TWO-TWISTOR DESCRIPTION OF TENSIONFUL STRINGS AND P-BRANES

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Our aim:

To introduce purely twistorial actions classically equivalent to the known space-time actions for standard strings and p-branes

Framework:

- only Minkowski metric and Minkowski twistors
- main tool: two-twistor geometry

  For Euclidean metric one can use single twistors

  $$T = CP(1) \text{ fibre over } S^4$$

  For Minkowski metric similar bundle construction requires $T \otimes T$
- in Sect. 1-4: $D=4$, no SUSY
- in Sect. 5: GS superstrings, $D=11$

We look for the interplay of three formulations:

A – "Conventional" space-time / phase space geometry

B – intermediate space-time / spinor geometry

C – purely twistorial geometry

Goal: to show that A $\simeq$ B $\simeq$ C

REFERENCES:

- particles $(m \neq 0,\ s \neq 0)$
  A. Bette, J. de Azcarraga, A. Frydryszak, S. Fedoruk, J.L. — hep-th/0405166, hep-th/0510266, hep-th/0510161
- strings
  S. Fedoruk, J.L. — hep-th/060624
- membranes, p-branes
  S. Fedoruk, J.L. — arXiv: 07062129

1. INTRODUCTION

a) Topological versus standard strings:

Topological (with $N=2$, $N=4$ SUSY):

- string field theory ⇒ finite multiplets of fields
- string oscillations absent from quantum-mechanical spectrum
- Classically topological B-model string similar to point-like structure (constant maps from world-sheet to target space)

Standard:

- string field theory ⇒ infinite higher spin fields multiplets, Regge towers
- classically string is an extended linear object
- classically appear dimensionfull parameters

Our aim: apply twistor framework to standard (super) strings, standard (super) p-branes.

b) Elements of twistor theory:

Primary geometry described by conformal $SU(2,2)$ spinors $\equiv$ twistors $Z_A \in T$ $(A = 1,2,3,4)$

T: $Z_A = (\pi_\alpha, \omega^{\dot\alpha})$
$(\bar Z_A = (\bar\pi_{\dot\alpha}, \bar\omega^{\alpha}))$

$$\langle Z, \bar Z \rangle = \pi_\alpha \bar\omega^\alpha + \omega^{\dot\alpha} \bar\pi_{\dot\alpha}$$

$\Uparrow$ $SU(2,2)$ norm – special choice

$T^n = T \otimes \cdots \otimes T$ $\quad Z_{A;i} = (\pi_{\alpha;i}, \omega^{\dot\alpha;}{}_i)$ $\quad (\pi_{\alpha;i})^* = \bar\pi_{\dot\alpha;}{}^i$, $(\omega^{\dot\alpha;}{}_i)^* = \bar\omega^{\alpha;i}$

Two basic formulae of twistor theory:

i) Incidence relation $T \longleftrightarrow CM^{3;1}$
twistors, complex Minkowski space

(Penrose 1962)
$$\bar\omega^{\alpha;i} = i Z^{\alpha\dot\beta} \bar\pi_{\dot\beta;}{}^i$$
$$\omega^{\dot\alpha;}{}_i = -i Z^{\dot\alpha\beta} \pi_{\beta;i}$$

$Z^{\alpha\dot\beta} \leftrightarrow Z_\mu = x_\mu + i y_\mu$
$Z^{\dot\alpha\beta} \leftrightarrow \bar Z_\mu = x_\mu - i y_\mu$

If $i = 1,2$ $(n=2)$:

$$Z^{\alpha\dot\beta} = \frac{i}{f} \bar\omega^{\alpha;i} \bar\pi^{\dot\beta;}{}_i \qquad f = \frac{1}{2} \bar\pi_{\dot\alpha;i} \bar\pi^{\dot\alpha;i}$$

(metric $a_i b^i = a_i \epsilon^{ij} b_j$)

ii) Composite formula for momenta:

$n=1$: $\quad P_{\alpha\dot\beta} = \pi_\alpha \bar\pi_{\dot\beta} \quad (P_{\alpha\dot\beta} P^{\alpha\dot\beta} \equiv 0 \Leftrightarrow P^2 = 0)$

$n>1$: $\quad P_{\alpha\dot\beta} = \pi_{\alpha;i} \bar\pi_{\dot\beta;}{}^i \quad (P^2 \geq 0)$

Canonical Liouville one-form (symplectic potential) on twistor space $\mathbb{T}^n$:

$$\Theta^{(1)} = \frac{i}{2}\sum_{i=1}^{n}(\bar{Z}^{A;i}\, dZ_{A;i} - d\bar{Z}^{A;i}\, Z_{A;i})$$

$$= \frac{i}{2}\sum_{i=1}^{n}(\bar{\omega}^{\alpha;i}\, d\pi_{\alpha;i} + \bar{\pi}_{\dot\alpha;}{}^{i}\, d\omega^{\dot\alpha}{}_{i} - \text{H.C.})$$

(corresponds to $\sum_{i=1}^{n} p_{\mu;i}\, dx^{\mu}{}_{;i}$)

Poisson brackets: (for $n=1$)

$$\{\pi_\alpha, \bar\omega^\beta\} = i\,\delta_\alpha{}^\beta \qquad \{\bar\pi_{\dot\alpha}, \omega^{\dot\beta}\} = -i\,\delta_{\dot\alpha}{}^{\dot\beta}$$

