Non-homogeneous random walks on a semi-infinite strip

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Joint work with Andrew Wade

Aspects of Random Walks
1st April 2014
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Simple random walk

Let $X_n$ be symmetric simple random walk (SRW) on $\mathbb{Z}^d$, i.e., given $X_1, \ldots, X_n$, the new location $X_{n+1}$ is uniformly distributed on the $2d$ adjacent lattice sites to $X_n$.

**Theorem (Pólya, 1921)**

*SRW is recurrent if $d = 1$ or $d = 2$, but transient if $d \geq 3$.*

Several proofs are available, typically using combinatorics or electrical network theory, but these classical approaches are of limited use if one wants to generalise or perturb the model slightly.
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Lamperti (1960) gave a very robust approach, based on the method of Lyapunov functions. Idea: reduce to a 1-dimensional problem by taking $Z_n = \|X_n\|$. 
Lamperti’s problem

$X_n = 0$ if and only if $Z_n = 0$.

But $Z_n$ is not homogeneous (and not even Markov). However, $Z_n$ is a stochastic process with asymptotically zero drift.

Lamperti investigated the asymptotic behaviour of these non-homogeneous random walks on $\mathbb{Z}_+$. He studied in detail how the asymptotic behaviour of the random walk is determined by the first two moment functions $\mu_1(z)$ and $\mu_2(z)$ of its increments.

Here, $\mu_k(z) = \mathbb{E}[(Z_{n+1} - Z_n)^k \mid Z_n = z]$. 
Lamperti’s problem

**Theorem (Lamperti)**

Let \((Z_n)\) be an irreducible time-homogeneous Markov chain on \(\mathbb{Z}_+\). Suppose that there exists \(\varepsilon > 0\) such that

\[
\sup_z \mathbb{E}[|Z_{n+1} - Z_n|^{2+\varepsilon} \mid Z_n = z] < \infty;
\]

\[
\liminf_{z \to \infty} \mathbb{E}[|Z_{n+1} - Z_n|^2 \mid Z_n = z] > 0.
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- If \(\liminf_{z \to \infty} (2z\mu_1(z) - \mu_2(z)) > 0\), then \(Z_n\) is transient.
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- If \(\lim_{z \to \infty} \inf (2z\mu_1(z) - \mu_2(z)) > 0\), then \(Z_n\) is transient.
- If \(2z\mu_1(z) \leq \mu_2(z) + O(z^{-\delta})\), for some \(\delta > 0\), then \(Z_n\) is null-recurrent.
Lamperti’s problem

Theorem (Lamperti)

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- If $\liminf_{z \to \infty} \left(2z\mu_1(z) - \mu_2(z)\right) > 0$, then $Z_n$ is transient.
- If $|2z\mu_1(z)| \leq \mu_2(z) + O(z^{-\delta})$, for some $\delta > 0$, then $Z_n$ is null-recurrent.
- If $\limsup_{z \to \infty} \left(2z\mu_1(z) + \mu_2(z)\right) < 0$, then $Z_n$ is positive-recurrent.
Lamperti’s classification

Typically, the result is applied when the drift \( \mu_1(x) \) is asymptotically zero, decaying as \( 1/z \) as \( z \to \infty \) and \( \mu_2(z) \) is asymptotically constant (and nonzero).

In particular, for \( \mu_1(z) = c/z + o(z^{-1}) \) and \( \mu_2(z) = s^2 + o(1) \), the results tell us that

- \( Z_n \) is transient for \( 2c > s^2 \),
- \( Z_n \) is null-recurrent for \( -s^2 < 2c < s^2 \),
- \( Z_n \) is positive-recurrent for \( 2c < -s^2 \).
Non-homogeneous RW on semi-infinite strip

\((X_n, \eta_n)\) — irreducible Markov chain on \(\mathbb{Z}_+ \times S\) for \(S\) finite
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- $(X_n, \eta_n)$ — irreducible Markov chain on $\mathbb{Z}_+ \times S$ for $S$ finite
- Chain is time-homogeneous, **non-homogeneous** in space
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We can view \(S\) as a set of internal states, influencing motion on \(\mathbb{Z}_+.\) E.g.,
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- regime-switching processes ($S$ contains market information)
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  - modulated queues (e.g., \(S =\) states of servers)
  - regime-switching processes (\(S\) contains market information)
  - physical processes with internal degrees of freedom (\(S =\) energy/momentum states of particle)
Model assumptions

