Rigorous Numerical Upscaling of Elliptic Multiscale Problems at High Contrast

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Joint work with **Daniel Peterseim** (Bonn)

based also on work with
C Pechstein (Linz), PS Vassilevski (LLNL), LT Zikatanov (Penn State)

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Outline – Take Away Points

- A **Model Problem** & Applications
- Two Competing Goals: **Solving** or **Upscaling**?
- The Zoo of **Multiscale Schemes** & their **Analysis**
- A Fully Robust **Variational Multiscale Method (VMM)**
  (for locally quasi-monotone high contrast coefficients)
- Robust **Quasi-Interpolation** Operators
- Uniform **Weighted Poincaré Inequalities**
- Generalised Multiscale Finite Elements (**GMsFEM**)
- An Abstract **Bramble–Hilbert Lemma**
- **Outlook:** Fully Robust VMM for General Coefficients

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Model Problem

- Elliptic PDE in bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$

$$-\nabla \cdot (\boldsymbol{\alpha} \nabla u) = f \qquad + \text{ suitable BCs on } \partial\Omega$$

Issues adressed even more pronounced in other equations, e.g. transport.

- **Strongly varying** coefficient $\boldsymbol{\alpha}(x) \geq 1$ (otherwise rescale eqn.)
(scalar $\boldsymbol{\alpha}$, or quasi-isotropic tensor $\boldsymbol{\alpha}$ with $\lambda_{\max}(\boldsymbol{\alpha}) \sim \lambda_{\min}(\boldsymbol{\alpha})$)

- FE discretisation (p.w. lin. V^h): $a(u_h, v_h) = (f, v_h) \; \forall v_h \in V_h$

- **Two possible aims:**
  - h-optimal, α-robust parallel **solver** (fine mesh $\mathcal{T}^h$, α **resolved**)
  - H-optimal(?), α-robust **approximation** in coarse space V^H
  (α **not resolved: "Upscaling"** – no scale separation!)

- **Key Question** (for both): **Robust coarsening**

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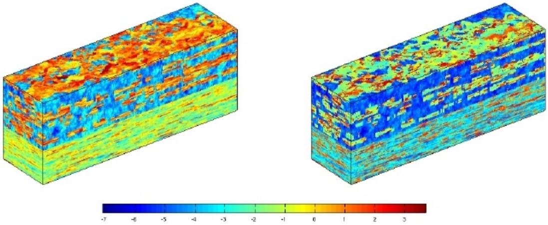
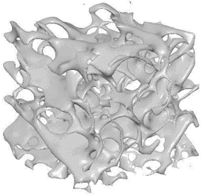
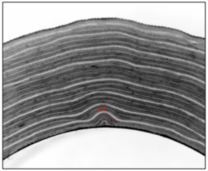
- **Subsurface flow**, e.g. in an oil reservoir
  (SPE10 benchmark)



- **Structural Mechanics**, e.g. in bone or carbon fibre composites





- ... many more ...

Goals (α resolved on fine mesh $\mathcal{T}^h$)

- Complicated variation of $\alpha(x)$ on many scales ($h \ll \mathrm{diam}(\Omega)$)
  Hard to **resolve** by "geometric" **coarse** mesh!
- **High contrast:** $\alpha_{\min} := \min_{\mathbf{x}} \alpha(\mathbf{x}) \ll \max_{\mathbf{x}} \alpha(\mathbf{x}) =: \alpha_{\max}$
- **Goal A:** Efficient & scalable multilevel parallel solver
  - **robust** w.r.t. mesh size h ($\Leftrightarrow$ w.r.t. problem size n)
  - **robust** w.r.t. coefficients $\alpha(x)$ **!**

+ underpinning theory that guides choice of components My background!

- **Goal B:** Simulate on coarse mesh where α is **not resolved!**
  - Discretisation in **"special"** coarse space V^H $\rightarrow$ **Upscaling**
  - **But:** Quality of approximation depends on (subgrid) variation & contrast in α **!** Strong links, but theory less developed.
- **Important. Goal B** not necessarily cheaper than **Goal A**
  (unless we have periodicity, scale separation, multiple RHSs, (mildly) nonlinear, or (slowly varying) time-dependent problem)

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Types of Multiscale Methods (incomplete list)

- Adaptive FEs ..., [Babuska, Rheinboldt, 1978]
- Generalised FEs [Babuska, Osborn, 1983]
- Numerical Upscaling ..., [Durlofsky, 1991]
- Multiscale Finite Elements [Hou, Wu, 1997], ...
- Variational Multiscale Method [Hughes et al, 1998]
- Multigrid Based Upscaling [Moulton, Dendy, Hyman, 1998]
- Multiscale Finite Volume Methods [Jenny, Lee, Tchelepi, 2003]
- Heterogeneous Multiscale Method [E, Engquist, 2003]
- Multiscale Mortar Spaces [Arbogast, Wheeler et al, 2007]
  (& other DD based methods)
- Adaptive Multiscale FVMs/FEs [Durlovsky, Efendiev, Ginting, 2007]
- Energy minimising bases [Dubois, Mishev, Zikatanov, 2009]
- Locally spectral (Generalised MsFEs) [Efendiev, Galvis, Wu, 2010]
- ... etc ...