$\downarrow$ Quantization $\downarrow$

$$[\hat\pi_\alpha, \hat{\bar\omega}^\beta] = -\hbar\,\delta_\alpha{}^\beta \qquad [\hat{\bar\pi}_{\dot\alpha}, \hat\omega^{\dot\beta}] = \hbar\,\delta_{\dot\alpha}{}^{\dot\beta}$$

Two polarizations $\Rightarrow$ two different realizations $\Rightarrow$ two twistor quantizations

a) conformal-covariant quantization with complex holomorphic phase space (Penrose 1968)

coordinate space: $Z_A = (\pi_\alpha, \omega^{\dot\alpha})$
momentum space: $\bar{Z}_A = (\bar\pi_{\dot\alpha}, \bar\omega^{\alpha})$

b) quantization with real spinorial phase space (Woodhouse 1975)

coordinate space: $(\pi_\alpha, \bar\pi_{\dot\alpha})$
momentum space: $(\omega^{\dot\alpha}, \bar\omega^{\alpha})$

a) $\Rightarrow$ provides space-time fields $(Z_A = (\pi_\alpha, x^{\alpha\dot\beta}))$

b) $\Rightarrow$ provides fourmomentum space fields

Advantage of two-twistor space: one can map

$$(\pi_{\alpha;i}, \omega^{\dot\alpha;}{}_{i}, \bar\pi_{\dot\alpha;}{}^{i}, \bar\omega^{\alpha;i}) \longrightarrow (x_\mu, \ldots\ldots)$$

real space-time

12 additional degrees of freedom (mass, spin, electric charge...)

But surprise: x_μ defined by standard Penrose incidence relation do not commute:

$$\underline{\{x_\mu, x_\nu\} = -\frac{1}{M^2}\epsilon_{\mu\nu\rho\tau} W^\rho P^\tau}$$

calculated from twistor PB

where

$$P^\mu = (\sigma^\mu)_{\alpha\dot\beta}\,\pi^{\alpha;}{}_{i}\,\bar\pi^{\dot\beta;i} = P^{(1)}_\mu + P^{(2)}_\mu \quad \text{– 2-twistor momentum}$$

$$W^\mu = (\sigma^\mu)_{\alpha\dot\beta}\,\pi^{\alpha;}{}_{i}\,\bar\pi^{\dot\beta;j}\underbrace{(\sigma^r)^i{}_j\, t^r}_{\text{traceless part of } t^i{}_j = \langle z^i, z^j\rangle} \quad \text{– 2-twistor Pauli-Lubanski fourvector}$$

where

$$M^2 = P_\mu P^\mu = 2|f|^2 = \frac{1}{2}\left|\bar\pi_{\dot\alpha;i}\,\bar\pi^{\dot\alpha;i}\right|^2$$

One can calculate from twistor PB

$r, s = 1, 2, 3$

nonvanishing norms $\Rightarrow$ $\{t^r, t^s\} = \epsilon^{rsu} t^u$ SU(2) PB algebra (spin)

Fourth generator $t^0 = t^1{}_1 + t^1{}_2$ $\quad t^r = \frac{1}{2}(\sigma^r)^i{}_j\, t_i{}^j$

electric charge

$\{t^0, t^r\} = 0$ extension of SU(2) to U(2)

2. TWISTORIAL DESCRIPTION OF PARTICLES

a) Massless ⟺ one-twistor description

Three equivalent Liouville one-forms $\theta^{(1)}$:

$$\theta^{(1)} = P_{\alpha\dot\beta}\, dx^{\alpha\dot\beta} \simeq \pi_\alpha \bar\pi_{\dot\beta}\, dx^{\alpha\dot\beta} \simeq \frac{i}{2}\underbrace{(\bar\omega^\alpha d\pi_\alpha + \bar\pi_\alpha d\omega^\alpha)}_{\bar Z^A dZ_A}$$

$P_{\alpha\dot\beta}$ composite

incidence relation

We get three classically equivalent models of massless relativistic particles with $h = 0$:

helicity

(A) $S_1^{d=1} = \int d\tau\,(p_\mu \dot x^\mu + \lambda p_\mu p^\mu)$ ← relativistic phase space (x^μ, p^μ)

(B) $S_2^{d=1} = \int \pi_\alpha \bar\pi_{\dot\beta}\, \dot x^{\alpha\dot\beta}$ ← intermediate space-time/spinor description

(C) $S_3^{d=1} = \frac{i}{2}\int d\tau\,\{(\bar Z_A \dot Z^A - Z_A \dot{\bar Z}^A) + \lambda \langle Z, \bar Z\rangle\}$

free null twistor-particle model

If helicity $h \neq 0$:

$$\lambda\langle Z, \bar Z\rangle \Rightarrow \lambda(\langle Z, \bar Z\rangle - 2h)$$

$SU(2,2|1)$ Supertwistors: $h = \bar\xi\xi$ $\quad\xi$ fermionic

$SU(2,3)$ "bosonic" supertwistors: $h = \bar u u$ $\quad u$ bosonic

(S. Fedoruk + JL hep-th/0506086)

b) Two-twistor description of massive charged particles with spin

Our derivation: C $\longrightarrow$ B $\longrightarrow$ A

C $$\Theta^{(1)} = \frac{i}{2}\left(Z_{A;j}\, d\bar{Z}^{A;j} - \bar{Z}^{A;j}\, dZ_{A;j}\right) + \lambda_a R_a$$

