Structure preservation in eigenvalue computation:
a challenge and a chance

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Outline

Nonlinear evp's with structure

Consider eigenvalue problem

$$P(\lambda)\,x = 0,$$

where

- $P(\lambda)$ is polynomial or rational matrix valued function;
- x is a real or complex eigenvector;
- λ is a real or complex eigenvalue;
- and $P(\lambda)$ has some further structure.

Definition

A nonlinear matrix function $P(\lambda)$ is called

- **T-even (H-even)** if $P(\lambda) = P(-\lambda)^T$ ($P(\lambda) = P(-\lambda)^H$);
- **T-palindromic (H-palindromic)** if $P(\lambda) = \mathrm{rev}P(\lambda)^T$ ($P(\lambda) = \mathrm{rev}P(\lambda)^H$).

In the following we often drop the prefix T and H.

Hamiltonians and symplectics

Let

$$J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

- A matrix H is called Hamiltonian if $(JH)^H = JH$ and skew-Hamiltonian if $(JH)^H = -JH$.
- Hamiltonian matrices from a Lie algebra, skew-Hamiltonian matrices form a Jordan algebra.
- A matrix S is called symplectic if $S^H JS = J$.
- Symplectic matrices form a Lie group.

Properties of even matrix polynomials.

Proposition

Consider a T-even eigenvalue problem $P(\lambda)x = 0$. Then $P(\lambda)x = 0$ if and only if $x^T P(-\lambda) = 0$, i.e., the eigenvalues occur in pairs $\lambda, -\lambda$.

Consider a H-even eigenvalue problem $P(\lambda)x = 0$. Then $P(\lambda)x = 0$ if and only if $x^H P(-\bar{\lambda}) = 0$, i.e., the eigenvalues occur in pairs $\lambda, -\bar{\lambda}$

Even matrix polynomials have Hamiltonian spectrum, they naturally generalize Hamiltonian problems $\lambda I + H$, where H is Hamiltonian.

Properties of palindromic matrix functions.

Proposition

Consider a T-palindromic eigenvalue problem $P(\lambda)x = 0$*. Then* $P(\lambda)x = 0$ *if and only if* $x^T P(1/\lambda) = 0$*, i.e., the eigenvalues occur in pairs* $\lambda, 1/\lambda$*.*

Consider a H-palindromic eigenvalue problem $P(\lambda)x = 0$*. Then* $P(\lambda)x = 0$ *if and only if* $x^T P(1/\bar{\lambda}) = 0$*, i.e., the eigenvalues occur in pairs* $\lambda, 1/\bar{\lambda}$*.*

Palindromic matrix polynomials have symplectic spectrum, they naturally generalize symplectic problems $\lambda I + S$*, where* S *is a symplectic matrix.*

Definition

Let $P(\lambda)$ be a matrix polynomial of degree k. Then the Cayley transformation of $P(\lambda)$ with pole at -1 is the matrix polynomial

$$\mathcal{C}_{-1}(P)(\mu) := (\mu + 1)^k P\left(\frac{\mu - 1}{\mu + 1}\right).$$

- The Cayley transformation creates a one-to-one map between palindromic and even polynomials (as it does between symplectic and Hamiltonian matrices).
- For the theory we only need to treat one structure, the results for the other follow automatically.
- For numerical methods one has to be careful.

Other structures

We will not discuss two other important structured classes.

- Real or complex symmetric nonlinear evp's. $P(\lambda) = P(\lambda)^T$
- Hermitian or real symmetric $P(\lambda)^H = P(\bar{\lambda})$.
- ...

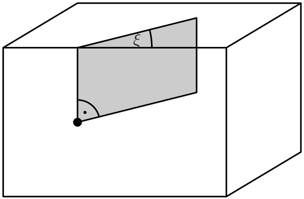
For more on these problems see work by Voss '03, Schreiber '08

Outline

Hilliges 04, Hilliges/Mehl/M. 04. Eigenvalues of $P(\lambda) = \lambda^2 A + \lambda B + A^T$, $B = B^T$, A low rank. Complex T-palindromic problem.

