Nonconforming mimetic methods for diffusion problems

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Outline

1 **The construction of an MFD method:**

- meshes;
- degrees of freedom;
- approximation of the bilinear form;
- approximation of the loading term.

2. **Consistency condition and degrees of freedom:**

- the conforming MFD formulation;
- the non-conforming MFD formulation.

3. **Building a bridge with VEM.**

4. **Convergence results and numerical experiments.**

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The linear diffusion problem

- Differential formulation:

$$-\text{div}(\mathsf{K}\nabla u) = f \quad \text{in} \;\; \Omega,$$
$$u = g \quad \text{on} \;\; \Gamma,$$

(this talk: constant K)

S. D. Poisson (1771-1840)

- Variational formulation:

Find $u \in H^1_g(\Omega)$ such that:

$$\int_\Omega \mathsf{K}\nabla u \cdot \nabla v \, dV = \int_\Omega fv \, dV \qquad \forall v \in H^1_0(\Omega),$$

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Scheme construction in five steps

Steps 1 and 2

1. We decompose Ω into a **mesh** Ω_h of polygons (2-D) or polyhedrons (3-D);
   - admissible meshes may contain ”crazy” cells (non-convex, ”singular” as in AMR);
   - we need some regularity assumptions to avoid pathological cases and perform the convergence analysis;

2. **degrees of freedom**: $\mathcal{V}_h$

$$u, v \in H^1_g(\Omega) \cap C^\alpha(\overline{\Omega}) \quad \longrightarrow \quad u_h, v_h \in \mathcal{V}_h, \qquad \textbf{numbers!}$$

(with $\alpha \geq 0$).

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Steps 3 and 4

3. **bilinear form**: $\mathcal{A}_h(\cdot,\cdot)\ :\ \mathcal{V}_h\times\mathcal{V}_h\rightarrow\mathbb{R}$

$$\mathcal{A}_h(u_h,v_h)\approx\int_\Omega \mathsf{K}\nabla u\cdot\nabla v\,dV,$$

   it is built by "mimicking" a fundamental relation of calculus (*integration by parts*);

4. **linear functional**: $(f,\cdot)_h\ :\ \mathcal{V}_h\rightarrow\mathbb{R}$

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MFD construction in five steps

Step 5

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Find $u \in H^1_g(\Omega)$ *such that:*

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becomes the “mimetic variational” formulation:

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Meshes: why polygonal/polyhedral?

- The meshes should be easily adaptable to the geometric characteristics of the domain, but also to the solution:
  - *non-conforming meshes (hanging nodes)*;
  - *(local) adaptive refinements (AMR)*;
  - *highly deformed cells*;
  - *non-convex cells*;
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- Growing interest to use them in scientific applications and commercial codes, SINTEF, CD-ADAPCO, ANSYS;

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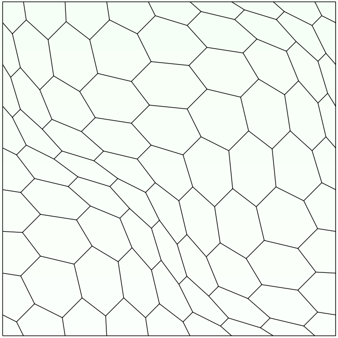
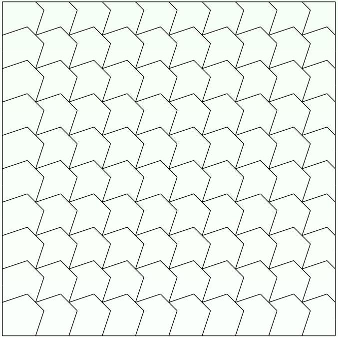
Meshes: academic examples

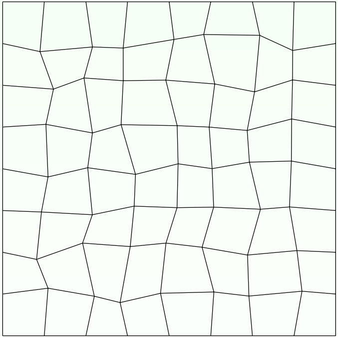
Examples: convex and non-convex polygonal cells

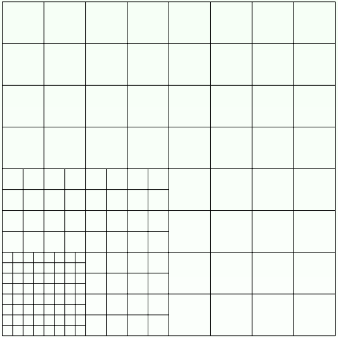




Meshes: academic examples

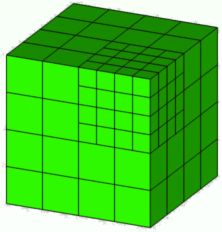
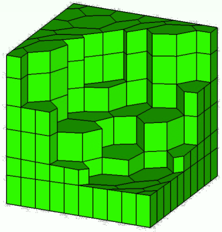
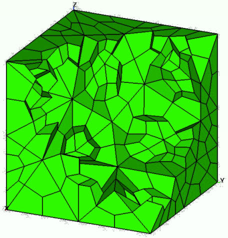
Examples: randomized quads and Adaptive Mesh Refinements (AMR)





Meshes: academic examples

Examples: locally refined, prismatic and random hexahedral meshes

Construction of $\mathcal{A}_h(u_h, v_h)$

- $\mathcal{A}_h(u_h, v_h)$ must be
  - **symmetric**, **bounded** and **semi-positive**;
  - **locally defined through an assembly process** (like FEM):

$$\mathcal{A}_h(u_h, v_h) = \sum_{\mathrm{P}} \mathcal{A}_{h,\mathrm{P}}(u_{h,\mathrm{P}}, v_{h,\mathrm{P}})$$

    where $u_{h,\mathrm{P}} = u_{h|\mathrm{P}}$, $v_{h,\mathrm{P}} = v_{h|\mathrm{P}}$;

- Any $\mathcal{A}_{h,\mathrm{P}}(u_{h,\mathrm{P}}, v_{h,\mathrm{P}})$ must be a **local approximation**:

$$\forall \mathrm{P} \in \Omega_h : \quad \mathcal{A}_{h,\mathrm{P}}(u_{h,\mathrm{P}}, v_{h,\mathrm{P}}) \approx \int_{\mathrm{P}} \mathsf{K}\nabla u \cdot \nabla v \, dV.$$

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Construction of $\mathcal{A}_h(u_h, v_h)$: consistency and stability

- PROBLEM: in MFD we do **not** have an approximation space (as in FEM, DG, VEM, etc)...only degrees of freedom!