$i,j = 1,2$
$a = 1 \cdots 4$

Free 2-twistor model with four physical constraints:

$$R_1 = 4|f|^2 - m^2 = 0 \quad \Leftarrow \text{mass}$$

$$\left.\begin{aligned} R_2 &= \vec{t}^{\,2} - s(s+1) = 0 \\ R_3 &= t_3 - m_3 = 0 \end{aligned}\right\} \Leftarrow \text{spin}$$

$$R_4 = t_0 - q = 0 \quad \Leftarrow \text{Abelian charge}$$

$s \neq 0,\ q \neq 0 \Rightarrow$ non-null twistors

We insert composite $z_\mu = x_\mu + i y_\mu$:

B $$\Theta^{(1)} = \pi_{\alpha;i}\,\bar{\pi}_{\dot\beta;}{}^{i}\, dx^{\alpha\dot\beta} + i y^{\alpha\dot\beta}\left(\pi_{\alpha;}{}^{i}\, d\bar{\pi}_{\dot\beta;i} - \bar{\pi}_{\dot\beta;i}\, d\pi_{\alpha;}{}^{i}\right) + \lambda_a R_a$$

where

$$y^{\alpha\dot\beta} = -\frac{1}{2|f|^2}\, t_i{}^{j}\, \pi^{\alpha;i}\, \bar{\pi}^{\dot\beta;}{}_{j}$$

imaginary part of CM^4

$\langle Z_i, \bar{Z}^j \rangle$

One gets one-form defining "intermediate" action:

(B) $$\Theta^{(1)} = \pi_{\alpha;i}\bar{\pi}_{\dot\beta;}{}^{i}\, dx^{\alpha\dot\beta} + \frac{i}{2} t_i{}^{j}\left(\frac{1}{f}\pi^{\beta;}{}_{j}\, d\bar{\pi}_{\dot\beta;}{}^{i} + \frac{1}{\bar f}\pi^{\beta;i}\, d\pi_{\beta;j}\right) + \lambda_a R_a$$

$\Uparrow$

$x^{\alpha\dot\beta}$ noncommutative!

Two-twistor generalization of Shirafuji model:

one twistor: $(x^{\alpha\dot\beta}, \pi_\alpha, \pi_{\dot\beta}) \Longrightarrow$ two twistors: $(x^{\alpha\dot\beta}, t_i{}^{j}, \pi_{\alpha;i}, \bar\pi_{\dot\beta;}{}^{i})$

Quantization:

Doubly infinite spin/charge multiplets:

$s = 0, \frac{1}{2}, 1 \cdots \quad ; \quad q = \cdots -1, 0, 1 \cdots$ (hep-th/0510266)

In order to obtain standard composite phase space (x_μ, p_μ) needed modification of Penrose incidence formula: (hep-th/0405166)

$$\bar\omega^{\alpha;i} = i\left(\tilde z^{\alpha\dot\beta}\bar\pi_{\dot\beta;}{}^{i} + \frac{t_1 - it_2}{\bar f}\pi^{\alpha;i}\right)$$

One gets $(\tilde z^{\alpha\dot\beta} = \tilde x^{\alpha\dot\beta} + i\tilde y^{\alpha\dot\beta})$

$$\tilde x^{\alpha\dot\beta} = x^{\alpha\dot\beta} - \frac{1}{2|f|^2}\epsilon_{3rs} t_r \pi^{\alpha;i}(\sigma_s)_i{}^{j}\bar\pi^{\dot\beta;}{}_{j}$$

Properties:

1) $\{\tilde x^{\alpha\dot\beta}, \tilde x^{\gamma\dot\delta}\} = 0 \quad \leftarrow$ commuting

2) "internal" $SU(2)$ broken to $O(2)$

Substituting $\tilde{X}^{\alpha\dot\beta}$ in $\Theta^{(1)}$ one gets

A $\quad \Theta^{(1)} = \underbrace{p_\mu dx^\mu}_{\text{standard phase space}} - i\underbrace{\left(\bar\sigma^{\dot\alpha;}{}_{1}\, d\bar\pi_{\dot\alpha;2} - \sigma^{\alpha;1} d\pi_{\alpha;}{}^{2}\right)}_{\text{spin sector}} + \underbrace{e\, d\varphi}_{\text{electric sector}} + \lambda_A R_A$

$A = 1, 2 \cdots 6$

Variables $\pi_\alpha, \bar\pi_{\dot\alpha}$ primary, remaining composite:

$$\sigma^{\alpha;i} = \frac{1}{\bar f}\, t_r (\sigma_r)^i{}_j\, \pi^{\alpha;j} \qquad \bar\sigma^{\dot\alpha;i} = -\frac{1}{f}\, t_r (\sigma_r)_i{}^j\, \bar\pi^{\dot\alpha}{}_{;j}$$

$$e = t_0 + t_3 \qquad \varphi = \frac{i}{2} \ln \frac{\bar f}{f}$$

We have 18 variables:

$x_\mu, p_\mu, \sigma_\alpha, \bar\sigma_{\dot\alpha}, \pi_\alpha, \bar\pi_{\dot\alpha}, e, \varphi \qquad \left(\begin{matrix}\sigma_\alpha \equiv \sigma_{\alpha;1} \\ \pi_\alpha \equiv \pi_{\alpha;1}\end{matrix}\right)$

Two identities encoding composite structure:

$$R_5 = \pi_\alpha p^{\alpha\dot\beta} \bar\pi_{\dot\beta} - \frac{1}{2} p_{\alpha\dot\beta} p^{\alpha\dot\beta} = 0$$

$$R_6 = \pi^\alpha \sigma_\alpha - \bar\pi^{\dot\alpha} \bar\sigma_{\dot\alpha} = 0$$

$18 - 2 = 16$

R_5, R_6 second class $\Rightarrow$ Dirac brackets

Quantization:

General solution (hep-th/0510161)

$$\Psi_{m,s,s_3,e}(p_\mu, \bar\pi_{\dot\alpha}, \pi_\alpha, \varphi) = \sum_{n,m=0}^{\infty} \sum_{\substack{(\alpha_1 \cdots \alpha_n) \\ (\beta_1 \cdots \beta_m)}} \pi_{\alpha_1} \cdots \pi_{\alpha_n} \cdot \bar\pi_{\dot\beta_1} \cdots \bar\pi_{\dot\beta_m} \cdot \psi^{(\alpha_1 \cdots \alpha_n)(\dot\beta_1 \cdots \dot\beta_m)}(p_\mu, \varphi)$$

satisfies Bargman-Wigner eq. for free higher spin fields

phase dependence $\exp(i e_0 \varphi)$

3. BOSONIC STRINGS FROM 2-TWISTORS

Order of derivation: C $\longrightarrow$ B $\longrightarrow$ A

a) Hamiltonian description

World-sheet momentum one-form (Siegel, 1996)

$$\theta^{(1)} = P_\mu \, dx^\mu \Longrightarrow \theta^{(2)} = P_\mu^{\ m} \underbrace{d\xi_m}_{\text{one-form}} \wedge dx^\mu \qquad m=0,1$$

P_μ^0, P_μ^1 – generalized string momenta $\leftarrow [P_\mu^m] = M^2$

The action:

C $$S_1^{d=2} = \int d^2\xi \left[P_\mu^{\ m} \partial_m X^\mu + \frac{1}{2T}(-g)^{-\frac{1}{2}} g_{mn} P_\mu^{\ m} P^{\mu n} \right] \qquad [T] = M^2$$

g_{mn}: world-sheet metric

Equations of motions:

(α) $\partial_m P_\mu^{\ m} = 0 \qquad \leftarrow \delta X$

(β) $P_\mu^{\ m} = -T(-g)^{\frac{1}{2}} g^{mn} \partial_n X_\mu \qquad \leftarrow \delta P$

(γ) $P_\mu^{\ m} P^{\mu n} = \frac{1}{2} g^{mn} g_{\kappa l} P_\mu^{\ \kappa} P^{l\mu} \qquad \leftarrow \delta g$

Inserting P_μ^1 one gets $(P_\mu^0 \equiv P_\mu)$

$$S_1^{d=2} = \int d^2\xi \left[P_\mu \dot{X}^\mu - \frac{\lambda}{2}\left(T^{-1} P_\mu^2 + T X'^2_\mu\right) - \rho \, P_\mu X'^\mu \right] \qquad \lambda = \frac{\sqrt{-g}}{g_{11}}, \quad \rho = \frac{g_{01}}{g_{11}}$$

This is Hamiltonian formulation of Nambu-Goto string (with two Virasoro constraints):

C $$S^{d=2} = -T \int d^2\xi \sqrt{-\det G^{(2)}} \qquad G^{(2)}_{mn} = \partial_m X^\mu \partial_n X_\mu$$

b) From space-time to twistorial formulation using Siegel formulation:

String generalization of composite momentum

$$P^m_{\alpha\dot\beta} = \sqrt{-g}\, e^m_{\ a}\, \bar\lambda^{\ i}_{\dot\beta} (\rho^a)_{ij} \lambda^{\dot j}_{\alpha}$$

$[\lambda^i_\alpha] = M$

zweibein

D=2 Dirac matrices (a=0,1)

D=2 fields $i,j=1,2$

One gets the action (Soroka, Sorokin, Tkach, Volkov 1989)

B

$$S^{d=2}_2 = \int d^2\xi \sqrt{-g}\,\Big[\bar\lambda^{\ i}_{\dot\beta}(\rho^a)_{ij}\lambda^{\dot j}_{\alpha}\partial_a X^{\alpha\dot\beta} + \\ + \frac{1}{2T}(\lambda^{\alpha i}\lambda_{\alpha i})(\bar\lambda_{\dot\alpha}{}^{j}\bar\lambda^{\dot\alpha}{}_{j})\Big]$$

Phase space description $((\lambda_{\alpha i})^* = \bar\lambda_{\dot\alpha}{}^{i}$ etc.):

$$(X_{\alpha\dot\beta}, \lambda_{\alpha i}, \bar\lambda_{\dot\alpha}{}^{i}, e^a_m) + (P_{\alpha\dot\beta}, \Pi^{\alpha i}, \bar\Pi^{\dot\alpha}{}_{i}, p^{(e)}{}_a)$$