Apel/M./Watkins 02 $P(\lambda) = \lambda^2 M(\alpha) + \lambda D(\alpha) - K(\alpha)$,
$M = M^T > 0$, $K = K^T \geq 0$, $D = -D^T$ for $\alpha \in [a, b]$ real even problem

Example: Crack in 3D Domain Ω



Discrete time optimal control

Minimize $$\sum_{j=0}^{\infty} \left(x_j^H Q x_j + x_j^H Y u_j + u_j^H Y^H x_j + u_j^H R u_j \right)$$

subject to the kth-order discrete-time control system

$$\sum_{i=0}^{k} M_i x_{j+i+1-k} = B u_j, \quad j = 0, 1, \ldots,$$

with starting values $x_0, x_{-1}, \ldots, x_{1-k} \in \mathbb{R}^n$ and coefficients $Q = Q^H \in \mathbb{R}^{n,n}$, $Y \in \mathbb{R}^{n,m}$, $R = R^H \in \mathbb{R}^{m,m}$, $M_i \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,m}$.

Classical case: $\hat{R} = \begin{bmatrix} Q & Y \\ Y^H & R \end{bmatrix}$ positive definite, $M_k = I$.

H_∞ control: $\hat{R}$ indef. or singular, descriptor case: M_k singular.

Discrete bvp with palindromic matrix polynomial

$$\hat{P}(\lambda) := \sum_{j=0}^{2k-2} \lambda^j \hat{\mathcal{M}}_j := \lambda^{2k-2} \begin{bmatrix} 0 & M_k & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$+\lambda^{2k-3} \begin{bmatrix} 0 & M_{k-1} & 0 \\ 0 & 0 & 0 \\ 0 & Y^H & 0 \end{bmatrix} + \lambda^{2k-4} \begin{bmatrix} 0 & M_{k-2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \cdots$$

$$+\lambda^{k-1} \begin{bmatrix} 0 & M_1 & -B \\ M_1^H & Q & 0 \\ -B^H & 0 & R \end{bmatrix} + \lambda^{k-2} \begin{bmatrix} 0 & M_0 & 0 \\ M_2^H & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \ldots$$

$$+\lambda^2 \begin{bmatrix} 0 & 0 & 0 \\ M_{k-2}^H & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 & 0 & 0 \\ M_{k-1}^H & 0 & Y \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ M_k^H & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Even matrix Polynomials.

- Passivation of linear control systems arising from model reduced semidisc. Maxwell equations Freund/Jarre ’02, Brüll ’08
- Optimal control of higher order DAEs M./Watkins ’02
- Gyroscopic systems Lancaster ’04, Hwang/Lin/M. ’03.
- Optimal Waveguide Design, Schmidt/Friese/Zschiedrich/Deuflhard ’03.
- H_∞ control for descriptor Benner/Byers/M./Xu ’04.

While Hamiltonian matrices cover only special cases, even matrix polynomials cover all the cases.

Palindromic Matrix Polynomials.

- Periodic surface acoustic wave filters Zaglmeyer 02.
- Computation of the Crawford number Higham/Tisseur/Van Dooren 02.
- H_∞ control for discrete time descriptor systems Losse/M./Poppe/Reis '08
- Passivation of discrete linear control systems arising from model reduced fully Maxwell equations Brüll '08

While symplectic matrices cover only special cases, palindromic matrix polynomials cover all the cases.

Outline

Definition

For a matrix polynomial $P(\lambda)$ of degree k, a matrix pencil $L(\lambda) = (\lambda\mathcal{E} + \mathcal{A})$ is called linearization of $P(\lambda)$, if there exist nonsingular unimodular matrices (i.e., of constant nonzero determinant) $S(\lambda)$, $T(\lambda)$ such that

$$S(\lambda)L(\lambda)T(\lambda) = \mathrm{diag}(P(\lambda), I_{(n-1)k}).$$

A linearization is called strong if also $\mathrm{rev}L$ is a linearization of $\mathrm{rev}P$.