Simplifying Assumptions & Theory (incomplete list of refs)

1. Periodic ⇒ Homogenisation theory ..., [Hou, Wu, 1997],... (most!)
2. Scale Separation ..., [Abdulle, 2005], ...
3. Inclusions and simple interfaces [Chu, Graham, Hou, 2010]
   (high contrast, no periodicity, no scale separation)
4. Bound in special flux norm [Berlyand, Owhadi, 2010]
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6. Weighted L^2-norm (using DD theory) **[RS, Zikatanov, in prep]**
   (weighted Poincaré, stable quasi-interpolant, weighted Bramble-Hilbert)
   - Uniform weighted Poincaré inequalities **[Pechstein, RS, 2011+]**
   - Stability and approximation of Clement-type quasi-interpolant **[RS, Vassilevski, Zikatanov, 2012]**
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A Variational Multiscale Method [Malqvist, Peterseim, 2012]

- (coarse) FE mesh $\mathcal{T}_H$ with mesh width H
- associated P1-FE space $V_H := \text{span}\{\Phi_j^H \mid j = 1, \ldots, N\}$
- Quasi-interpolation operator $\mathfrak{I}_H : V_h \to V_H$ [Carstensen, 1999] with

$$\mathfrak{I}_H v := \sum_j \frac{(v, \Phi_j^H)_{L^2(\Omega)}}{(1, \Phi_j^H)_{L^2(\Omega)}} \Phi_j^H$$

($\mathfrak{I}_H$ invertible on V_H!)

Decomposition

$$V_h = V_H \oplus V_h^{\mathrm{f}} \quad \text{with} \quad V_h^{\mathrm{f}} := \text{kernel}\, \mathfrak{I}_H = \{v \in V_h \mid \mathfrak{I}_H v = 0\}$$

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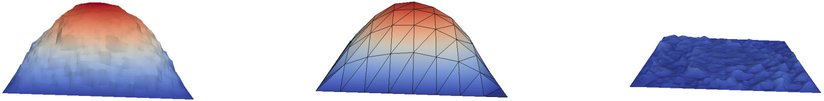
Example



$$\underbrace{u}_{\in V_h} = \underbrace{(\mathfrak{I}_H|_{V_H})^{-1}\mathfrak{I}_H u}_{\in V_H} + \underbrace{u - (\mathfrak{I}_H|_{V_H})^{-1}\mathfrak{I}_H u}_{\in V_h^{\mathrm{f}}}$$

Localizable Orthogonal Decomposition

- For each $v \in V_h$ define the fine scale projection $P^{\mathrm{f}} v \in V_h^{\mathrm{f}}$ by

$$a(P^{\mathrm{f}} v, w) = a(v, w) \quad \text{for all } w \in V_h^{\mathrm{f}}$$

a–Orthogonal Decomposition

$$V_h = V_H^{\mathrm{ms}} \oplus V_h^{\mathrm{f}} \quad \text{and} \quad a(V_H^{\mathrm{ms}}, V_h^{\mathrm{f}}) = 0 \quad \text{with } V_H^{\mathrm{ms}} := (1 - P^{\mathrm{f}}) V_H$$

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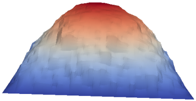
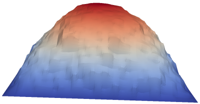




$$\underbrace{u}_{\in V_h} = \underbrace{u_H^{\mathrm{ms}}}_{\in V_H^{\mathrm{ms}}} + \underbrace{u^{\mathrm{f}}}_{\in V_h^{\mathrm{f}}}$$

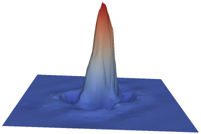
Modified (multiscale) nodal basis

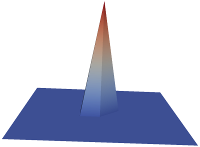
- $\{\Phi_j^H \mid j = 1, \ldots, N\} \subset V_H$ denotes classical nodal basis
- $\varphi_j^f := P^f \Phi_j^H \in V_h^f$ denotes the fine scale correction of Φ_j^H

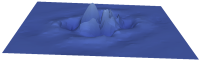
Ideal multiscale FE space

$$V_H^{ms} = \text{span}\left\{\Phi_j^H - \varphi_j^f \mid j = 1, \ldots, N\right\}$$

Example

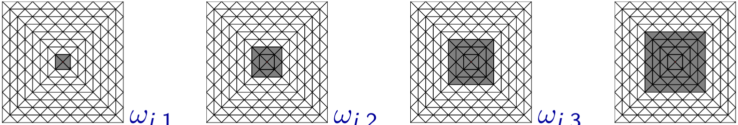






$$\underbrace{\Phi_j^H - \varphi_j^f}_{\in V_H^{ms}} = \underbrace{\Phi_j^H}_{\in V_H} + \underbrace{\varphi_j^f}_{\in V_h^f}$$

- Define nodal patches $\omega_{j,k}$ of k-th order around vertex x_j^H of $\mathcal{T}_H$



$\omega_{j,1}$ $\omega_{j,2}$ $\omega_{j,3}$ $\omega_{j,4}$

Lemma

There exists a $\gamma < 1$ *such that* $|\varphi_j^f|_{H^1(\Omega\setminus\omega_{j,k})} \lesssim \gamma^k |\varphi_j^f|_{H^1(\Omega)}$.