"Coordinates" "momenta"

One can derive two first class constraints

$$F = \lambda_{\alpha i}\Pi^{\alpha i} + \bar\lambda_{\dot\alpha}{}^{i}\bar\Pi^{\dot\alpha}{}_{i} - 2e^a_m p^{(e)m}{}_a = 0$$

$$G = i(\lambda_{\alpha i}\Pi^{\alpha i} - \bar\lambda_{\dot\alpha}{}^{i}\bar\Pi^{\dot\alpha}{}_{i})$$

generating two local gauge invariances of SSTV model:

$$\lambda'_{\alpha i} = e^{i(b+ic)}\lambda_{\alpha i} \quad \bar\lambda'_{\dot\alpha}{}^{i} = e^{-i(b-ic)}\bar\lambda_{\dot\alpha}{}^{i} \quad e'^a_m = e^{2c} e^a_m$$

One can fix the granges (b,c) as follows:

$$A = \lambda_{\alpha i}\lambda^{\alpha i} - T = 0 \qquad \bar{A} = \bar{\lambda}_{\dot{\alpha}}{}^{i}\bar{\lambda}^{\dot{\alpha}}{}_{i} - T = 0 \qquad T \text{ real}$$

i.e. the last term in SSTV action is like a D=2 cosmological term

$$T^2 \cdot \frac{1}{2T}\sqrt{-g} \qquad \sqrt{-g} = \det e \equiv e$$

We use world-sheet Penrose incidence relations

$$\mu_i^{\dot{\alpha}} = \lambda_{\alpha i} X^{\alpha\dot{\alpha}} \qquad \bar{\mu}_i^{\alpha} = X^{\alpha\dot{\alpha}}\bar{\lambda}_{\dot{\alpha} i} \qquad [\mu_i^{\dot{\alpha}}] = M^0$$

and observe that the reality of X_μ requires

$$t^i{}_j \equiv \bar{\lambda}_{\dot{\alpha}}{}^{i}\mu^{\dot{\alpha}}{}_{j} - \bar{\mu}^{\alpha i}\lambda_{\alpha j} = 0 \Leftarrow \text{two null twistors!}$$

We get after eliminating $X^{\alpha\dot{\beta}}$

$$(B') \quad S_3^{d=2} = \int d^2\xi \Big\{ \tfrac{1}{2} e\, e^m_a \big[\bar{\lambda}_{\dot{\alpha}}{}^{i}(\rho^a)_{ij}\partial_m \mu^{\dot{\alpha} j} - \mu^{\dot{\alpha} i}(\rho^a)_{ij}\partial_m \lambda_{\dot{\alpha}}^{j} + \text{c.c.}\big] + \frac{T}{2} e + \Lambda_i{}^{j} t^i{}_j + \Lambda A + \bar{\Lambda}\bar{A} \Big\}$$

Lagrange multipliers

Further step: we solve algebraic eq. for zweibein

$$e_m^a = -\frac{1}{T}\left[\partial_m \bar{Z}^{Ai}(\rho^a)_{ij} Z_A^j - \bar{Z}^{Ai}(\rho^a)_{ij}\partial_m Z_A^j\right]$$

where $Z_A^i = (\lambda_\alpha^i, \mu^{\dot{\alpha} i})$, $\bar{Z}^{Ai} = (\bar{\mu}^{\alpha i}, -\bar{\lambda}^{\dot{\alpha} i})$

Further we recall that

$$t_i{}^{\dot{\gamma}} = Z_{Ai} \bar{Z}^{A}{}_{\dot{\gamma}}$$

After eliminating zweibein $e^m{}_a$ one gets

Indices i,j suppressed

$$S_3^{d=2} = \int d^2\xi \left\{ \frac{1}{4T} \epsilon_{ab} \left[\partial_m Z^A \rho^a Z_A - \bar{Z}^A \rho^a \partial_m Z_A \right] \cdot \right.$$

purely twistorial!

$$\left. \cdot \left[\partial_m \bar{Z}^A \rho^b Z_A - \bar{Z}^A \rho^b \partial_m Z_A \right] + \Lambda_{\dot{\gamma}}{}^{i} (Z_{Ai} \bar{Z}^{A\dot{j}}) + \Lambda A + \bar{\Lambda} \bar{A} \right\}$$

We get basic fourlinear twistor string action.