Moments bound on jumps of $X_n$

$$(B_p) \quad \exists \ C_p < \infty \text{ s.t. } \mathbb{E}[|X_{n+1} - X_n|^p \mid \mathcal{F}_n] \leq C_p$$
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For this talk, we assume $(B_p)$ holds for some $p > 2$. 
Model assumptions

$\eta_n$ is “close to being Markov” when $X_n$ is large

Define

\[ p(x, i, y, j) = \mathbb{P}[(X_{n+1}, \eta_{n+1}) = (y, j) \mid (X_n, \eta_n) = (x, i)] \]

\[ q_x(i, j) = \sum_{y \in \mathbb{Z}_+} p(x, i, y, j) \]
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$$q_x(i, j) = \sum_{y \in \mathbb{Z}_+} p(x, i, y, j)$$

$$(Q_{\infty}) \quad q(i, j) = \lim_{x \to \infty} q_x(i, j)$$

exists for all $i, j \in S$ and $(q(i, j))$ is irreducible

Markov chain with transition probabilities $q(i, j)$ is irreducible on finite state space $S$, so it has a stationary distribution $\pi$ satisfying

$$\pi(j) = \sum_{i \in S} \pi(i)q(i, j)$$

for all $j \in S$. 
Model assumptions

Lamperti-type moment conditions

Define

$$\mu_k(x, i) = \mathbb{E}[(X_{n+1} - X_n)^k \mid (X_n, \eta_n) = (x, i)]$$

(ML)  \exists \ c_i, s_i \in \mathbb{R} \text{ for all } i \in S \text{ (at least one } s_i \text{ nonzero) such that}

$$\mu_1(x, i) = \frac{c_i}{x} + o(x^{-1}); \quad \mu_2(x, i) = s_i^2 + o(1).$$
Recurrence/transience of $X_n$

With these three assumptions $(B_p)$, $(Q_{\infty})$, $(M_L)$, we can give conditions that imply the recurrence or transience of $X_n$.

Note: $X_n$ not assumed to be Markov — need to define what we mean by recurrence/transience of $X_n$. Here, finiteness of $S$ helps.
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\((X_n, \eta_n)\) is an irreducible Markov chain, so is either recurrent or transient. Moreover,

**Lemma**

(i) If \((X_n, \eta_n)\) is recurrent, then $\mathbb{P}[X_n = 0 \text{ i.o.}] = 1$.

(ii) If \((X_n, \eta_n)\) is transient, then $\mathbb{P}[X_n = 0 \text{ i.o.}] = 0$, and $X_n \rightarrow \infty$ a.s.
Null- vs. positive-recurrence of $X_n$

We can also define null- and positive-recurrence of $X_n$:

**Lemma**

*There exists a (unique) measure $\nu$ on $\mathbb{Z}_+$ such that*

$$
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} 1\{X_k = x\} = \nu(x) \text{ a.s.,}
$$

*for all $x \in \mathbb{Z}_+$.*

(i) *If $(X_n, \eta_n)$ is null, then $\nu(x) = 0$ for all $x \in \mathbb{Z}_+$.*

(ii) *If $(X_n, \eta_n)$ is positive-recurrent, then $\nu(x) > 0$ for all $x \in \mathbb{Z}_+$ and $\sum_{x \in \mathbb{Z}_+} \nu(x) = 1$.***
Recurrence classification of $X_n$

**Theorem (G., Wade, 2014)**

Suppose that $(B_p)$ holds for some $p > 2$ and conditions $(Q_\infty)$ and $(M_L)$ hold. The following sufficient conditions apply.

- If $\sum_{i \in S} (2c_i - s_i^2)\pi(i) > 0$, then $X_n$ is transient.
- If $|\sum_{i \in S} 2c_i\pi(i)| < \sum_{i \in S} s_i^2\pi(i)$, then $X_n$ is null-recurrent.
- If $\sum_{i \in S} (2c_i + s_i^2)\pi(i) < 0$, then $X_n$ is positive-recurrent.

[With better error bounds in $(Q_\infty)$ and $(M_L)$ we can also show that the boundary cases are null-recurrent.]

This generalises Lamperti’s results for walks on $\mathbb{Z}_+$ (the case of $S$ a singleton).
Label an arbitrary state $0 \in S$.
Define $\tau_0 = \min \{ n \in \mathbb{Z}_+ : \eta_n = 0 \}$ and for $m \geq 0$ set $\tau_{m+1} = \min \{ n > \tau_m : \eta_n = 0 \}$.
(Conditions $(B_p)$ and $(Q_\infty)$ imply $\tau_m < \infty$ for all $m$.)