- **Consistency**: **exactness property on polynomials** $\rightarrow$ **accuracy**
  Let $u, v \in \mathbb{P}_k(\mathrm{P})$, $u_{h,\mathrm{P}}, v_{h,\mathrm{P}}$ their dofs:

$$\mathcal{A}_{h,\mathrm{P}}(u_{h,\mathrm{P}}, v_{h,\mathrm{P}}) = \int_{\mathrm{P}} \mathsf{K}\nabla u \cdot \nabla v \, dV.$$

- **Stability**: **well-posedness property** $\rightarrow$ **continuity and coercivity**
  There exist two constants $\sigma_\star, \sigma^\star$ such that

$$\sigma_\star ||v_{h,\mathrm{P}}||^2_{1,h,\mathrm{P}} \leq \mathcal{A}_{h,\mathrm{P}}(v_{h,\mathrm{P}}, v_{h,\mathrm{P}}) \leq \sigma^\star ||v_{h,\mathrm{P}}||^2_{1,h,\mathrm{P}}$$

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Low order: towards a local consistency condition

The low-order setting, $m = 1$, $d = 2$

Let K be constant on P. We **integrate by parts** on the polygonal cell P.

- IF u is a **linear polynomial** on P $\implies$ $\mathsf{K}\nabla u$ is a **constant vector**;

THEN

$$\int_{\mathsf{P}} \mathsf{K}\nabla u \cdot \nabla v \, dV = -\underbrace{\int_{\mathsf{P}} \mathrm{div}(\mathsf{K}\nabla u)\, v \, dV}_{\textbf{equal to zero!}} + \sum_{\mathsf{e}\in\partial\mathsf{P}} \underbrace{\mathsf{K}\nabla u \cdot \mathbf{n}_{\mathsf{P,e}}}_{\textbf{constant}} \int_{\mathsf{e}} v \, dS$$

THUS,

$$\int_{\mathsf{P}} \mathsf{K}\nabla u \cdot \nabla v \, dV = \sum_{\mathsf{e}\in\partial\mathsf{P}} \mathsf{K}\nabla u \cdot \mathbf{n}_{\mathsf{P,e}} \int_{\mathsf{e}} v \, dS.$$

The local consistency condition: two options

The low-order setting, $m = 1$, $d = 2$

1. we use a numerical integration rule on each edge $\mathsf{e} = (\mathsf{v}', \mathsf{v}'')$, we require the **exactness for linear polynomials**:

$$\sum_{\mathsf{e}\in\partial\mathsf{P}} \mathsf{K}\nabla u \cdot \mathbf{n}_{\mathsf{P},\mathsf{e}} \int_{\mathsf{e}} v\, dS \approx \sum_{\mathsf{e}\in\partial\mathsf{P}} \mathsf{K}\nabla u \cdot \mathbf{n}_{\mathsf{P},\mathsf{e}} \underbrace{|\mathsf{e}|\,\frac{v(\mathbf{x}_{\mathsf{v}'}) + v(\mathbf{x}_{\mathsf{v}''})}{2}}_{\textbf{trapezoidal rule}}.$$

2. we introduce the **0-th order moment** of v as a degree of freedom:

$$\sum_{\mathsf{e}\in\partial\mathsf{P}} \mathsf{K}\nabla u \cdot \mathbf{n}_{\mathsf{P},\mathsf{e}} \int_{\mathsf{e}} v\, dS = \sum_{\mathsf{e}\in\partial\mathsf{P}} \mathsf{K}\nabla u \cdot \mathbf{n}_{\mathsf{P},\mathsf{e}} |\mathsf{e}|\, \mu_{\mathsf{e},0}(v)$$

where:

$$\mu_{\mathsf{e},0}(v) = \frac{1}{|\mathsf{e}|}\int_{\mathsf{e}} v\, dS.$$

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1. Conforming mimetic discretization

The low-order setting, $m = 1$, $d = 2$

1. According to

$$\sum_{\mathrm{e}\in\partial\mathrm{P}} \mathrm{K}\nabla u \cdot \mathbf{n}_{\mathrm{P,e}} \int_{\mathrm{e}} v\, dS \approx \sum_{\mathrm{e}\in\partial\mathrm{P}} \mathrm{K}\nabla u \cdot \mathbf{n}_{\mathrm{P,e}} |\mathrm{e}| \, \frac{v(\mathbf{x}_{\mathrm{v}'}) + v(\mathbf{x}_{\mathrm{v}''})}{2}.$$

   we require that

$$\mathcal{A}_{h,\mathrm{P}}\big(u_{h,\mathrm{P}}, v_{h,\mathrm{P}}\big) = \sum_{\mathrm{e}\in\partial\mathrm{P}} \mathrm{K}\nabla u \cdot \mathbf{n}_{\mathrm{P,e}} |\mathrm{e}| \, \frac{v_{\mathrm{v}'} + v_{\mathrm{v}''}}{2}.$$

   when

   - $u_{h,\mathrm{P}}$ is a discrete representation of the linear polynomial u on P;
   - $v_{\mathrm{v}'}, v_{\mathrm{v}''}$ are the degrees of freedom of $v_{h,\mathrm{P}}$ at $\mathrm{v}', \mathrm{v}''$.

The dofs represent the vertex values of $u_{h,\mathrm{P}}, v_{h,\mathrm{P}}$

2. Non-conforming mimetic discretization

The low-order setting, $m = 1$, $d = 2$

2. As $|e|\,\mu_{e,0}(v) = \int_e v\,dS$, and according to:

$$\sum_{e\in\partial P} K\nabla u \cdot \mathbf{n}_{P,e} \int_e v\,dS = \sum_{e\in\partial P} K\nabla u \cdot \mathbf{n}_{P,e} |e|\,\mu_{e,0}(v)$$

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when

- $u_{h,P}$ is a discrete representation of the linear polynomial u on P;
- $v_{e,0}$ is the degree of freedom of $v_{h,P}$ associated with edge e.