The action can be derived from the following Liouville two-form:

$$\Theta^{(2)} = \Theta_1^{(1)} \wedge \Theta_2^{(1)}$$

$$\left(\begin{array}{l} \Theta_1^{(1)} \leftrightarrow Z_A^1 \\ \Theta_2^{(1)} \leftrightarrow Z_A^2 \end{array} \right)$$

by introducing the world sheet embedding

$$(\sigma, \tau) \longrightarrow Z_A^i(\sigma, \tau)$$

Extension to bosonic twistor p-brane as composite of p+1 twistor fields $Z_A^i(\sigma_1 \dots \sigma_p, \tau)$

(p+1)-dim. fields

$$\Theta^{(p+1)} = \Theta_1^{(1)} \wedge \Theta_2^{(1)} \wedge \dots \wedge \Theta_{p+1}^{(1)}$$

p = 2 - membrane ⇒ Sect. 4

c) Quantization:

The fourlinear twistor string action is **linear in time derivative**:

$m,n = 0,1$
$A,B = 1 \ldots 4$

$$\frac{1}{T}\epsilon^{mn}\left(\bar{Z}^{A1}\partial_m Z_{A1} - \partial_m \bar{Z}^{A1} Z_{A1}\right)\cdot$$

$$\cdot\left(\bar{Z}^{B2}\partial_n Z_{B2} - \partial_n \bar{Z}^{B2} Z_{B2}\right) =$$

$$= Q_2\left(\bar{Z}^{A1}\dot{Z}_{A1} - \dot{\bar{Z}}^{A1} Z_{A1}\right) -$$

$$- Q_1\left(\bar{Z}^{A2}\dot{Z}_{A2} - \dot{\bar{Z}}^{A2} Z_{A2}\right)$$

where

$$\left(\dot{Z} \equiv \frac{\partial Z}{\partial \tau} \qquad Z' \equiv \frac{\partial Z}{\partial \sigma}\right)$$

space derivatives

$$Q_i = \frac{1}{T}\left(\bar{Z}^{Bi} Z'_{Bi} - \bar{Z}'^{Bi} Z_{Bi}\right)$$

(no summation over i!)

Constraints:

a) $D^{Ai} = P^{Ai} - \epsilon^{ij} Q_j \bar{Z}^{Ai} \simeq 0$

b) $\bar{D}_{Ai} = \bar{P}_{Ai} + \epsilon_{ij} \bar{Q}^j Z_{Ai} \simeq 0$

$Q_j = Q_j(Z)$

c) $V_i{}^j = Z_{Ai}\bar{Z}^{Aj} \simeq 0$ **null string twistors**

d) $I_{AB} Z^{A1} Z^{B2} = \frac{T}{2}$ $\left(\leftarrow \lambda_{\alpha i}\lambda^{\alpha i} = T\right)$

e) $I_{AB} \bar{Z}^{A1}\bar{Z}^{B2} = \frac{T}{2}$ $\left(\leftarrow \lambda_{\dot{\alpha} i}\lambda^{\dot{\alpha} i} = T\right)$

(I_{AB}- projectors infinity twistors)

DIRAC QUANTIZATION of a)–e) ⇒ under consideration

4. TWO-TWISTOR DESCRIPTION OF MEMBRANE AND P-BRANES

The presented scheme for strings can be generalized in two-twistor space to membranes. For p-branes – one needs $2^{\left[\frac{p+1}{2}\right]}$ twistors

a) Extension of Siegel momentum formulation to arbitrary p

p-form

$$\theta^{(1)} = p_\mu dx^\mu \longrightarrow \theta^{(p+1)} = P_\mu^{(p)} \wedge dX^\mu$$

$$P_\mu^{(p)} = P_\mu^m \epsilon_{m n_1 \ldots n_p} d\xi^{n_1} \wedge \ldots \wedge d\xi^{n_p}$$

Siegel action

(A)
$$S = \int d^{p+1}\xi \left(P_\mu^m \partial_m X^\mu + \frac{1}{2T}(-g)^{-\frac{1}{2}} g_{mn} P_\mu^m P^{\mu n} + \right.$$
$$\left. + \frac{T}{2}(p-1)(-g)^{\frac{1}{2}} \right)$$

$$P_\mu^m = -T(-g)^{\frac{1}{2}} g^{mn} \partial_n X_\mu$$

Polyakov-type action

$$S = -\frac{T}{2}\int d^{p+1}\xi\, (-g)^{\frac{1}{2}} \left[g^{mn} \partial_m X^\mu \partial_n X_\mu - (p-1) \right]$$

Putting $P_\mu^0 \equiv P_\mu$, eliminating $P_\mu^{\underline{m}}$ $\quad m = 1, 2 \ldots p$

Hamiltonian

$$S = P_\mu \dot{X}^\mu - \frac{\sqrt{-g}}{2} \det(g^{\underline{mn}}) \left[\frac{1}{2} P_\mu P^\mu + T \det g_{\underline{mn}} \right]$$
$$- g_{0\underline{n}}\, g^{\underline{nm}} (P_\mu \partial_{\underline{m}} X^\mu)$$

(p+1) Virasoro conditions

b) Intermediate spinor/space-time formulation

In case of arbitrary p Cartan-Penrose relation

$$P_\mu^m = \sqrt{-g}\, e_a^{\ m}\, \bar\lambda^{\dot{\hat\alpha} i} (\rho^a)_i{}^j \lambda_{\hat\beta j} (\gamma_\mu)_{\hat\alpha}{}^{\hat\beta}$$

det e — $\sqrt{-g}$; (p+1)-bein — $e_a^{\ m}$; world-volume Dirac matrices — ρ^a; D-dimensional Dirac matrices — γ_μ

! number of twistors! ⟹ $i = 1 \cdots 2^{[\frac{p+1}{2}]}$ $\quad \hat\alpha = 1 \cdots 2^{[\frac{D}{2}]}$