- **Practical multiscale method:** Fix k and define the localised correction $\varphi_{j,k}^f \in V_h^f(\omega_{j,k}) := \{v \in V_h^f \mid \operatorname{supp} v \subset \omega_{j,k}\}$ s.t.

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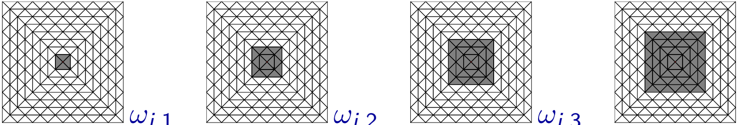
Localized multiscale FE spaces

$$V_{H,k}^{\mathrm{ms}} := \operatorname{span}\{\Phi_j^H - \varphi_{j,k}^f \mid j = 1, \ldots, N\}$$

Exponential decay and localisation

- Define nodal patches $\omega_{j,k}$ of k-th order around vertex x_j^H of $\mathcal{T}_H$



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The Multiscale Coarse Problem

Multiscale approximation

Seek $u^{\text{ms}}_{H,k} \in V^{\text{ms}}_{H,k}$ such that

$$a(u^{\text{ms}}_{H,k}, v) = (f, v) \quad \text{for all} \quad v \in V^{\text{ms}}_{H,k}$$

- $\dim V^{\text{ms}}_{H,k} = \dim V_H = N$ & basis functions have local support
- Overlap of the supports is proportional to the parameter k

Theorem (Malqvist & Peterseim, 2012)

$$|u - u^{\text{ms}}_{H,k}|_{H^1(\Omega)} \lesssim k^d H^{-1} \gamma^k \|f\|_{H^{-1}(\Omega)} + H\|f\|_{L_2(\Omega)} + |u - u_h|_{H^1(\Omega)}$$

Thus, provided $k \gtrsim \log_\gamma(\frac{1}{H})$ and h is suff'ly small we have **optimal** $\mathcal{O}(H)$ convergence without any assumptions on scales or regularity.

Similarly, $\mathcal{O}(H^2)$ convergence in L^2-norm using an Aubin-Nitsche argument.

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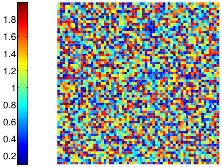
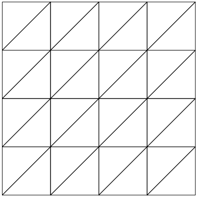
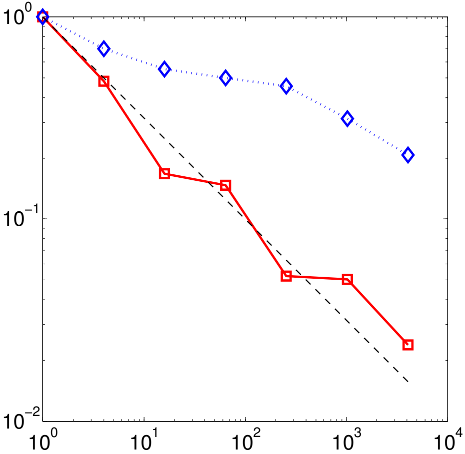


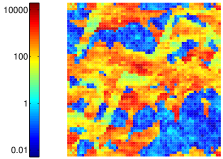


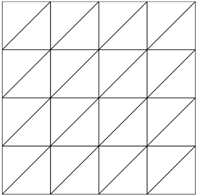
$f \equiv 1$ and $u|_{\partial\Omega} \equiv 0$
$H = 2^{-1}, 2^{-2}, \ldots, 2^{-7}$
$h = 2^{-9}$, $k = \lceil 2\log(1/H) \rceil$



$\frac{|u - u^{\mathrm{ms}}_{H,k}|_{H^1(\Omega)}}{|u|_{H^1(\Omega)}}$ vs. #dofs

Numerical Experiment (high contrast)





$f \equiv 1$ and $u|_{\partial\Omega} \equiv 0$
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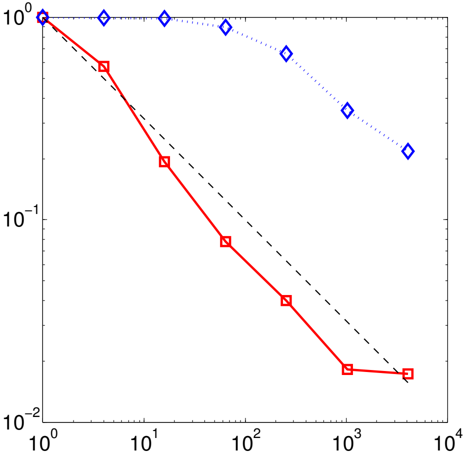

$$\frac{|u - u^{\text{ms}}_{H,k}|_{H^1(\Omega)}}{|u|_{H^1(\Omega)}} \text{ vs. } \#\text{dofs}$$

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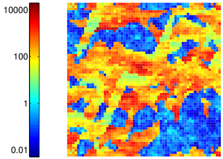