The dofs represent the zero-th order moments of $u_{h,P}, v_{h,P}$

Algebraic consistency: matrices $\mathbb{N}$ and $\mathbb{R}$

Low order setting, $m = 1$, $d = 2$

- **basis of** $\mathbb{P}_1(\mathrm{P}) = \Big\{1,\ (x - x_{\mathrm{P}}),\ (y - y_{\mathrm{P}})\Big\} = \{u_1, u_2, u_3\}$
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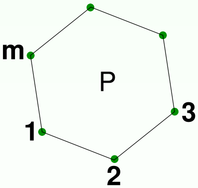

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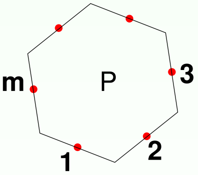
$$\mathbb{N} = \begin{pmatrix} 1 & (x_1 - x_{\mathrm{P}}) & (y_1 - y_{\mathrm{P}}) \\ 1 & (x_2 - x_{\mathrm{P}}) & (y_2 - y_{\mathrm{P}}) \\ \vdots & \vdots & \vdots \\ 1 & (x_m - x_{\mathrm{P}}) & (y_m - y_{\mathrm{P}}) \end{pmatrix}$$



Algebraic consistency: matrices $\mathbb{N}$ and $\mathbb{R}$

Low order setting, $m = 1$, $d = 2$

- **basis of** $\mathbb{P}_1(\mathrm{P}) = \Big\{1,\ (x - x_{\mathrm{P}}),\ (y - y_{\mathrm{P}})\Big\} = \{u_1, u_2, u_3\}$
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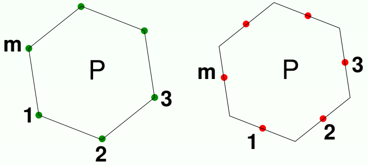

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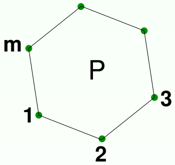

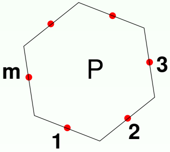

- **matrix $\mathbb{R}$: integration-by-parts for the polynomials u_i:**

$$\mathcal{A}_{h,\mathrm{P}}\big(u_{i\,h,\mathrm{P}}, v_{h,\mathrm{P}}\big) = \sum_{\mathrm{f}\in\mathrm{P}} \mathsf{K}\nabla u_i \cdot \mathbf{n}_{\mathrm{P,e}} \int_{\mathrm{e}} v\, dS = \mathbf{v}^T \mathbb{R}_i$$

Algebraic consistency: $\mathbb{M}\,\mathbb{N} = \mathbb{R}$

Low order setting, $m = 1$, $d = 2$

RECALL THAT

$$\mathcal{A}_{h,\mathrm{P}}\big(u_{i\,h,\mathrm{P}}, v_{h,\mathrm{P}}\big) = \sum_{\mathrm{f}\in\mathrm{P}} \mathsf{K}\nabla u_i \cdot \mathbf{n}_{\mathrm{P,e}} \int_{\mathrm{e}} v\, dS = \mathbf{v}^T \mathbb{R}_i$$

SINCE

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THEN

$$\mathbb{M}\,\mathbb{N}_i = \mathbb{R}_i \qquad i = 1, 2, 3.$$

EQUIVALENTLY,

$$\mathbb{M}\,\mathbb{N} = \mathbb{R}$$

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EQUIVALENTLY,

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Algebraic consistency: $\mathbb{M}\,\mathbb{N} = \mathbb{R}$

Low order setting, $m = 1$

- The formula $\mathbb{M}\,\mathbb{N} = \mathbb{R}$ is **ubiquitous** in the MFD method.

- Also,

$$\mathbb{N}^T\mathbb{R}_{|ij} = \int_{\mathsf{P}} \mathsf{K}\nabla u_i \cdot \nabla u_j \, dV \quad \text{where} \quad u_i, u_j \in \left\{1,\, x - x_{\mathsf{P}},\, y - y_{\mathsf{P}}\right\}$$

- **The (one-parameter) formula for the stiffness matrix**:

$$\mathbb{M} = \underbrace{\mathbb{R}(\mathbb{N}^T\mathbb{R})^{\dagger}\mathbb{R}^T}_{\mathbb{M}\mathbb{N}=\mathbb{R}} + \underbrace{\mu\left(\mathbb{I} - \mathbb{N}(\mathbb{N}^T\mathbb{N})^{-1}\mathbb{N}^T\right)}_{\textbf{stability}} \mathbb{M}_0 + \mathbb{M}_1$$

The second term depends on the parameter μ and gives a (one-parameter) family of methods.

The stiffness matrix formula

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Remarks:

- The consistency term $\mathbb{M}_0$ is responsible of the accuracy of the method.
- The stability term $\mathbb{M}_1$ ensures the well-posedness of the method.
- The bilinear form $\mathcal{A}_{h,\mathrm{P}}$ contains a **stabilization term** that depends on a set of parameters $\Rightarrow$ **family of schemes!**
- Both terms can be given the same (algebraic) form of the corresponding terms in the VEM.

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Three-dimensional case: conforming MFD

The low-order setting, $m = 1$, $d = 3$

- Recall that $v_{h|\mathrm{v}} := v_{\mathrm{v}} \approx v(\mathbf{x}_{\mathrm{v}'})$ and

$$\int_{\mathrm{P}} \mathsf{K}\nabla u \cdot \nabla v \, dV = \sum_{\mathrm{f}\in\partial\mathrm{P}} \mathsf{K}\nabla u \cdot \mathbf{n}_{\mathrm{P},\mathrm{f}} \int_{\mathrm{f}} v \, dS$$

- we assume that there exists a **quadrature rule** $\{(\mathbf{x}_{\mathrm{f},\mathrm{v}}, \omega_{\mathrm{f},\mathrm{v}})_{\mathrm{v}\in\partial\mathrm{f}}\}$ on each face $\mathrm{f} \in \partial\mathrm{P}$ such that

$$\int_{\mathrm{f}} v \, dS \approx \sum_{\mathrm{v}\in\partial\mathrm{f}} \omega_{\mathrm{f},\mathrm{v}} v(\mathbf{x}_{\mathrm{f},\mathrm{v}})$$

  is exact when v is a linear polynomial;

- we require that for every **linear polynomial** u and every discrete field v_h the bilinear form satisfies

$$\mathcal{A}_{h,\mathrm{P}}\big(u_{h,\mathrm{P}}, v_{h,\mathrm{P}}\big) := \sum_{\mathrm{f}\in\partial\mathrm{P}} \mathsf{K}\nabla u \cdot \mathbf{n}_{\mathrm{P},\mathrm{f}} \sum_{\mathrm{v}\in\partial\mathrm{f}} \omega_{\mathrm{f},\mathrm{v}} v_{\mathrm{v}} \qquad \big[v_{\mathrm{v}} \text{ represents } v(\mathbf{x}_{\mathrm{f},\mathrm{v}})\big].$$

Three-dimensional case: non-conforming MFD

The low-order setting, $m = 1$, $d = 3$

Let K be constant on P, u a linear polynomial, and integrate by parts.