Companion form and structure.

Example The quadratic even eigenvalue problem

$$(\lambda^2 M + \lambda G + K)x = 0$$

with $M = M^T, K = K^T, G = -G^T$ has Hamiltonian spectrum but the companion linearization

$$\begin{bmatrix} O & I \\ -K & -G \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} I & O \\ O & M \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

does not preserve this structure.

- Numerical methods destroy eigenvalue symmetries in finite arithmetic !
- Perturbation theory requires structured perturbations for stability near imaginary axis. Ran/Rodman 1988.
- **Can we find structure preserving linearizations.**

Vector space of linearizations

Notation: $\Lambda := [\lambda^{k-1}, \lambda^{k-2}, \ldots, \lambda, 1]^T$, $\otimes$ - Kronecker product.

Definition (Mackey[2]/Mehl/M. '06.)

For a given $n \times n$ matrix polynomial $P(\lambda)$ of degree k define the sets:

$$
\begin{aligned}
\mathcal{V}_P &= \{v \otimes P(\lambda) \,:\, v \in \mathbb{F}^k\}, \; v \text{ is called right ansatz vector,} \\
\mathcal{W}_P &= \{w^T \otimes P(\lambda) : w \in \mathbb{F}^k\}, \; w \text{ is called left ansatz vector,} \\
\mathbb{L}_1(P) &= \left\{ L(\lambda) = \lambda\mathcal{E} + \mathcal{A} : \mathcal{E}, \mathcal{A} \in \mathbb{F}^{kn\times kn}, L(\lambda)\cdot(\Lambda \otimes I_n) \in \mathcal{V}_P \right\}, \\
\mathbb{L}_2(P) &= \left\{ L(\lambda) = \lambda\mathcal{E} + \mathcal{A} : \mathcal{E}, \mathcal{A} \in \mathbb{F}^{kn\times kn}, (\Lambda^T \otimes I_n)\cdot L(\lambda) \in \mathcal{W}_P \right\} \\
\mathbb{DL}(P) &= \mathbb{L}_1(P) \cap \mathbb{L}_2(P).
\end{aligned}
$$

Are there structured linearizations in these classes?

Even linearization

Lemma

Consider an $n \times n$ even matrix polynomial $P(\lambda)$ of degree k. For an ansatz vector $v = (v_1, \dots, v_k)^T \in \mathbb{F}^k$ the linearization $L(\lambda) = \lambda X + Y \in \mathbb{DL}(P)$ is even, i.e. $X = X^T$ and $Y = -Y^T$, (or $X = X^H$ and $Y = -Y^H$,) if and only if the v-polynomial

$$p(v; x) := v_1 x^{k-1} + \dots + v_{k-1} x + v_k$$

is even.

What are appropriate even polynomials $p(v; x)$.

Real even quadratics

If P is real, quadratic, even and has ev ∞, then there is no even linearization.

Example: Let

$$P(\lambda) = \lambda^2 M + \lambda D + K$$

be even, i.e. $M = M^T$, $D = -D^T$, $K = K^T$.

If M is singular, then the even linear pencil (obtained with $v = e_2$)

$$\lambda \begin{bmatrix} 0 & -M \\ M & G \end{bmatrix} + \begin{bmatrix} M & 0 \\ 0 & K \end{bmatrix}$$

is not a linearization, since $L(\lambda)$ is not regular.

We can alternatively look at odd problems, then the ev. 0 is 'bad'.