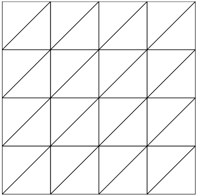

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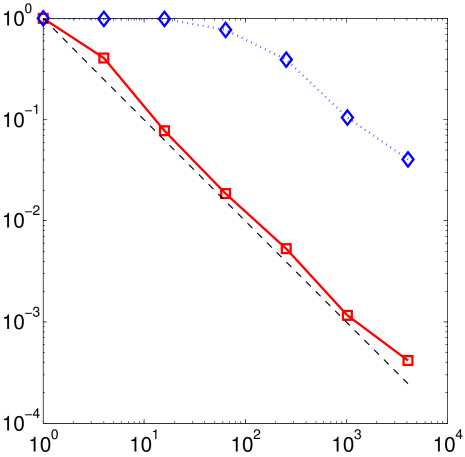

$$\frac{\|u - u^{\text{ms}}_{H,k}\|_{L^2(\Omega)}}{\|u\|_{L^2(\Omega)}} \text{ vs. } \#\text{dofs}$$

But unfortunately $\gamma := \exp\left(\sqrt{\frac{\alpha_{\min}}{\alpha_{\max}}}\right)$ and so $\gamma \to 1$ as the contrast $\frac{\alpha_{\max}}{\alpha_{\min}} \to \infty$. The hidden constant depends also on $\frac{\alpha_{\max}}{\alpha_{\min}}$.

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Theorem useless for high contrast !

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$$\Downarrow$$

Theorem useless for high contrast !

A contrast-robust theory

Now, instead of

- working in standard H^1 and L^2-norm
- and using the simple norm equivalence

$$\alpha_{\min}|v|_{H^1(\Omega)} \leq \|v\|_a \leq \alpha_{\max}|v|_{H^1(\Omega)}$$

we want to work

- directly in the **energy norm** $\|v\|_{a,\omega} := (\int_\omega \alpha|\nabla v|^2 \,dx)^{1/2}$ and the **weighted** L^2**-norm** $\|v\|_{0,\alpha,\omega} := (\int_\omega \alpha v^2 \,dx)^{1/2}$
- and use a **coefficient–weighted quasi-interpolant**
- as well as a **weighted Poincaré type inequality** and a **weighted inverse type inequality**

Main Result (Peterseim & RS, 2013+)

If there exists a linear, continuous quasi-interpolation operator $\mathfrak{I}_H : V_h \to V_H$ and two generic constants C_2 and C_3 such that

(QI1) $(\mathfrak{I}_H|_{V_H})^{-1}\mathfrak{I}_H v_H = v_H$, for all $v_H \in V_H$

(QI2) $H_T^{-2}\|v - \mathfrak{I}_H v\|_{0,\alpha,T}^2 + \|v - \mathfrak{I}_H v\|_{a,T}^2 \le C_2\|v\|_{a,\omega_T}^2$, for all $v \in V_h$ and $T \in \mathcal{T}_H$

(QI3) for all $v_H \in V_H$ there exists a $v \in V_h$, s.t. $\mathfrak{I}_H v = v_H$, $\operatorname{supp} v \subset \operatorname{supp} v_H$ and $\|v\|_a \le C_3\|v_H\|_a$.

then (with some universal constant $m \lesssim 1$)

$$\|u - u_{H,k}^{\mathrm{ms}}\|_a \lesssim \left(\frac{\alpha_{\max}}{\alpha_{\min}}\right)^m \frac{e^{-k}}{H}\|f\|_{H^{-1}(\Omega)} + \frac{H}{\alpha_{\min}^{-1/2}}\|f\|_{L_2(\Omega)} + \|u - u_h\|_a$$

Thus, provided $k \gtrsim \ln(\frac{\alpha_{\max}}{\alpha_{\min}}\frac{1}{H})$ and h suff'ly small we have **optimal** $\mathcal{O}(H)$ convergence without assumptions on regularity or contrast.

Again, $\mathcal{O}(H^2)$ convergence in L^2-norm follows by an Aubin-Nitsche argument.

A suitable quasi-interpolation operator – Assumption (QI2)

- Now adapt theory developed for 2-level Schwarz to prove (QI2)
- For simplicity assume α p.w. constant w.r.t. some grid $\mathcal{T}_\eta$, with $h < \eta < H$, but not by $\mathcal{T}_H$ ($\mathcal{T}_H \subset \mathcal{T}_\eta \subset \mathcal{T}_H$ nested)
- For every $T \in \mathcal{T}_H$ define $\omega_T := \bigcup\{T' : T \cap T' \neq \emptyset\}$.

Lemma (Old) [RS, Vassilevski, Zikatanov, SINUM 2012]

For all $T \in \mathcal{T}_H$, let $C_K^P > 0$ be the best constant s.t. for all $v \in V_h$ the following **weighted Poincaré inequality** holds:

$$\inf_{\xi \in \mathbb{R}} \|v - \xi\|^2_{0,\alpha,\omega_T} \leq C_T^P \, \mathrm{diam}(\omega_T)^2 \|\nabla v\|^2_{a,\omega_T} \qquad \textbf{(WPI)}$$

(with a slight variation near Dirichlet boundaries). Then

$$H_T^{-2}\|v - \mathfrak{I}_H v\|^2_{0,\alpha,T} + \|v - \mathfrak{I}_H v\|^2_{a,T} \lesssim C_2 \|v\|^2_{a,\omega_T} \qquad \text{(QI2)}$$

with $\mathfrak{I}_H v = \sum_j \frac{\int_{\mathrm{supp}(\Phi_j^H)} \alpha v \, dx}{\int_{\mathrm{supp}(\Phi_j^H)} \alpha \, dx} \Phi_j^H$ and $C_2 = \max_{T \in \mathcal{T}_H} C_T^P$.