- We use the **0-th order moment** of v as a degree of freedom:

$$\int_{\mathrm{P}} \mathrm{K}\nabla u \cdot \nabla v \, dV = \sum_{\mathrm{f}\in\partial\mathrm{P}} \mathrm{K}\nabla u \cdot \mathbf{n}_{\mathrm{P,f}} \int_{\mathrm{f}} v \, dS = \sum_{\mathrm{f}\in\partial\mathrm{P}} \mathrm{K}\nabla u \cdot \mathbf{n}_{\mathrm{P,e}} |\mathrm{e}| \, \mu_{\mathrm{f},0}(v)$$

  where:

$$\mu_{\mathrm{f},0}(v) = \frac{1}{|\mathrm{f}|} \int_{\mathrm{f}} v \, dS.$$

- The local consistency condition is:

$$\mathcal{A}_{h,\mathrm{P}}\big(u_{h,\mathrm{P}}, v_{h,\mathrm{P}}\big) = \sum_{\mathrm{f}\in\partial\mathrm{P}} \mathrm{K}\nabla u \cdot \mathbf{n}_{\mathrm{P,e}} |\mathrm{f}| \, v_{\mathrm{f},0} \qquad \big[v_{\mathrm{f},0} \text{ represents } \mu_{\mathrm{f},0}(v)\big]$$

For both formulations, we do the same as in 2D!

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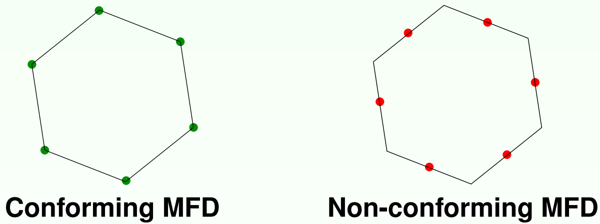
Summarizing the low-order formulation:

Low order setting, $m = 1$

- Degrees of freedom:



- exactness for linear polynomials;
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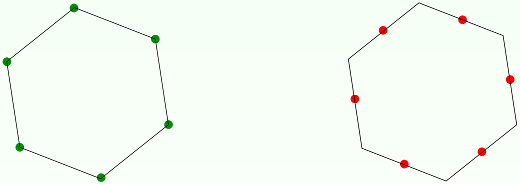

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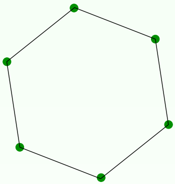

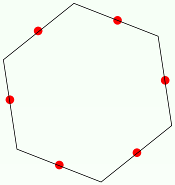

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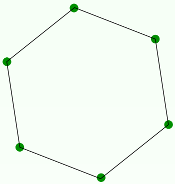

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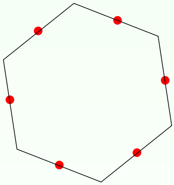

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High order: towards a local consistency condition (2D)

The high-order setting, $m > 1$, $d = 2$

Let K be constant and integrate by parts on the polygonal cell P:

$$\int_P \mathsf{K}\nabla u \cdot \nabla v \, dV = -\int_P \underbrace{\mathrm{div}(\mathsf{K}\nabla u)}_{\textbf{not zero!}} \, v \, dV + \sum_{e \in \partial P} \int_e \underbrace{\mathsf{K}\nabla u \cdot \mathbf{n}_{P,e}}_{\textbf{not constant!}} \, v \, dS.$$

If u is a polynomial of degree m on P:

- $\mathrm{div}(\mathsf{K}\nabla u)$ is a polynomial of degree $m-2$;
- $\mathsf{K}\nabla u \cdot \mathbf{n}_{P,e}$ is a polynomial of degree $m-1$;

Divergence term

Internal degrees of freedom, $m > 1$, $d = 2$

- For the **conforming** and **non-conforming case**, we use the **moments of v** to express the integral over P:

  if

  $$\mathrm{div}(\mathsf{K}\nabla u) = a_0 1 + a_1 x + a_2 y + \ldots \in \mathbb{P}_{m-2}(\mathrm{P})$$

  then

  $$\int_{\mathrm{P}} \mathrm{div}(\mathsf{K}\nabla u)\, v\, dV = a_0 \underbrace{\int_{\mathrm{P}} 1\, v\, dV}_{\hat{v}_{\mathrm{P},0}} + a_1 \underbrace{\int_{\mathrm{P}} x\, v\, dV}_{\hat{v}_{\mathrm{P},1,x}} + a_2 \underbrace{\int_{\mathrm{P}} y\, v\, dV}_{\hat{v}_{\mathrm{P},1,y}} + \ldots$$

  $$= a_0 \hat{\mathbf{v}}_{\mathrm{P},\mathbf{0}} + a_1 \hat{\mathbf{v}}_{\mathrm{P},\mathbf{1},\mathbf{x}} + a_2 \hat{\mathbf{v}}_{\mathrm{P},\mathbf{1},\mathbf{y}} + \ldots$$

  This choice suggests us to define

  - $m(m-1)/2$ **internal** degrees of freedom $\approx \hat{\mathbf{v}}_{\mathrm{P},\mathbf{0}}, \hat{\mathbf{v}}_{\mathrm{P},\mathbf{1},\mathbf{x}}, \hat{\mathbf{v}}_{\mathrm{P},\mathbf{1},\mathbf{y}}, \ldots$

Edge terms: conforming MFD

Nodal degrees of freedom, $m > 1$, $d = 2$

- We use a **Gauss-Lobatto formula** with $m+1$ nodes and weights $\{(\mathbf{x}_{\mathrm{e},q}, w_{\mathrm{e},q})\}$ on every (2D) edge $\mathrm{e} \in \partial\mathrm{P}$ for:

$$\int_{\mathrm{e}} \mathsf{K}\nabla u \cdot \mathbf{n}_{\mathrm{P,e}}\, v\, dS \approx \sum_{q=1}^{m+1} w_{\mathrm{e},q} \mathsf{K}\nabla u(\mathbf{x}_{\mathrm{e},q}) \cdot \mathbf{n}_{\mathrm{P,e}}\, v(\mathbf{x}_{\mathrm{e},q}).$$

This choice suggests us to define:

- one degree of freedom per vertex,
  $v_{\mathrm{e},1} = v_{\mathrm{v}'} \approx v(\mathbf{x}_{\mathrm{v}'})$, $v_{\mathrm{e},m+1} = v_{\mathrm{v}''} \approx v(\mathbf{x}_{\mathrm{v}''})$;
- $(m-1)$ nodal degrees of freedom per edge of P,
  $v_{\mathrm{e},q} \approx v(\mathbf{x}_{\mathrm{e},q})$ for $q = 2, \ldots m$.

High-order conforming MFD

The high-order setting, $m > 1$, $d = 2$

Local Consistency Condition:

Let K be constant.

