Proof of the 1-factorization and Hamilton decomposition conjectures

Allan Lo
University of Birmingham

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Joint work with
Béla Csaba (Szeged), Daniela Kühn (Birmingham),
Deryk Osthus (Birmingham) and Andrew Treglown (QMUL)
A 1-factorization of a graph $G$ is a decomposition into edge-disjoint perfect matchings.

If $G$ contains a 1-factorization, then $|G|$ is even and $G$ is $D$-regular. 
$\Rightarrow \chi'(G) = D$. 
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$\Rightarrow \chi'(G) = D$.

Question
Does every $D$-regular graph $G$ with $|G| = n$ even and $D \geq n/2$ contain a 1-factorization?
Every $D$-regular graph $G$ with $|G| = n$ even and

$$D \geq \begin{cases} 
  n/2 - 1 & \text{if } n = 0 \pmod{4} \\
  n/2 & \text{if } n = 2 \pmod{4},
\end{cases}$$

contains a 1-factorization. Equivalently, $\chi'(G) = D$. 

"An odd component contains no perfect matching."
1-factorization conjecture (Chetwynd and Hilton 1985, Dirac 1950s)

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Some partial results:

- True for $D = n - 1$, i.e. complete graphs.
- Chetwynd and Hilton (1989), and independently Niessen and Volkmann (1990), for $D \geq (\sqrt{7} - 1)n/2 \approx 0.82n$.
- Perkovic and Reed (1997) for $D \geq (1/2 + \varepsilon)n$ with $\varepsilon > 0$. 

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Theorem (Csaba, Kühn, L, Osthus, Treglown 2013$^+$)

The 1-factorization conjecture holds for large $n$. 
A **Hamilton decomposition** of a graph $G$ is a decomposition into edge-disjoint Hamilton cycles.

If $G$ contains a Hamilton decomposition, then $G$ is $D$-regular and $D$ is even.
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**Question**

Does every $D$-regular graph on $n$ vertices with $D \geq n/2$ even contain a Hamilton decomposition?
### Hamilton decomposition conjecture (Nash-Williams 1970)

Every $D$-regular graph on $n$ vertices with $D \geq \lceil n/2 \rceil$ can be decomposed into edge-disjoint Hamilton cycles and at most one perfect matching.

The bound is best possible.

"No disconnected graph contains a Hamilton cycle."

The same bound for the existence of a single Hamilton cycle in $D$-regular graphs.

Observation: If $n$ is even, then 'a Hamilton cycle = two perfect matchings'.

Hamilton decomposition $\Rightarrow$ 1-factorization.

But Hamilton decomposition conjecture $\not\Rightarrow$ 1-factorization conjecture.
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Some partial results:

- Walecki (1890) $K_n$ has a Hamilton decomposition.
- Nash-Williams (1969), $D \geq \lfloor n/2 \rfloor$ implies a Hamilton cycle.
- Jackson (1979), $D/2 - n/6$ edge-disjoint Hamilton cycles
- Christofides, Kühn and Osthus (2012) if $D \geq n/2 + \varepsilon n$, then $G$ contains $(1 - \varepsilon')D/2$ edge-disjoint Hamilton cycles
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**Theorem (Csaba, Kühn, L, Osthus, Treglown 2013)\(^{\dagger}\)**

The Hamilton decomposition conjecture holds for large $n$. 

\(^{\dagger}\) Proof of the 1-factorization and Hamilton decomposition conjectures
Robust expander

**Definition**

Given $0 < \nu < \tau < 1$, we say that a graph $G$ on $n$ vertices is a robust $(\nu, \tau)$-expander, if for all $S \subseteq V(G)$ with $\tau n \leq |S| \leq (1 - \tau)n$ the number of vertices that have at least $\nu n$ neighbours in $S$ is at least $|S| + \nu n$.