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Lemma (New) Proof analogous! [Peterseim, RS, 2013+])

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with $\mathfrak{I}_H v := \sum_{j=1}^{N} \frac{(\alpha v, \Phi_j^H)_{L^2(\Omega)}}{(\alpha, \Phi_j^H)_{L^2(\Omega)}} \Phi_j^H$ and $C_2 \eqsim \frac{H}{\eta} \max_{T \in \mathcal{T}_H} C_T^P$

(price to pay to also get (QI3))

Corollary [RS, Zikatanov, in prep]

Assume that the PDE solution $u \in H^{1+s}(\Omega)$, for some $s > 0$. Then (under the same assumptions as above)

$$\inf_{v_H \in V_H} \|u - v_H\|_{0,\alpha} \lesssim C_* H \|f\|_{H^{-1}(\Omega)} .$$

- Possibly **not sharp** (w.r.t. H), but needs **minimal** regularity
- **Sharp** w.r.t. coefficient variation. We can show lower bound: i.e. $C_* \gg H^{-1} \Rightarrow$ no approximation!
- Constant C_* can be independent of α (local quasi-monotonicity; see below)
- Extends readily to other "nodal" spaces, such as MsFEs

Approximation result in the weighted L^2–norm (p.w. linears)

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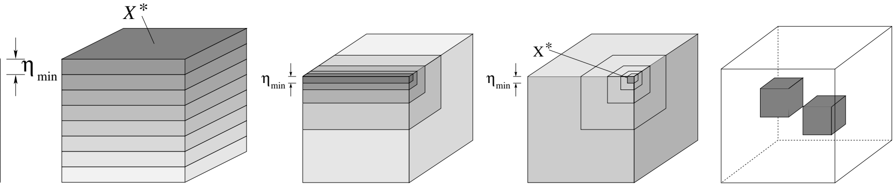
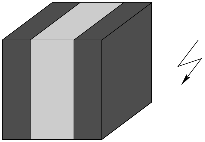
When is Poincaré constant independent of contrast in α?

- Careful theory in [Pechstein, RS, IMAJNA 2012] linking robustness to **quasi-monotonicity**!
- **Bounds** for the <u>effective Poincaré constant</u> C_T^P in 3D :

  Darker colour means higher permeability.



$\mathcal{O}(1)$ $\quad$ $\mathcal{O}(1 + \log(\frac{H}{\eta}))$ $\quad$ $\mathcal{O}(\frac{H}{\eta})$ $\quad$ $\mathcal{O}(\frac{\eta}{h})$



$C_T^P \gtrsim \frac{\alpha_2}{\alpha_1}$ **(i.e. not robust!)**

Poincaré's inequality

Domain $\Omega \subset \mathbb{R}^d$ (open, bounded, connected set). $\exists C > 0$ s.t.

$$\inf_{\gamma \in \mathbb{R}} \|u - \gamma\|^2_{L^2(\Omega)} \leq C \operatorname{diam}(\Omega)^2 \, |u|^2_{H^1(\Omega)} \qquad \forall u \in H^1(\Omega).$$

- C depends only on shape of Ω, **not** on $\operatorname{diam}(\Omega)$
- Infimum attained at

$$\gamma^* = \overline{u}^{\Omega} := \frac{1}{|\Omega|} \int_\Omega u \, dx$$

- Inequality (with different constant C) also works for

$$\gamma = \overline{u}^{X} := \frac{1}{|X|} \int_X u \, dx$$

where $X \subset \Omega$ subset or $(d-1)$-dimensional manifold

(with positive volume/surface measure)

Weighted Poincaré type inequality

For $\alpha \in L^\infty(\Omega)$ uniformly positive, we define

$$\|v\|^2_{L^2(\Omega),\alpha} := \int_\Omega \alpha\,|v|^2 dx \quad \text{and} \quad |v|^2_{H^1(\Omega),\alpha} := \int_\Omega \alpha\,|\nabla v|^2 dx$$

Clearly,

$$\|u - \overline{u}^\Omega\|^2_{L^2(\Omega),\alpha} \;\leq\; C \max_{x,y\in\Omega} \frac{\alpha(x)}{\alpha(y)}\, \mathrm{diam}\,(\Omega)^2\, |u|^2_{H^1(\Omega),\alpha}$$

Question

Can we find C^P **independent** of variation & contrast in α such that

$$\inf_{\gamma\in\mathbb{R}} \|u - \gamma\|^2_{L^2(\Omega),\alpha} \;\leq\; C^P\, |u|^2_{H^1(\Omega),\alpha}$$

for some class of weights $\alpha : \Omega \to \mathbb{R}^+$ **?**

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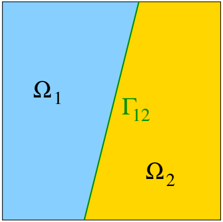
Model Case #1