- For every $u \in \mathbb{P}_m(\mathrm{P})$ ($m \geq 1$) and every discrete field $v_{h,\mathrm{P}} \in \mathcal{V}_h$ we require that:

$$\mathcal{A}_{h,\mathrm{P}}\big(u_{h,\mathrm{P}}, v_{h,\mathrm{P}}\big) := \underbrace{-\sum_{j=0}^{m(m-1)/2-1} a_j \hat{v}_{\mathrm{P},j}}_{\textbf{divergence}} + \underbrace{\sum_{\mathrm{e}\in\partial\mathrm{P}} \sum_{q=1}^{m+1} w_{\mathrm{e},q} \mathsf{K}\nabla u(\mathbf{x}_{\mathrm{e},q})\cdot\mathbf{n}_{\mathrm{P},\mathrm{e}}\, v_{\mathrm{e},q}}_{\textbf{boundary}}.$$

($u_{h,\mathrm{P}}$ are the dofs of u for P; terms $a_j \hat{v}_{\mathrm{P},j}$ are conveniently renumbered).

Edge terms: non-conforming MFD

Edge degrees of freedom, $m > 1$, $d = 2$

- We use the **moments of v** to express the integral over $\mathrm{e} \in \partial\mathrm{P}$:

if

$$(\mathsf{K}\nabla u)_{|\mathrm{e}} \cdot \mathbf{n}_{\mathrm{P},\mathrm{e}} = b_0 1 + b_1 \xi + b_2 \xi^2 + \ldots \in \mathbb{P}_{m-1}(\mathrm{e})$$

then

$$\int_{\mathrm{e}} \mathsf{K}\nabla u \cdot \mathbf{n}_{\mathrm{P},\mathrm{e}}\, v\, dS = b_0 \underbrace{\int_{\mathrm{e}} 1\, v\, dS}_{\hat{v}_{\mathrm{f},0}} + b_1 \underbrace{\int_{\mathrm{e}} \xi\, v\, dS}_{\hat{v}_{\mathrm{f},1}} + b_2 \underbrace{\int_{\mathrm{e}} \xi^2\, v\, dS}_{\hat{v}_{\mathrm{f},2}} + \ldots$$

$$= b_0 \hat{\mathbf{v}}_{\mathrm{e},\mathbf{0}} + b_1 \hat{\mathbf{v}}_{\mathrm{e},\mathbf{1}} + b_2 \hat{\mathbf{v}}_{\mathrm{e},\mathbf{2}} + \ldots$$

This choice suggests us to define

- m degrees of freedom per **edge** $\approx \hat{\mathbf{v}}_{\mathrm{e},\mathbf{0}}, \hat{\mathbf{v}}_{\mathrm{e},\mathbf{1}}, \hat{\mathbf{v}}_{\mathrm{e},\mathbf{2}}, \ldots$

High-order non-conforming MFD

The high-order setting, $m > 1$, $d = 2$

Local Consistency Condition:

Let K be constant.

- For every $u \in \mathbb{P}_m(\mathrm{P})$ ($m \geq 1$) and every discrete field $v_{h,\mathrm{P}} \in \mathcal{V}_h$ we require that:

$$\mathcal{A}_{h,\mathrm{P}}\big(u_{h,\mathrm{P}}, v_{h,\mathrm{P}}\big) := -\underbrace{\sum_{j=0}^{m(m-1)/2-1} a_j \hat{v}_{\mathrm{P},j}}_{\textbf{divergence}} + \underbrace{\sum_{\mathrm{e}\in\partial\mathrm{P}} \sum_{j=0}^{m-1} b_j \hat{v}_{\mathrm{e},j}}_{\textbf{boundary}}.$$

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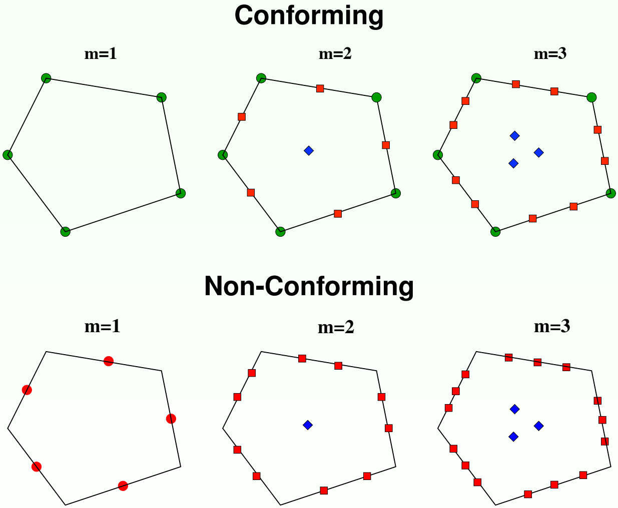
Degrees of freedom

Conforming/non-conforming case



Algebraic consistency condition: $\mathbb{M}\,\mathbb{N} = \mathbb{R}$

Let M be a **symmetric** and **semi-positive definite** matrix such that

$$\mathcal{A}_{h,\mathrm{P}}\big(u_{h,\mathrm{P}}, v_{h,\mathrm{P}}\big) = v_{h,\mathrm{P}}^T \mathbb{M} u_{h,\mathrm{P}}.$$

- For any $u \in \{1, x, y, x^2, xy, y^2, \dots\}$ and any discrete field $v_{h,\mathrm{P}}$

  - we write

  $$\mathcal{A}_{h,\mathrm{P}}\big(v_{h,\mathrm{P}}, u_{h,\mathrm{P}}\big) = \mathbf{v}^T \mathbb{M}\,\mathbb{N}_u \quad \text{where} \quad \mathbb{N}_u = \big[u_{h,\mathrm{P}}\big] \quad (\text{"dofs" of } u);$$

  - we impose the *local consistency condition*:

  $$\mathcal{A}_{h,\mathrm{P}}\big(u_{h,\mathrm{P}}, v_{h,\mathrm{P}}\big) = \dots = \mathbf{v}^T\,\mathbb{R}_u$$

  - we obtain by comparison:

  $$\mathbb{M}\,\mathbb{N}_u = \mathbb{R}_u$$

A family of schemes

- Using $\mathbb{N} = [\mathbb{N}_1, \mathbb{N}_2, \ldots]$, $\mathbb{R} = [\mathbb{R}_1, \mathbb{R}_2, \ldots]$, we have:

  $\mathbb{M}\,\mathbb{N} = \mathbb{R}$ and

  $$(\mathbb{R}^T\mathbb{N})_{ij} = \int_{\mathrm{P}} \mathsf{K}\nabla u_i \cdot \nabla u_j \, dV \qquad \text{where } u_i, u_j \in \{1, x, y, x^2, \ldots\}.$$

- $\mathbb{M}$ (symmetric and semi-positive definite) is given by

  $$\mathbb{M} = \underbrace{\mathbb{R}(\mathbb{R}^T\mathbb{N})^{-1}\mathbb{R}^T}_{\mathbb{M}\mathbb{N}=\mathbb{R}} + \underbrace{\delta\mathbb{M}}_{\textbf{stability}} \qquad \text{with} \qquad \delta\mathbb{M}\,\mathbb{N} = 0,$$

  where $\delta\mathbb{M}$ is a **symmetric matrix of parameters**.