Lemma (Mackey/Mackey/Mehl/M. '06)

Consider an $n \times n$ palindromic matrix polynomial $P(\lambda)$ of degree k.
Then, for a vector $v = (v_1, \ldots, v_k)^T \in \mathbb{F}^k$ the linearization $L(\lambda) = \lambda X + Y \in \mathbb{DL}(P)$ is (the permutation of) a palindromic pencil, if and only if $p(v; x)$ is palindromic, which is the case iff v is a palindromic vector.

What are appropriate palindromic polynomials $p(v; x)$.

Real palindromic quadratics

Example: For the palindromic polynomial

$$P(\lambda)y = (\lambda^2 A + \lambda B + A^T)y = 0,\ B = B^T$$

all palindromic vectors have the form $v = [\alpha, \alpha]^T$, $\alpha \neq 0$ leads to a palindromic pencil

$$\kappa Z^T + Z,\ Z = \begin{bmatrix} A & B - A^T \\ A & A \end{bmatrix}.$$

This is a linearization iff if -1 is not an eigenvalue of $P(\lambda)$.
We can alternatively look at anti-palindromic linearizations, then the ev. 1 is 'bad'.

Outline

Trimmed linearization

To get good numerical results it is essential to deflate 'bad' ev's from the polynomial problem.

- Compute appropriate (structured) staircase form associated with the eigenvalues $1, -1, 0, \infty$ and the singular part directly for matrix polynomial.
- Remove parts associated with eigenvalues $1, -1, 0, \infty$ and singular parts. This can (at least in principle) be done exactly.
- Perform (structured) linearization on the resulting 'trimmed' matrix polynomial.
- 'Near bad' eigenvalues, however, lead to ill-conditioning.

Theorem (Byers/M./Xu 07)

Let $A_i \in \mathbb{C}^{m,n}$ $i = 0, \ldots, k$. *Then, the tuple* $(A_k, \ldots, A_0)$ *is unitarily equivalent to a matrix tuple* $(\hat{A}_k, \ldots, \hat{A}_0) = (UA_kV, \ldots, UA_0V)$, *all terms* $\hat{A}_i$, $i = 0, \ldots, k$ *have form*

$$\left[\begin{array}{cccc|c|cccc}
A & A & A & \cdots & \cdots & \cdots & A & A & A_l^{(i)} \\
A & A & A & \cdots & \cdots & \cdots & \iddots & A_{l-1}^{(i)} & 0 \\
A & A & A & \cdots & \cdots & \iddots & \iddots & 0 & 0 \\
\vdots & \vdots & \vdots & \iddots & \iddots & A_1^{(i)} & \iddots & \vdots & \vdots \\
\hline
\vdots & \vdots & \iddots & \iddots & A_0^{(i)} & 0 & \cdots & \vdots & \vdots \\
\hline
\vdots & \iddots & \iddots & \tilde{A}_1^{(i)} & \iddots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \iddots & \iddots & \iddots & \iddots & \vdots & \vdots & \vdots & \vdots \\
A & \tilde{A}_{l-1}^{(i)} & 0 & \iddots & \iddots & \vdots & \vdots & \vdots & 0 \\
\tilde{A}_l^{(i)} & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0
\end{array}\right].$$

Properties of this staircase form

- Each of the blocks $A_j^{(i)}$ $i = 0, \ldots, k,\ j = 1, \ldots, l$ either has the form $\begin{bmatrix} \Sigma & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 0 \end{bmatrix}$,
- Each of the blocks $\tilde{A}_j^{(i)}$ $i = 1, \ldots, k,\ j = 1, \ldots, l$ either has the form $\begin{bmatrix} \Sigma \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.
- For each j only of the $A_j^{(i)}$ and $\tilde{A}_j^{(i)}$ is nonzero.
- In the tuple of middle blocks $(A_0^{(k)}, \ldots, A_0^{(k)})$ (essentially) no k of the coefficients have a common nullspace.