Assume $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$ (Ω_k "well-shaped")
with **interface** $\Gamma_{12} := \partial\Omega_1 \cap \partial\Omega_2$

and $\boldsymbol{\alpha}_{|\Omega_k} = \alpha_k = \text{const}$



Apply standard Poincaré type inequality on Ω_1 and Ω_2, i.e.

$$\|u - \overline{u}^{\Gamma_{12}}\|^2_{L^2(\Omega_k)} \leq C \,\text{diam}\,(\Omega_k)^2 \,|u|^2_{H^1(\Omega_k)} \qquad \forall\, u \in H^1(\Omega_k)$$

Then multiplying by α_k and adding implies

$$\|u - \overline{u}^{\Gamma_{12}}\|^2_{L^2(\Omega),\boldsymbol{\alpha}} \leq C \,\text{diam}\,(\Omega)^2 \,|u|^2_{H^1(\Omega),\boldsymbol{\alpha}}$$

with C depending on (the shape of) Ω_k and Γ_{12} but **not** on $\boldsymbol{\alpha}$ **!**

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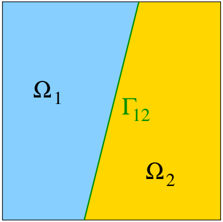

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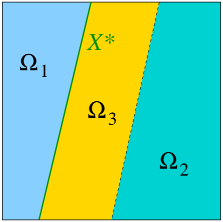

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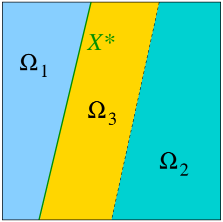
Define manifold $X^* := \partial\Omega_1 \cap \partial\Omega_3$

Treat Ω_1 and Ω_3 as before, and

$$\begin{aligned} \|u - \overline{u}^{X^*}\|^2_{L^2(\Omega_2),\boldsymbol{\alpha}} &= \alpha_2 \|u - \overline{u}^{X^*}\|^2_{L^2(\Omega_2 \cup \Omega_3)} \\ &\leq \alpha_2 \, C \, \text{diam}\,(\Omega)^2 \, |u|^2_{H^1(\Omega_2 \cup \Omega_3)} \\ &\leq C \, \text{diam}\,(\Omega)^2 \left\{ \int_{\Omega_2} \alpha_2 |\nabla u| dx + \int_{\Omega_3} \underbrace{\alpha_2}_{\leq \alpha_3} |\nabla u| dx \right\} \\ &\leq C \, \text{diam}\,(\Omega)^2 \, |u|^2_{H^1(\Omega_2 \cup \Omega_3),\boldsymbol{\alpha}} \end{aligned}$$

Again C depends on (the shape of) Ω_k and X^*, but **not** on $\boldsymbol{\alpha}$!

Model Case #2



Assume $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2 \cup \overline{\Omega}_3$ (Ω_k "well-shaped")
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Define manifold $X^* := \partial\Omega_1 \cap \partial\Omega_3$

Treat Ω_1 and Ω_3 as before, and

$$\begin{aligned}
\| u - \overline{u}^{X^*} \|^2_{L^2(\Omega_2),\boldsymbol{\alpha}} &= \alpha_2 \, \| u - \overline{u}^{X^*} \|^2_{L^2(\Omega_2 \cup \Omega_3)} \\
&\leq \alpha_2 \, C \, \text{diam}\,(\Omega)^2 \, |u|^2_{H^1(\Omega_2 \cup \Omega_3)} \\
&\leq C \, \text{diam}\,(\Omega)^2 \left\{ \int_{\Omega_2} \alpha_2 \, |\nabla u| dx + \int_{\Omega_3} \underbrace{\alpha_2}_{\leq \alpha_3} |\nabla u| dx \right\} \\
&\leq C \, \text{diam}\,(\Omega)^2 \, |u|^2_{H^1(\Omega_2 \cup \Omega_3),\boldsymbol{\alpha}}
\end{aligned}$$

Again C depends on (the shape of) Ω_k and X^*, but **not** on $\boldsymbol{\alpha}$!

Assume $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2 \cup \overline{\Omega}_3$ (Ω_k "well-shaped")
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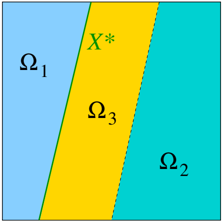
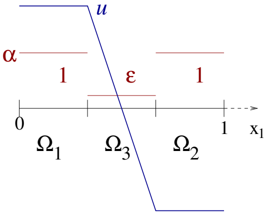


However, if $\alpha_1, \alpha_2 \gg \alpha_3$ then such an inequality **cannot** exist:



Counter example:

$\alpha_1 = \alpha_2 = 1$ and $\alpha_3 = \varepsilon \ll 1$

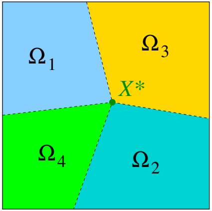
$\|u\|^2_{L^2(\Omega),\boldsymbol{\alpha}} \sim 1$

$|u|^2_{H^1(\Omega),\boldsymbol{\alpha}} \sim \varepsilon$

Model Case #3

Assume $\overline{\Omega} = \overline{\Omega}_1 \cup \cdots \cup \overline{\Omega}_4$ (Ω_k "well-shaped")
s.t. $\alpha_{|\Omega_k} = \alpha_k = \text{const}$ **(arbitrary!)**

Define **"manifold"** $X^* := \bigcup_{k=1}^4 \partial\Omega_k$
(non-empty!)