- A one-parameter (γ) choice for $\delta\mathbb{M}$ is given by:

  $$\delta\mathbb{M} = \gamma\big(\mathbb{I} - \mathbb{N}(\mathbb{N}^T\mathbb{N})^{-1}\mathbb{N}^T\big).$$

The linear functional $(f, \mathbf{v}_h)_h$

The low-order case $m = 1$

Recall that $\left(f, v_h\right)_h \approx \int_\Omega fv\, dV$.

- We assemble $\left(f, v_h\right)_h$ from **local contribution**:

$$\left(f, v_h\right)_h := \sum_{\mathrm{P}} \left(f, v_h\right)_{h,\mathrm{P}} \quad \text{where} \quad \left(f, v_h\right)_{h,\mathrm{P}} \approx \int_{\mathrm{P}} fv\, dV$$

- We approximate the forcing term by its average on P:

$$f \approx \frac{1}{|\mathrm{P}|} \int_{\mathrm{P}} f\, dV =: \bar{f}_{\mathrm{P}};$$

- We use a (first-order) **quadrature** based on **vertex** *(conforming)* or **edge** *(non-conforming)* values. Example: let $\{(\mathbf{x}_{\mathrm{v}}, w_{\mathrm{P,v}})\}$:

$$\int_{\mathrm{P}} fv\, dV \approx \bar{f}_{\mathrm{P}} \int_{\mathrm{P}} v\, dV \approx |\mathrm{P}|\, \bar{f}_{\mathrm{P}} \sum_{\mathrm{v}\in\partial\mathrm{P}} w_{\mathrm{P,v}} v(\mathbf{x}_{\mathrm{v}}) \qquad [\text{conforming}]$$

The linear functional $(f, \mathbf{v}_h)_h$

The low-order case $m = 1$

- Recall that $\left(f, v_h\right)_h := \sum_{\mathrm{P}} \left(f, v_h\right)_{h,\mathrm{P}}$, where

$$\left(f, v_h\right)_{h,\mathrm{P}} \approx \int_{\mathrm{P}} fv \, dV, \quad \text{and} \quad \int_{\mathrm{P}} fv \, dV \approx |\mathrm{P}| \, \overline{f}_{\mathrm{P}} \sum_{\mathrm{v} \in \partial \mathrm{P}} w_{\mathrm{P},\mathrm{v}} v_{\mathrm{v}}$$

- Thus, for every cell P we define

$$\left(f, v_h\right)_{h,\mathrm{P}} := |\mathrm{P}| \, \overline{f}_{\mathrm{P}} \sum_{\mathrm{v} \in \partial \mathrm{P}} w_{\mathrm{P},\mathrm{v}} v_{\mathrm{v}} \qquad \forall v_h \in \mathcal{V}_h$$

$$|\mathrm{P}| \, \overline{f}_{\mathrm{P}} = \int_{\mathrm{P}} f \, dV$$

$w_{\mathrm{P},\mathrm{v}}$ 1-st order integration weights.

The linear functional $(f, \mathbf{v}_h)_h$

High-order case $m > 1$

- Again,

$$\left(f, v_h\right)_h := \sum_{\mathrm{P}} \left(f, v_h\right)_{h,\mathrm{P}} \quad \text{where} \quad \left(f, v_h\right)_{h,\mathrm{P}} \approx \int_{\mathrm{P}} f v \, dV.$$

- For $m > 1$ we consider the **orthogonal projection** of f onto the polynomials of degree $m - 2$:

$$f \approx c_0 1 + c_1 x + c_2 y + \ldots \in \mathbb{P}_{m-2}(\mathrm{P})$$

- and use the moments of v to express the r.h.s. integral:

$$\int_{\mathrm{P}} f\, v\, dV \approx c_0 \underbrace{\int_{\mathrm{P}} 1\, v\, dV}_{\hat{v}_{\mathrm{P},0}} + c_1 \underbrace{\int_{\mathrm{P}} x\, v\, dV}_{\hat{v}_{\mathrm{P},1,x}} + c_2 \underbrace{\int_{\mathrm{P}} y\, v\, dV}_{\hat{v}_{\mathrm{P},1,y}} + \ldots$$

$$= c_0 \hat{\mathbf{v}}_{\mathrm{P},\mathbf{0}} + c_1 \hat{\mathbf{v}}_{\mathrm{P},\mathbf{1},\mathbf{x}} + c_2 \hat{\mathbf{v}}_{\mathrm{P},\mathbf{1},\mathbf{y}} + \ldots$$

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High-order case $m > 1$

- Recall that

$$\left(f, v_h\right)_h := \sum_{\mathrm{P}} \left(f, v_h\right)_{h,\mathrm{P}} \quad \text{where} \quad \left(f, v_h\right)_{h,\mathrm{P}} \approx \int_{\mathrm{P}} fv \, dV.$$

- thus, for every cell P we define

$$\left(f, v_h\right)_{h,\mathrm{P}} := \sum_j c_j \hat{v}_{\mathrm{P},j} \qquad \forall v_h \in \mathcal{V}_h$$

$$f \approx c_0 1 + c_1 x + c_2 y + \ldots \in \mathbb{P}_{m-2}(\mathrm{P})$$

(c_j) projection coefficients

$\hat{v}_j$ moments, degrees of freedom of v_h

(The terms $c_j \hat{v}_{\mathrm{P},j}$ are conveniently renumbered).

Extension to 3D and variable coefficients

3D formulation

- The **3D conforming** formulation should have degrees of freedom associated to *vertices*, *edges*, *faces* and *cells*: too many!
- For the **3D non-conforming** formulation: we use moments on the *faces* and on the *cells* as for the VEM method.

Variable coefficients (conforming/non-conforming)

- **Modified consistency condition.**
  If $u \in \mathbb{P}_m(\mathrm{P})$ and $\mathrm{K}(X)$ is variable in P:

$$\int_{\mathrm{P}} \mathrm{K}(\mathbf{x})\nabla u \cdot \nabla v \, dV \approx \int_{\mathrm{P}} \Pi_{m-1}(\mathrm{K}(\mathbf{x})\nabla u) \cdot \nabla v \, dV = \dots$$

  There exists a VEM counterpart using a **modified projector** $\tilde{\Pi}^{\nabla}$.

