)) and not Euclidean minimal

\( \leadsto \) Finite type Cases I or II
Willmore Case

- Let \( f : T^2 \rightarrow S^4 \) be Willmore and not Euclidean minimal. Then either
  - \( \deg(\perp_f) = 0 \) and \( f \) belongs to Case I (and is of finite type with \( \Sigma \xrightarrow{\mu} \mathbb{CP}^1 \)) or
  - \( \deg(\perp_f) \neq 0 \) and \( f \) belongs to Case IIIa (and is super–conformal).

- Euclidean minimal tori belong to Case IIIa or IIIb. In case that the normal bundle is trivial (as it is for minimal tori with planar ends in \( \mathbb{R}^3 = S^3 \setminus \{\infty\} \))
  - one cannot define \( \Sigma_{hol} \) using \( \nabla^\mu \),
  - but Taimanov–Schmidt spectral curve \( \Sigma_{mult} \) is well defined.

  Question: can \( \Sigma_{mult} \) be compactified? is \( \Sigma_{mult} \) reducible?

- Case II does not occur for Willmore tori (with \( \eta = 0 \)).
Tori with Harmonic Normal Vectors

**Theorem**

If a conformal immersion \( f : T^2 \to S^4 = \mathbb{HP}^1 \) has the property that, for some point \( \infty \in S^4 \), one factor of the Gauss map

\[
M \to Gr^+(2, 4) = S^2 \times S^2
\]

is harmonic, then \( f \) is constrained Willmore and belongs to

- **Case II** of the Main Theorem if the harmonic factor is not holomorphic and to
- **Case III** if the factor is holomorphic.

In Case III of the Main Theorem, there always exists \( \infty \in S^4 \) such that one factor of the Gauss map is holomorphic. In Case II, if \( \mathcal{W} < 8\pi \), there always exists \( \infty \in S^4 \) such that one factor of the Gauss map is harmonic.
Examples of Tori with Harmonic Normal Vectors

- CMC tori in $\mathbb{R}^3$
  ($H \neq 0$ case: Bobenko $\leadsto$ arbitrary genus)

- CMC tori in $S^3$
  (Bobenko $\leadsto$ arbitrary genus)

- Hamiltonian stationary tori
  (Helein, Romon $\leadsto$ harmonic map takes values in a great circle $\leadsto g = 0$)

- Lagrangian tori with conformal Maslov form
  (Castro, Urbano $\leadsto$ harmonic map is equivariant $\leadsto g \leq 1$)