Here we can use discrete Poincaré (or Sobolev) inequalities:

Let V^h be p.w. linear FEs (quasi-uniform $\mathcal{T}^h$) and Ω_k union of a few (coarse) simplices (quasi-uniform of size $\mathcal{O}(\eta)$). Then (in 2D):

$$\|u - \overline{u}^{X^*}\|^2_{L^2(\Omega_k)} \leq C \left(1 + \log\left(\frac{\eta}{h}\right)\right) \eta^2 \, |u|^2_{H^1(\Omega_k)} \quad \forall u \in V^h(\Omega_k)$$

where $\eta := \max_k \operatorname{diam}(\Omega_k)$ and $\overline{u}^{X^*} := u(X^*)$.

Adding up $\rightsquigarrow$ **robust weighted** discrete Poincaré type inequality

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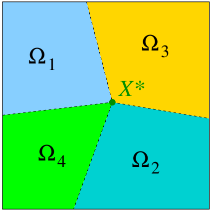


Here we can use **discrete** Poincaré (or Sobolev) inequalities:

Let V^h be p.w. linear FEs (quasi-uniform $\mathcal{T}^h$) and Ω_k union of a few (coarse) simplices (quasi-uniform of size $\mathcal{O}(\eta)$). Then (in 2D):

$$\|u - \overline{u}^{X^*}\|^2_{L^2(\Omega_k)} \leq C\left(1 + \log\left(\frac{\eta}{h}\right)\right)\eta^2 \, |u|^2_{H^1(\Omega_k)} \quad \forall u \in V^h(\Omega_k)$$

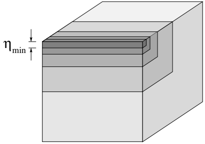
where $\eta := \max_k \operatorname{diam}(\Omega_k)$ and $\overline{u}^{X^*} := u(X^*)$.

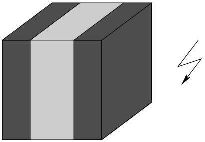
Adding up $\rightsquigarrow$ **robust weighted discrete** Poincaré type inequality

Theorem (Weighted Poincaré Ineq.) [Pechstein, RS, IMAJNA'12]

Let $x_{\max} \in \overline{\omega}$ be the point where $k(x)$ attains its maximum on $\overline{\omega}$.
If there exists a path P from every point $x \in \omega$ to $x_{\max}$ such that k never decreases along P (**quasi-monotonicity**), then there exists a constant $C^P > 0$ independent of h, $k(x)$ and $\mathrm{diam}(\omega)$ such that

$$\inf_{\gamma \in \mathbb{R}} \int_\omega \alpha(x)(v-\gamma)^2 \le C^P \, \mathrm{diam}(\omega)^2 \int_\omega \alpha(x)|\nabla v|^2 \quad \forall v \in V_h \, .$$





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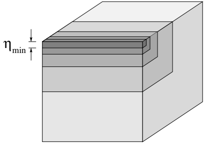

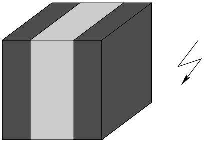

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RECALL: Main Theorem (Peterseim & RS, 2013+)

If there exists a linear, continuous quasi-interpolation operator $\mathfrak{I}_H : V_h \to V_H$ and two generic constants C_2 and C_3 such that

(QI1) $(\mathfrak{I}_H|_{V_H})^{-1}\mathfrak{I}_H v_H = v_H$, for all $v_H \in V_H$

(QI2) $H_T^{-2}\|v - \mathfrak{I}_H v\|_{0,\alpha,T}^2 + \|v - \mathfrak{I}_H v\|_{a,T}^2 \le C_2\|v\|_{a,\omega_T}^2$, for all $v \in V_h$ and $T \in \mathcal{T}_H$

(QI3) for all $v_H \in V_H$ there exists a $v \in V_h$, s.t. $\mathfrak{I}_H v = v_H$, $\operatorname{supp} v \subset \operatorname{supp} v_H$ and $\|v\|_a \le C_3\|v_H\|_a$.

then (with some universal constant $m \lesssim 1$)

$$\|u - u_{H,k}^{\mathrm{ms}}\|_a \lesssim \left(\frac{\alpha_{\max}}{\alpha_{\min}}\right)^m \frac{e^{-k}}{H}\|f\|_{H^{-1}(\Omega)} + \frac{H}{\alpha_{\min}^{-1/2}}\|f\|_{L_2(\Omega)} + \|u - u_h\|_a$$

Thus, provided $k \gtrsim \ln(\frac{\alpha_{\max}}{\alpha_{\min}}\frac{1}{H})$ and h suff'ly small we have **optimal** $\mathcal{O}(H)$ convergence without assumptions on regularity or contrast.

Again, $\mathcal{O}(H^2)$ convergence in L^2-norm follows by an Aubin-Nitsche argument.