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Building a bridge with the VEM

Conforming/non-conforming MFD, $m \leq 1$

- Let $\mathbb{N} = \big[\mathbf{1}, \hat{\mathbb{N}}\big]$, $\mathbb{R} = \big[\mathbf{0}, \hat{\mathbb{R}}\big]$;

$$\mathbb{N}^T\mathbb{R} = \begin{pmatrix} 0 & 0 \\ 0 & \hat{\mathbb{N}}^T\hat{\mathbb{R}} \end{pmatrix} \quad \text{and} \quad \big(\mathbb{N}^T\mathbb{R}\big)^\dagger = \begin{pmatrix} 0 & 0 \\ 0 & (\hat{\mathbb{N}}^T\hat{\mathbb{R}})^{-1} \end{pmatrix}$$

  where $\hat{\mathbb{N}}^T\hat{\mathbb{R}}$ is symmetric and positive definite.

- Let $\mathbb{G} = -\big[\,|\mathrm{P}|\,\hat{\mathbb{N}}^T\hat{\mathbb{R}}\big]^{-\frac{1}{2}}\mathbb{R}^T$. Then,

$$\mathbb{M}_0 = \mathbb{R}(\mathbb{N}^T\mathbb{R})^\dagger\mathbb{R}^T = \hat{\mathbb{R}}(\hat{\mathbb{N}}^T\hat{\mathbb{R}})^{-1}\hat{\mathbb{R}}^T = \mathbb{G}^T\mathbb{G}\,|\mathrm{P}|\,.$$

- $\mathbb{G}u_{h,\mathrm{P}} \approx -\mathsf{K}\nabla u$ is the **flux operator** such that

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Similarities and differences:

For both the conforming and the non-conforming MFD and VEM formulations we can prove that:

- the **degrees of freedom** are the same;
- the **consistency** term is the same:
  - in the MFD setting it relates to an exactness property;
  - in the VEM setting it is the projection of the bilinear form on polynomials;
- the **stabilization** term of VEM forms a **subset** of those of MFD:
  - in the MFD setting it gives the proper rank of the stiffness matrix;
  - in the VEM setting it relates to the non-computable part of the bilinear form;
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MFD and VEM: much more than a bridge!

For the Poisson equation (in primal form) we have:

- **Conforming MFD**

  2009 *low-order, 2D-3D*: Brezzi, Buffa, Lipnikov (M2AN);
  2011 *high-order, 2D*: Beirao da Veiga, Lipnikov, M. (SINUM);

- **Conforming VEM**

  2013 *any order, 2D*: ”volley” team (M3AS);

- **Non-conforming MFD**

  2014 *any order, 2D-3D*: Lipnikov, M., (JCP);

- **Non-conforming VEM**

  2014 *any order, 2D-3D*: Ayuso, Lipnikov, M. (submitted).

A mesh-dependent norm

Conforming case

We consider the mesh-dependent norm

$$||v_h||^2_{1,h} = \sum_{\mathrm{P}\in\Omega_h} ||v_h||^2_{1,h,\mathrm{P}}$$

that mimics the $|\cdot|_{1,\Omega}$ semi-norm;

- for the low-order method ($m = 1$, $d = 2, 3$), $\mathrm{e} = (\mathrm{v}', \mathrm{v}'')$ being an edge,

$$||v_h||^2_{1,h,\mathrm{P}} = ||\mathcal{GRAD}_h(v_h)||^2_{h,\mathrm{P}} = h_\mathrm{P} \sum_{\mathrm{e}\in\partial\mathrm{P}} |v_{\mathrm{v}''} - v_{\mathrm{v}'}|^2\,;$$

- for the high-order method ($m > 1$, $d = 2$), $\mathrm{e} = (\mathrm{v}', \mathrm{v}'')$ being an edge,

$$||v_h||^2_{1,h,\mathrm{P}} = h_\mathrm{P} \sum_{\mathrm{e}\in\partial\mathrm{P}} \left\|\frac{\partial v_{h,\mathrm{f}}}{\partial s}\right\|^2_{L^2(\mathrm{e})} + \left[\text{"moments"}\right]$$

Convergence results

Conforming case

The ***consistency*** and the ***stability*** conditions allow us to determine a ***family of mimetic schemes:***

- for the **low-order** method $m = 1$:

$$||u^I - u_h||_{1,h} < Ch\big(\,|f|_{0,\Omega} + |u|_{1,\Omega} + |u|_{2,\Omega}\,\big);$$

(Brezzi, Buffa, Lipnikov, M2AN (2009)),

- for the **high-order** method $m > 1$:

$$||u^I - u_h||_{1,h} < Ch^{\mathbf{m}}||u||_{m+1,\Omega};$$

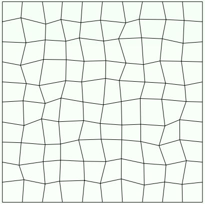
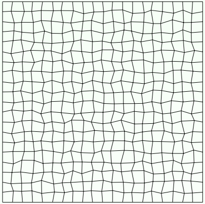
(Beirao da Veiga, Lipnikov, M., SINUM (2011); VEM, Brezzi et. al. M3AS ("volley" paper)

(For the non-conforming case refer to the talk of Blanca A.).

Conforming MFD method

Meshes with randomized quadrilaterals

- **Meshes:**

- **Exact solution:** $u(x,y) = \left(x - e^{2(x-1)}\right)\left(y^2 - e^{3(y-1)}\right)$
- **Diffusion tensor**

$$\mathsf{K} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Conforming MFD method

Randomized quadrilaterals, $|| \cdot ||_{1,h}$ errors, constant K

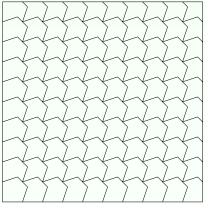
		$\mathbf{m} = \mathbf{2}$		$\mathbf{m} = \mathbf{3}$	
n	h	Error	Rate	Error	Rate
0	$1.922\,10^{-1}$	$1.416\,10^{-1}$	$--$	$7.454\,10^{-2}$	$--$
1	$9.705\,10^{-2}$	$2.441\,10^{-2}$	2.57	$8.632\,10^{-3}$	3.15
2	$4.838\,10^{-2}$	$5.366\,10^{-3}$	2.18	$1.536\,10^{-3}$	2.48
3	$2.467\,10^{-2}$	$1.399\,10^{-3}$	1.99	$1.739\,10^{-4}$	3.23
4	$1.263\,10^{-2}$	$3.524\,10^{-4}$	**2.06**	$2.227\,10^{-5}$	**3.07**