For D=4 membrane ($i,j = 1,2$, $a,m = 0,1,2$)

$$P^m_{\alpha\dot\beta} = \sqrt{-g}\, \bar\lambda_{\dot\beta}^{\ i} (\rho^a)_i{}^j \lambda_{\alpha j}$$

One gets

Lagrange multiplier — Λ

$$\text{(B)} \quad S = \int d^3\xi \sqrt{-g} \left(\bar\lambda_{\dot\beta}\, \rho^m \lambda_\alpha \partial_m X^{\dot\beta\alpha} + 2T + \Lambda A \right)$$

The constraint: $\quad A = (\lambda_{\alpha i} \lambda^{\alpha i})(\bar\lambda_{\dot\alpha i} \bar\lambda^{\dot\alpha i}) - 2T^2 = 0$

This "mass shell condition" can be derived in Siegel formulation as expressing the following constraint:

$$g_{mn} P_\mu^m P^{\mu n} = -\sqrt{-g}\, T^2$$

· Difference with string case: for string $A=0$ can be only achieved as local gauge fixing – for membrane $A=0$ follows from field eq.

c) Purely twistorial membrane action (D=4)

$$\mu_i^{\dot\alpha} = X^{\dot\alpha\beta}\lambda_{\beta i} \qquad \bar\mu^{\alpha i} = \bar\lambda^i_{\dot\beta} X^{\dot\beta\alpha}$$

$$\underbrace{X^{\dot\alpha\beta}}_{\text{hermitean}} \Longleftrightarrow V_i{}^{\dot\gamma} = \lambda_{\alpha i}\bar\mu^{\alpha\dot\gamma} - \mu_i{}^{\dot\alpha}\bar\lambda^{\dot\gamma}{}_{\dot\alpha} = 0$$

null twistors

If
$$Z_{Ai} = (\lambda_{\alpha i}, \mu_i^{\dot\alpha}), \quad \tilde Z^{Ai} = (\bar\mu^{\alpha i}, -\bar\lambda^i_{\dot\alpha})$$
one gets

det e

$$\text{B}' \quad S = \int d^3\xi \left[\tfrac{1}{2}\sqrt{-g}\, e_a^m (\partial_m \tilde Z^A \rho^a Z_A - \tilde Z^A \rho^a \partial_m Z_A) + \right.$$
$$\left. + 2\sqrt{-g}\,T + \Lambda A + \Lambda_i{}^{\dot\gamma} V_{\dot\gamma}{}^i \right]$$

Eliminating e_a^m (dreibein) one gets purely twistorial action:

$$\text{C} \quad S = -\frac{1}{48T^2}\int d^2\xi \left(\epsilon_{abc}\epsilon^{mnk}\theta_{(1)m}^a \theta_{(1)n}^b \theta_{(1)k}^c + \right.$$
$$\left. + \Lambda A + \Lambda_i{}^{\dot\gamma} V_{\dot\gamma}{}^i \right)$$

where

$$\theta_{(1)m}^a = \frac{\partial \tilde Z^{Ai}}{\partial \xi_m}(\rho^a)_i{}^j Z_{Aj} - \tilde Z^{Ai}(\rho^a)_i{}^j \frac{\partial Z_{Aj}}{\partial \xi_m}$$

Action induced on world-volume by 3-form:

$$\theta_{(3)} = \epsilon_{abc}\,\theta_{(1)}^a \wedge \theta_{(1)}^b \wedge \theta_{(1)}^c$$

5. SUPEREXTENSIONS

twistors → supertwistors

complex Grassmanns

$$Z_A = (\pi_\alpha, \omega^{\dot\alpha}) \Rightarrow Z_R = (Z_A, \xi_1 \cdots \xi_N)$$

repr. of SU(2,2) ⇒ repr. of SU(2,2|N)

Incidence relations superextended

$$Z_R \leftrightarrow (x_\mu, \theta^i_\alpha, \bar\theta^i_{\dot\alpha}) \quad i = 1 \cdots N \quad \text{(Ferber, 1978)}$$

a) Superparticles → Liouville one-form

C $$\Theta^{SUSY}_{(1)} = \frac{i}{2}(Z_R d\bar Z^R - dZ_R \bar Z^R) \left(+ \lambda \underset{\substack{\uparrow \\ SU(2,2|N) \\ \text{norm}}}{\langle Z_R, Z^R \rangle} \right)$$

If $N=1$ – helicity operator

$$\hat h = \frac{1}{2}\xi^+ \xi \to \text{helicity values } (0, \tfrac{1}{2})$$

After quantization free WZ multiplet

Supercovariant extension:

$$dx^{\alpha\dot\beta} \to \omega^{\alpha\dot\beta} = dx^{\alpha\dot\beta} - i(\theta^\alpha d\bar\theta^{\dot\beta} - d\theta^\alpha \bar\theta^{\dot\beta})$$

Inserting SUSY incidence relations one gets

B $$\Theta^{SUSY}_{(1)} = \pi_\alpha \bar\pi_{\dot\beta}\, \omega^{\alpha\dot\beta}$$ ← Shirafuji model

Inserting composite $p_{\alpha\dot\beta}$ one gets

A $$\Theta^{SUSY}_{(1)} = p_\mu \omega^\mu - \frac{1}{2} e p^2 \leftrightarrow \underset{\substack{\uparrow \\ \text{Brink-Schwarz} \\ \text{superparticle}}}{\frac{1}{2e}\omega_\mu \omega^\mu}$$

b) Superstrings → Liouville two-forms

C) How looks purely supertwistorial level?