Assumptions (QI1) and (QI3)

- (QI1): Let $v_H := \sum_j \gamma_j \Phi_j^H \in V_H$. Then $\mathfrak{I}_H v_H = \sum_j (\tilde{M}\gamma)_j \Phi_j^H$ where $\tilde{M}$ is a scaled mass matrix on V_H which is invertible.
- (QI3) is more difficult, but under the above assumptions on the coefficient (i.e. p.w. const. w.r.t. $\mathcal{T}_\eta$), it can be shown similar to Lemma 1 in [Malqvist, Peterseim '12] with $C_3 \eqsim \left(\frac{H}{\eta}\right)^2$.

In summary, we do get **optimal, contrast independent** convergence rates, but so far only under **fairly stringent** assumptions on the type of coefficient variation (i.e. locally quasi-monotone & p.w. constant w.r.t. $\mathcal{T}_\eta$ for moderate H/η)

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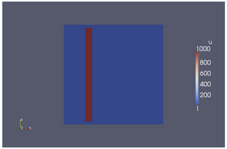

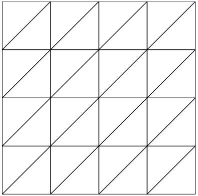

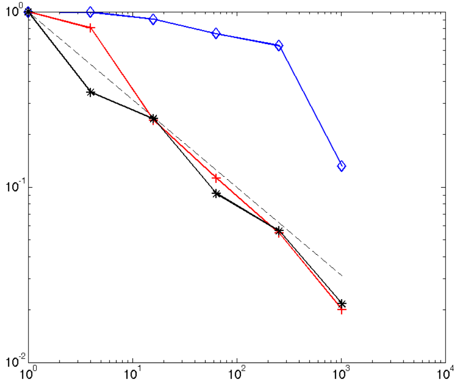

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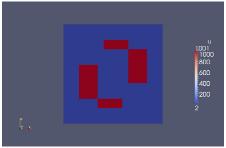
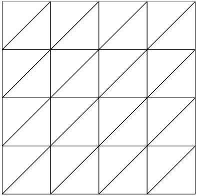
$h = 2^{-7}$, $k = 2$

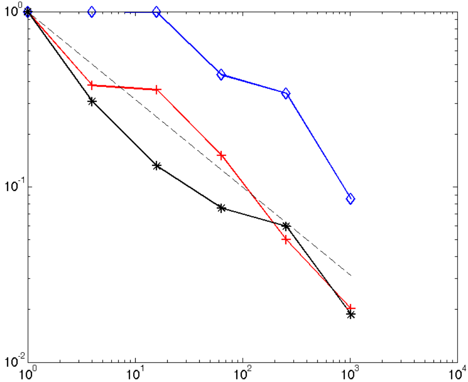
$$\frac{|u - u^{\text{ms}}_{H,k}|_{H^1(\Omega)}}{|u|_{H^1(\Omega)}} \text{ vs. } \#\text{dofs}$$

(black = unweighted; red = weighted)

Numerical Experiment II







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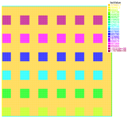

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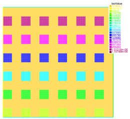

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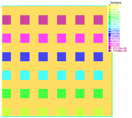

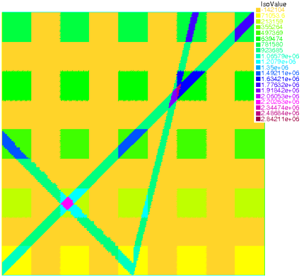

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Local energy minimising coarse spaces (incl. GMsFEM)

- Suppose $\{\Omega_\ell\}_{\ell=1}^L$ is overlapping partition of Ω.

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For each subdomain Ω_ℓ, assume that we have a collection of **linear functionals** $\{f_{\ell,j}\}_{j=1}^{m_\ell} \subset V_h(\Omega_\ell)'$ and let

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- Now define global coarse space

$$V_H = \mathrm{span}\left\{\Phi_{\ell,j} := I_h\left(\chi_\ell \Psi_{\ell,j}\right) : \ell = \overline{1,L},\ j = \overline{1,m_\ell}\right\}$$

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An abstract Bramble–Hilbert Lemma

Suppose $V \subset \mathcal{H}$ with **Hilbert** space $(\mathcal{H}, \|\cdot\|)$, $a(\cdot,\cdot)$ an abstract **symmetric** continuous bilinear form on $V \times V$ and $\{f_k\}_{k=1}^m \subset V'$.

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(Note that this is **independent** *of the constants* c_q *and* c_f *in* **A2** *and* **A3***.)*

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Conclusions & Outlook

- Upscaling for model elliptic problem at high contrast
  (extends to more complicated problems)
- Analysis difficult without scale separation and at high contrast
- Robust variational multiscale method
  (analysis for locally quasi-monotone coefficients)
- **Philosophy & New theoretical tools:**
  - Start with V_H that has uniform, stable L^2-approx. properties
  - Find stable, weighted quasi-interpolator (weighted Poincaré)
  - a-orthogonalise basis fcts. (local) $\mapsto$ new multiscale space V_H^{ms}
  - Leads to uniform H-optimal convergence (in energy & L^2)
  - Bramble-Hilbert: uniform L^2-approx. with V_H = GMsFEM
    (for general coefficients)
  - Question is how to (formulate and) prove (QI2-4) in that case.
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References 5-7 are also available as preprints on my website:

`http://people.bath.ac.uk/∼masrs/publications.html`