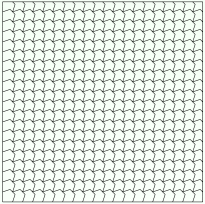
		$\mathbf{m} = \mathbf{4}$		$\mathbf{m} = \mathbf{5}$	
n	h	Error	Rate	Error	Rate
0	$1.922\,10^{-1}$	$1.031\,10^{-2}$	$--$	$4.567\,10^{-3}$	$--$
1	$9.705\,10^{-2}$	$1.690\,10^{-3}$	2.65	$2.674\,10^{-4}$	4.15
2	$4.838\,10^{-2}$	$1.273\,10^{-4}$	3.71	$1.336\,10^{-5}$	4.30
3	$2.467\,10^{-2}$	$8.279\,10^{-6}$	4.06	$4.586\,10^{-7}$	**5.01**
4	$1.263\,10^{-2}$	$5.545\,10^{-7}$	**4.04**	–	–

Conforming MFD method

Meshes with non-convex polygons

- **Meshes:**





- **Exact solution:** $u(x, y) = e^{-2\pi y} \sin(2\pi x)$
- **Diffusion tensor**

$$K = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad K(x, y) = \begin{pmatrix} (x+1)^2 + y^2 & -xy \\ -xy & (x+1)^2 \end{pmatrix}$$

Conforming MFD method

Non-convex polygons, $||\cdot||_{1,h}$ errors, constant K

		$\mathbf{m}=\mathbf{2}$		$\mathbf{m}=\mathbf{3}$	
n	h	Error	Rate	Error	Rate
0	$1.458\,10^{-1}$	2.858	$--$	1.007	$--$
1	$7.289\,10^{-2}$	$7.867\,10^{-1}$	1.86	$2.819\,10^{-1}$	1.84
2	$3.644\,10^{-2}$	$2.049\,10^{-1}$	1.94	$5.597\,10^{-2}$	2.33
3	$1.822\,10^{-2}$	$5.289\,10^{-2}$	**1.95**	$8.897\,10^{-3}$	**2.65**

		$\mathbf{m}=\mathbf{4}$		$\mathbf{m}=\mathbf{5}$	
n	h	Error	Rate	Error	Rate
0	$1.458\,10^{-1}$	$1.943\,10^{-1}$	$--$	$2.282\,10^{-2}$	$--$
1	$7.289\,10^{-2}$	$1.276\,10^{-2}$	3.93	$1.128\,10^{-3}$	4.34
2	$3.644\,10^{-2}$	$7.075\,10^{-4}$	4.17	$4.406\,10^{-5}$	**4.68**
3	$1.822\,10^{-2}$	$3.950\,10^{-5}$	**4.16**	–	–

Conforming MFD method

Non-convex polygons, $|| \cdot ||_{1,h}$ errors, non-constant K

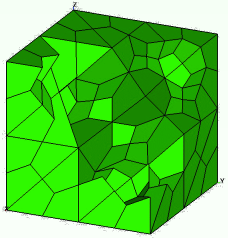
		$\mathbf{m} = \mathbf{2}$		$\mathbf{m} = \mathbf{3}$	
n	h	Error	Rate	Error	Rate
0	$1.458\,10^{-1}$	3.007	$--$	$9.873\,10^{-1}$	$--$
1	$7.289\,10^{-2}$	$8.081\,10^{-1}$	1.89	$2.760\,10^{-1}$	1.84
2	$3.644\,10^{-2}$	$2.071\,10^{-1}$	1.96	$5.621\,10^{-2}$	2.29
3	$1.822\,10^{-2}$	$5.303\,10^{-2}$	**1.97**	$9.083\,10^{-3}$	**2.63**

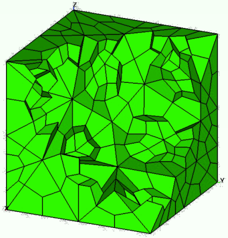
		$\mathbf{m} = \mathbf{4}$		$\mathbf{m} = \mathbf{5}$	
n	h	Error	Rate	Error	Rate
0	$1.458\,10^{-1}$	$2.059\,10^{-1}$	$--$	$1.988\,10^{-2}$	$--$
1	$7.289\,10^{-2}$	$1.367\,10^{-2}$	3.92	$1.016\,10^{-3}$	4.29
2	$3.644\,10^{-2}$	$7.562\,10^{-4}$	4.18	$3.924\,10^{-5}$	**4.69**
3	$1.822\,10^{-2}$	$4.210\,10^{-5}$	**4.17**	–	–

Non-conforming MFD method

Meshes with random hexahedra

- **Meshes:**



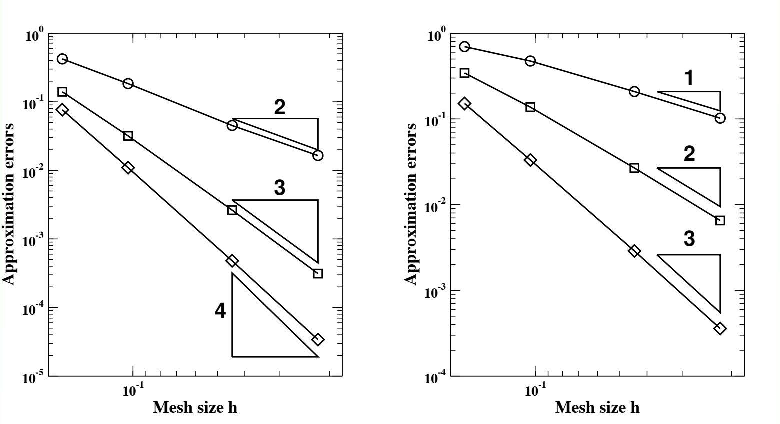


- **Exact solution:** $u(x, y, z) = x^3y^2z + x\sin(2\pi xy)\sin(2\pi yz)\sin(2\pi z)$
- **Diffusion tensor**

$$\mathsf{K} = \begin{pmatrix} 1 + y^2 + z^2 & -xy & -xz \\ -xy & 1 + x^2 + z^2 & -yz \\ -xz & -yz & 1 + x^2 + y^2 \end{pmatrix}$$

Non-conforming MFD method

Meshes with random hexahedra



$L^2(\Omega)$ **relative error** $\qquad$ $H^1(\Omega)$ **relative error**

The error is given by $u - \Pi_m^{\nabla}(u_h)$

Conclusions

- The conforming and non-conforming MFD methods are such that:
  (i) the low-order formulation uses either vertex or edge values to represent linear polynomials; it works in 2-D and 3-D;
  (ii) the high-order formulation uses edge nodal values and moments to represent *m*-degree polynomials; it works in 2-D and 3-D (only non-conforming).
  (iii) a reformulation as finite element exists in the virtual element framework.

- Possible future developments:
  (i) more complex operators (+convection, +reaction);
  (ii) exploit the strong connection with the VEM;
  (iii) curved faces;
  (iv) ...

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- Possible future developments:
  - (i) more complex operators (+convection, +reaction);
  - (ii) exploit the strong connection with the VEM;
  - (iii) curved faces;
  - (iv) ...

A few references...

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