Structured staircase forms

- Structured staircase forms for even and palindromic polynomials and pencils under congruence Byers/M./Xu '07.
- There exist exceptional cases where the 'bad' ev's cannot be removed.
- In many cases exactly 'bad' eigenvalues can be deflated ahead in a structure preserving way. This leads to 'trimmed linearizations'.
- In all the industrial examples the ev's 0 and ∞, ± 1 can be removed without much computational effort, just using the structure of the model.
- Singular parts can be removed altogether.

Structured vs. unstructured

Example Consider a 3×3 even pencil with matrices

$$N = Q \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} Q^T, \qquad M = Q \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} Q^T,$$

where Q is a random real orthogonal matrix. The pencil is congruent to

$$\lambda \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For different randomly generated orthogonal matrices Q the QZ algorithm in MATLAB produced all variations of eigenvalues that are possible in a general 3×3 pencil.

Structured Staircase

Example revisited Our implementation of the structured staircase Algorithm determined that in the cloud of rounding-error small perturbations of even $\lambda N + M$, there is an even pencil with structured staircase form

$$\lambda \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Outline

pencils

- Structure preserving QR/QZ like methods for even pencils Benner/M./Xu '97, '98, Chu/Liu/M. '04, Byers/Kressner '07
- Structure preserving Arnoldi method and JD methods for even pencils M./Watkins '01, Apel/M./Watkins '02, Hwang/Lin/Mehrmann '03
- Palindromic Jabobi and Laub trick Mackey^2,Mehl,M. '07;
- Palindromic QR/QZ algorithm and URV algorithm Dissertation Schröder 07;
- Recursive doubling Chu/Lin/Wang/Wu '05

Outline

Even linearization for crack problem

For the even quadratic $P(\lambda) = \lambda^2 M + \lambda G - K$ with $M = M^T > 0, K = K^T > 0, G = -G^T$ we have the even linearization:

$$\lambda N - W = \lambda \begin{bmatrix} 0 & -M \\ M & G \end{bmatrix} - \begin{bmatrix} -M & 0 \\ 0 & K \end{bmatrix}$$

with $N = -N^T$, $W = W^T$.

This can be transformed to a Hamiltonian matrix.

- ▷ $N = Z_1 Z_2, \quad Z_2^T J = \pm J Z_1$, sparse J-Cholesky factorization.
- ▷ Transform to $\lambda I - H = \lambda I - Z_1^{-1} W Z_2^{-1}$, where $H = Z_1^{-1} W Z_2^{-1}$ is Hamiltonian.

Sparse Representation of H and H^{-1}

$$
\begin{aligned}
H &= Z_1^{-1} W Z_2^{-1} \\
&= \begin{bmatrix} I & -\frac{1}{2}G \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & -K \\ M^{-1} & 0 \end{bmatrix} \begin{bmatrix} I & -\frac{1}{2}G \\ 0 & I \end{bmatrix}
\end{aligned}
$$

$$
H^{-1} = \begin{bmatrix} I & \frac{1}{2}G \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & M \\ (-K)^{-1} & 0 \end{bmatrix} \begin{bmatrix} I & \frac{1}{2}G \\ 0 & I \end{bmatrix}
$$

Multiplication with H or H^{-1} only needs solves with mass matrix M or stiffness matrix K, respectively.
Note that ’bad’ eigenvalues are removed.

General Arnoldi method

Form Krylov basis

$$[q_1, Aq_1, A^2q_1, \ldots, A^\ell q_1]$$

and orthogonalize vectors.
Ordinary Arnoldi process

$$q_{j+1} = Aq_j - \sum_{i=1}^{j} q_i h_{ij}.$$

With $Q_\ell = [q_1, q_2, \ldots, q_\ell]$ we have

$$AQ_\ell = Q_\ell H_\ell + f_\ell e_\ell^T$$

and use eigenvalues of the Hessenberg matrix H_ℓ as approximations to eigenvalues of A.