“$G$ is still an expander after removing a sparse subgraph.”
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```
\begin{align*}
S & \quad \geq \nu n \\
& \quad RN_\nu(S)
\end{align*}
```

“$G$ is still an expander after removing a sparse subgraph.”

**Theorem (Kühn and Osthus 2013)**

For $1/n \ll \nu \ll \tau \ll \alpha$, every $\alpha n$-regular robust $(\nu, \tau)$-expander $G$ on $n$ vertices can be decomposed into edge-disjoint Hamilton cycles and at most one perfect matching.
Let $G$ be a $D$-regular graph with $|G| = n$ and $D \geq n/2 - 1$. Then either

(i) $G$ is a robust expander;

(ii) $G$ is $\varepsilon$-close to complete bipartite graph $K_{n/2,n/2}$;

(iii) $G$ is $\varepsilon$-close to union of two complete graphs $K_{n/2}$.

$G$ is $\varepsilon$-close to $H$ if $G$ can be transformed to $H$ by adding/removing at most $\varepsilon n^2$ edges.

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Proof of the 1-factorization and Hamilton decomposition conjectures
Extremal structure

Structural Lemma

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First attempt: Finding many Hamilton cycles

\[ A \approx K_{n/2} \quad \text{and} \quad B \approx K_{n/2} \]

Find many Hamilton cycles in \( G[A] \) and \( G[B] \).

'Easy', because \( G[A] \) and \( G[B] \) are almost complete.

'Connect' cycles on to get a Hamilton decomposition.

'Hard', because \( G[A, B] \) is sparse.

Actually, we need to first construct the connections, then extend each connection into a Hamilton cycle.

Proof of the 1-factorization and Hamilton decomposition conjectures
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$A \approx K_n/2 \approx K_n/2$ ???


2. 'Connect' cycles on to get a Hamilton decomposition.

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Proof of the 1-factorization and Hamilton decomposition conjectures
First attempt: Finding many Hamilton cycles


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First attempt: Finding many Hamilton cycles


2. ‘Connect’ cycles on to get a Hamilton decomposition.
   Hard, because $G[A, B]$ is sparse.

Actually, we need to first construct the connections, then extend each connection into a Hamilton cycle.
$G[A]$ and $G[B]$ are almost complete.

- $V_0 = \{\text{bad vertices}\}$ and $|V_0| \leq \varepsilon n$.
  
  - e.g. If $v$ has $\varepsilon n$ neighbours in $A$ and in $B$, then $v$ is bad.
Vertex partition

- $G[A]$ and $G[B]$ are almost complete.
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  e.g. If $v$ has $\varepsilon n$ neighbours in $A$ and in $B$, then $v$ is bad.

A connecting subgraph $J$ will cover $V_0$, connect $A$ and $B$, has endpoints in $A \cup B$. 
A key property of connecting subgraphs

Contract all paths incident with \( V_0 \).
Replace \( AB \)-edges with \( AA \)-edges and \( BB \)-edges.
Call resulting green graph \( J^* \).

Proof of the 1-factorization and Hamilton decomposition conjectures
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Properties of $J^*$
Let $C_A$ be a spanning cycle on $A$ and $C_B$ be a spanning cycle on $B$. Suppose that $J^* \subseteq C_A + C_B$ (in a suitable order). If we replace $J^*$ with $J$, then we get a Hamilton cycle.
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Finding a Hamilton decomposition

Decompose edges not in $A$ and not in $B$ into connecting subgraphs, $J_1, J_2, \ldots, J_{D/2}$.

Replace each $J_i$ with $J^*_i$.


Note that $G^*[A]$ and $G^*[B]$ are $D$-regular multigraphs.

'Hamilton decompose' $G^*[A]$.

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$G$ contains a Hamilton decomposition. □
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Proof of the 1-factorization and Hamilton decomposition conjectures
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Replace each \( J_i \) with \( J_i^* \).

Let \( G^* = G[A] + G[B] + J_1^* + \cdots + J_{D/2}^* \).

Note that \( G^*[A] \) and \( G^*[B] \) are \( D \)-regular multigraphs.

‘Hamilton decompose’ \( G^*[A] \) and \( G^*[B] \).

Replace each \( J_i^* \) with \( J_i \).
Finding a Hamilton decomposition

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Decompose edges not in $A$ and not in $B$ into connecting subgraphs, $J_1, J_2, \ldots, J_{D/2}$.

2. Replace each $J_i$ with $J_i^\ast$.
Let $G^\ast = G[A] + G[B] + J_1^\ast + \cdots + J_{D/2}^\ast$.
Note that $G^\ast[A]$ and $G^\ast[B]$ are $D$-regular multigraphs.

3. ‘Hamilton decompose’ $G^\ast[A]$ and $G^\ast[B]$.

4. Replace each $J_i^\ast$ with $J_i$. 

□
Finding a Hamilton decomposition

1. Decompose edges not in $A$ and not in $B$ into connecting subgraphs, $J_1, J_2, \ldots, J_{D/2}$.
2. Replace each $J_i$ with $J^*_i$.
   Note that $G^*[A]$ and $G^*[B]$ are $D$-regular multigraphs.
4. Replace each $J^*_i$ with $J_i$.
5. $G$ contains a Hamilton decomposition.
Finding a Hamilton decomposition

1. Decompose edges not in $A$ and not in $B$ into connecting subgraphs, $J_1, J_2, \ldots, J_{D/2}$. Hard


4. Replace each $J_i^*$ with $J_i$.

5. $G$ contains a Hamilton decomposition.
Question

Suppose that $G$ is a graph on $n$ vertices with $\delta(G) \geq n/2$ (not necessarily regular). How many edge-disjoint Hamilton cycles are contained in $G$?

- Dirac (1957) one Hamilton cycle.
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Conjecture (Nash-Williams 1971)

Every graph $G$ on $n$ vertices with $\delta(G) \geq n/2$ contains $(n - 2)/8$ edge-disjoint Hamilton cycles.
Babai’s construction:

|A| = 4k + 2 and |B| = |A| − 2

G[A] is a matching of size 2k + 1
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|A| = 4k + 2 and |B| = |A| - 2

G[A] is a matching of size 2k + 1

Observation

Every Hamilton cycles contains at least 2 edges from A
⇒ G has at most e(A)/2 edge-disjoint Hamilton cycles
⇒ G has ≤ k edge-disjoint Hamilton cycles
Packing Hamilton cycles with large $\delta(G)$

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**Conjecture (Nash-Williams 1971)**

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**Theorem (Csaba, Kühn, Lapinskas, L, Osthus, Treglown 2013+)**

The Nash-Williams conjecture is true for large $n$. 

Proof of the 1-factorization and Hamilton decomposition conjectures
Open problem

Let $\text{reg}_{\text{even}}(G)$ be the degree of the largest even-regular spanning subgraph in $G$. 
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**Observation**

Every graph $G$ on $n$ vertices contains $\leq \frac{\text{reg}_{\text{even}}(G)}{2}$ edge-disjoint Hamilton cycles.

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**Some partial results by Ferber, Krivelevich and Sudakov (2013)**

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