For $N=1$ two possible extensions:

i) Both twistors supersymmetrized → $N=(1,1)$ (nonchiral) SUSY

$$\theta^{1}_{(1)} \wedge \theta^{2}_{(1)} \longrightarrow \theta^{1\,\mathrm{SUSY}}_{(1)} \wedge \theta^{2\,\mathrm{SUSY}}_{(1)}$$

One should add $\langle Z_i, Z^+ \rangle = \hat{t}_i \, t = 0$ in order to obtain real (nonchiral) superspace

ii) One twistor supersymmetrized

$$\theta^{1}_{(1)} \wedge \theta^{2}_{(1)} \longrightarrow \theta^{1\,\mathrm{SUSY}}_{(1)} \wedge \theta^{2}_{(1)} \qquad N=(1,0)$$

$$\longrightarrow \theta^{1}_{(1)} \wedge \theta^{2\,\mathrm{SUSY}}_{(1)} \qquad N=(0,1)$$

If one relates Z^1_A (Z^2_A) with left (right) moving string modes one obtains purely (super)twistorial action for heterotic string

The relations with B and A level (e.g. GS superstring) is valid under particular fixing of x-transformation.

conjecture

iii) general case

$$\theta^{1}_{(1)} \wedge \theta^{2}_{(1)} \rightarrow \theta^{1\,(p\text{-SUSY})}_{(1)} \wedge \theta^{2\,(q\text{-SUSY})}_{(1)}$$

Supertwistorial composite $N=(p,q)$ superstring

The role of κ-transformations in supertwistorization procedure:

κ-transformations occur in super-p-brane models on levels A, B, not C

Example: N-extended D=4 superparticle

real degrees:

A $(p_\mu, x_\mu ; \theta_{\alpha;i}, \bar{\theta}_{\dot\alpha;i})$ — $4+4$, $4N$ — 8 bosonic, 4N fermionic

B $(p_\mu, \pi_\alpha, \bar{\pi}_{\dot\alpha} ; \theta_{\alpha;i}, \bar{\theta}_{\dot\alpha;i})$ — $4+4$, $4N$ — 8 bosonic, 4N fermionic

C $(Z_A, \xi_1 \cdots \xi_N) + \text{c.c.}$ — 8, $2N$ — 8 bosonic, 2N fermionic

κ-transformations reduce the fermionic degrees of freedom by half (by 2N) in order that A B match with C!

c) BPS preons – Penrose approach to M-theory?

M-theory is a D=11 supersymmetric theory with generalized supersymmetries:

$$\{Q_K, Q_L\} \equiv P_{KL} = (\Gamma_\mu C)_{KL} P^\mu + (\Gamma_{\mu\nu} C)_{KL} P^{[\mu\nu]} + (\Gamma_{[\mu_1 \cdots \mu_5]} C)_{KL} P^{[\mu_1 \cdots \mu_5]}$$

$K, L = 1 \cdots 32$
$\mu = 0, 1 \cdots 10$

"M-algebra"

$P_{KL} = P_{LK} \leftarrow 528$ generalized momenta

BPS preons ansatz (D=11 generalized Cartan-Penrose relation)

$$P_{KL} = \sum_{i=1}^{n} \lambda_K^i \lambda_L^i \qquad i = 1 \ldots n \lesssim 32$$

(Bandos, de Azcarraga, Izquierdo, J.L., PRL 2001)

BPS states:

$$\det P_{KL} = 0 \iff n < 32$$

Number of BPS preons determine breaking of N=1 D=11 SUSY:

$$n \text{ preons} \longleftrightarrow \frac{32-n}{32} \text{ fractional SUSY}$$

$$\frac{32-n}{32} = \nu, \qquad 0 \leq \nu \leq 1$$

In particular for standard M-superbranes

$\nu = \frac{1}{2} \longleftrightarrow n = 16$ (x-transformation here 16 param.)

Question: How looks dynamics permitting $\nu = \frac{31}{32}$ fractional SUSY corresponding to single preon?

It appears that D=11 SUGRA does not provide preonic soliton solutions

Conjecture: one-preon theory impossible in space-time framework?

Completing of supertwistorial picture:

SUSY phase space for BPS preon: $(\underset{\substack{\uparrow \\ 32}}{\lambda_K}, \underset{\substack{\uparrow \\ 32}}{\omega_K}, \xi)$

Fundamental representation of generalized D=11 superconformal group OSp(1|64)

Application of supersymmetric incidence relations:

- without constraints leads to 528 coordinates
- suitably constrained D=11 generalized supertwistors can reduce $528 \to 11$.

6. FINAL REMARKS

- recalling main difference with Witten twistor string:
  - in our approach we use incidence relations in classical string theory ("classical mechanics of string")
  - Witten uses incidence relations on the level of string field theory ("second-quantized string")
- in presented approach important further studies
  - → symmetries
  - → quantization
  - → anomalies
  - → spectrum
  - → string in curved (super)spaces
- microscopic dynamics of M-theory not known ⇒ is the twistorial philosophy (primary spinorial geometry) applicable?

  Twistorial "Theory of Everything"?