Problems with Arnoldi iteration

- Loss of orthogonality in the process leads to spurious eigenvalues, i.e. the same eigenvalues converge again and again.
- To avoid this, we can reorthogonalize or restart. But this expensive, so to make this feasible: Implicit restart. ARPACK, Lehoucq, Sorensen Yang 1998.
- Typically we get convergence of exterior eigenvalues.
- Only in the symmetric case a complete convergence theory is avaliable.
- To get interior eigenvalues we can use shift-and-invert
- Arnoldi does not respect the structure.

Start: Build a length l Arnoldi process.

$$AQ_\ell = Q_\ell H_\ell + f_\ell e_\ell^T$$

For $i = 1, 2, \ldots$ until satisfied:

1. Compute eigenvalues of H_ℓ and split them into a wanted set $\lambda_1, \ldots, \lambda_k$ and an unwanted set $\lambda_{k+1}, \ldots, \lambda_\ell$.
2. Perform $p = \ell - k$ steps of the QR-iteration with the unwanted eigenvalues as shifts and obtain $H_\ell V_\ell = V_\ell \tilde{H}_\ell$.
3. Restart: Postmultiply by the matrix V_k consisting of the k leading columns of V_ℓ.

   $$AQ_\ell V_k = Q_\ell V_k \tilde{H}_k + \tilde{f}_k e_k^T,$$

   where $\tilde{H}_k$ is the leading $k \times k$ principal submatrix of $\tilde{H}_\ell$.
4. Set $Q_k = Q_\ell V_k$ and extend Arnoldi factorization to length ℓ.

Shift-and-invert

To obtain interior eigenvalues we use shift-and-invert, i.e., we apply the implicitly restarted Arnoldi to a rational function of the matrix.
Goals:

- Pick shift point near the region where the desired eigenvalues are.
- Use a rational transformation that retains the eigenvalue symmetry.
- Transformation must be cheaply computable.

Exploitable Structures

- Hamiltonian

$$H^{-1}$$

- skew-Hamiltonian

$$H^{-2}$$

$$(H - \tau I)^{-1}(H + \tau I)^{-1}$$

$$(H - \tau I)^{-1}(H + \tau I)^{-1}(H - \overline{\tau} I)^{-1}(H + \overline{\tau} I)^{-1}$$

- symplectic

$$(H - \tau I)^{-1}(H + \tau I)$$

$$(H - \tau I)^{-1}(H + \overline{\tau} I)(H - \overline{\tau} I)^{-1}(H + \tau I)$$

$\tau =$ target shift.

Three different structures, three different methods.

$$\mathcal{W} = (H - \tau I)^{-1}(H + \overline{\tau} I)^{-1}(H - \overline{\tau} I)^{-1}(H + \tau I)^{-1}$$

Each factor has the form

$$(H - \tau I)^{-1} =$$

$$\begin{bmatrix} I & \frac{1}{2}G + \tau M \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & -M \\ Q(\tau)^{-1} & 0 \end{bmatrix} \begin{bmatrix} I & \frac{1}{2}G + \tau M \\ 0 & I \end{bmatrix}$$

- $Q(\tau) = \tau^2 M + \tau G + K$
- $Q(\tau) = P_q L_q U_q$ (sparse *LU* decomposition)
- One decomposition for all four factors.

Skew-Hamiltonian Arnoldi Process SHIRA

- Isotropic Arnoldi process

$$\tilde{q}_{j+1} = \mathcal{W} q_j - \sum_{i=1}^{j} q_i h_{ij} - \sum_{i=1}^{j} J q_i t_{ij}$$

- produces *isotropic* subspaces:
  $Jq_1, \ldots, Jq_k$ are orthogonal to $q_1, \ldots, q_k$.
- Theory $t_{ij} = 0$. Practice $t_{ij} = \epsilon$ (roundoff)
- Enforcement of isotropy is crucial.
- Consequence: get each eigenvalue only once.

Input: H and $\tau = \alpha$ or $\tau = i\alpha$, $\alpha \in \mathbb{R}$.
Output: Approx. inv. subspace of H ass. with p ev's near τ.

- ▷ Generate Arnoldi vectors $Q_k = [q_1, \ldots, q_k]$ and upper Hessenberg $H_k \in \mathbb{R}^{k\times k}$ such that $(H^2 - \tau^2 I)^{-1} Q_k = Q_k H_k$. Compute $\Omega_k = H_k^{-1} + \lambda_0^2 I$.
- ▷ Compute the real Schur decomposition $\Omega_k = U_k T_k U_k^T$.
- ▷ Reorder the p desired stable eigenvalues of T_k to the top of T_k, i.e., $T_k = V_k \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} V_k^T$, where $T_{11} \in \mathbb{R}^{p\times p}$ has desired eigenvalues.
- ▷ Set $\tilde{Q}_p := Q_k U_k V_k \begin{bmatrix} I_p \\ 0 \end{bmatrix}$.
- ▷ Compute the unique positive square root ${T_{11}}^{1/2}$ of T_{11}.
- ▷ Compute the stable invariant subspace $V_p = H\tilde{Q}_p - \tilde{Q}_p {T_{11}}^{1/2}$.

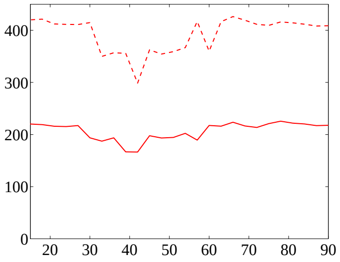
Numerical Results: Fichera Corner

Discretized Problem $n = 2223$, asking for 6 ev's in right half-pl.

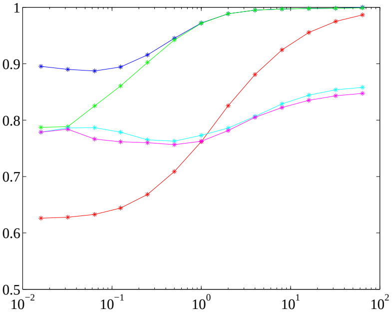
$$
\begin{aligned}
\lambda_1 &= 0.96269644895 \\
\lambda_2 &= 0.98250961158 + 0.00066849814i \\
\lambda_3 &= 0.98250961158 - 0.00066849814i \\
\lambda_4 &= 1.35421843051 \\
\lambda_5 &= 1.39562564903 \\
\lambda_6 &= 1.49830518846.
\end{aligned}
$$

	flops (10^7)	
τ	SHIRA	unstructured
0	32.6	140.1
0.3	32.6	79.6
0.6	25.7	69.8
0.9	23.0	50.7
1.2	17.5	31.5

Crack example: CPU in s for 15 ev's in $[0, 2)$, $h = \pi/120$; $\tau = 0$, solid line: SHIRA, dashed line: IRA with $(H - \tau I)^{-1}$;

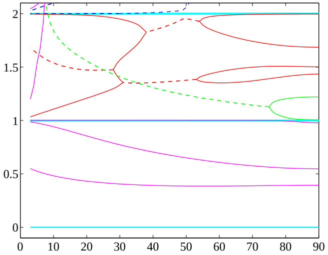


Eigenvalues for various material parameters



Results of SHIRA for Crack problem

Ev's with real part in $(0.1, 2.1)$. Dashed: nonreal eigenvalues. Triple ev's $\alpha = 0$ and $\alpha = 1$, 3 simple real ev's.



Summary second talk

- Palindromic and even polynomial eigenvalue problems are important in many applications.
- Structured linearization methods are available.
- Structured staircase forms are available.
- New trimmed linearization techniques are available.
- Structure preserving numerical methods for small even and palindromic pencils have been constructed.
- Structure preserving numerical methods for large sparse even and palindromic pencils have been constructed (provided we can still factor).

Thank you very much
for your attention.

information, papers, codes etc

http://www.math.tu-berlin.de/~mehrmann

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