Embedded process: $Y_n = X_{\tau_n}$ on state space $\mathbb{Z}_+$
Properties of $Y_n$ and $\tau_n$

$Y_n$ is an irreducible Markov chain.

$\tau_{n+1} - \tau_n$ conditional on $Y_n = x$ is independent of $n$. 
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Set $\tau = \min\{n > 0 : \eta_n = 0\}$.
Then $\tau_{n+1} - \tau_n$ conditional on $Y_n = x$ has the same distribution as $\tau$ conditional on $(X_0, \eta_0) = (x, 0)$.
This random variable is “well-behaved”: it has exponential tails and all moments of $\tau$ are finite.
Properties of $Y_n$ and $\tau_n$

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$(X_n)$ recurrent if and only if $(Y_n)$ recurrent.

$(X_n)$ positive-recurrent if and only if $(Y_n)$ positive-recurrent.
Hence our recurrence classification will follow from an application of Lamperti’s result to \( Y_n \).

We need to calculate \( \mathbb{E}[(Y_{n+1} - Y_n)^k \mid Y_n = x] \), for \( k = 1, 2 \).

Enough to calculate \( \mathbb{E}[(X_\tau - X_0)^k \mid (X_0, \eta_0) = (x, 0)] \). For this we use the Doob decomposition of \( X_n \).
Doob decomposition of $X_n$

Write

$$X_n - X_0 = M_n + \sum_{k=0}^{n-1} \mathbb{E}[X_{k+1} - X_k \mid X_k, \eta_k],$$

where $M_n$ is a martingale with $M_0 = 0$. Using the definition of $\mu_1(x, i)$,

$$X_n - X_0 = M_n + \sum_{k=0}^{n-1} \mu_1(X_k, \eta_k)$$

$$= M_n + \sum_{i \in S} \sum_{k=0}^{n-1} \mu_1(X_k, i) \mathbf{1}\{\eta_k = i\}$$
Moment calculations

So,

\[ X_\tau - X_0 = M_\tau + \sum_{i \in S} \sum_{k=0}^{\tau-1} \mu_1(X_k, i) 1\{\eta_k = i\} \]

Optional Stopping Theorem: \( \mathbb{E}[M_\tau \mid (X_0, \eta_0) = (x, 0)] = M_0 = 0 \).
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\[ \mathbb{E}_{x,0}[X_\tau - X_0] = \sum_{i \in S} \mathbb{E}_{x,0} \left[ \sum_{k=0}^{\tau-1} \mu_1(X_k, i)1\{\eta_k = i\} \right] \]

where \( \mathbb{E}_{x,0}[\cdot] \) is short for \( \mathbb{E}[\cdot \mid (X_0, \eta_0) = (x, 0)] \).
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\[ = \sum_{i \in S} \mathbb{E}_{x,0} \left[ \sum_{k=0}^{\tau-1} \mu_1(x, i) 1\{\eta_k = i\} \right] + o(x^{-1}), \]

where \( \mathbb{E}_{x,0}[\cdot] \) is short for \( \mathbb{E}[\cdot \mid (X_0, \eta_0) = (x, 0)] \).
Moment calculations

We need one more approximation:

\[ \mathbb{E}_{x,0} \left[ \sum_{k=0}^{\tau-1} 1\{\eta_k = i\} \right] = \frac{\pi(i)}{\pi(0)} + o(1). \]
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Combining these with \( \mu_1(x, i) = c_i/x + o(x^{-1}) \) we get

\[
\mathbb{E}[X_\tau - X_0 \mid (X_0, \eta_0) = (x, 0)] = \frac{1}{\pi(0)} \sum_{i \in S} \frac{c_i \pi(i)}{x} + o(x^{-1}).
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Combining these with $\mu_1(x, i) = c_i/x + o(x^{-1})$ we get

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\mathbb{E}[X_{\tau} - X_0 \mid (X_0, \eta_0) = (x, 0)] = \frac{1}{\pi(0)} \sum_{i \in S} \frac{c_i \pi(i)}{x} + o(x^{-1}).
$$

Similar reasoning using the Doob decomposition for $X_n^2$ yields the second moment:

$$
\mathbb{E}[(X_{\tau} - X_0)^2 \mid (X_0, \eta_0) = (x, 0)] = \frac{1}{\pi(0)} \sum_{i \in S} s_i^2 \pi(i) + o(1).
$$
Moments for $Y_n$

In terms of $Y_n$ we have:

**Lemma**

\[
\mathbb{E}[Y_{n+1} - Y_n \mid Y_n = x] = \frac{1}{\pi(0)} \sum_{i \in S} \frac{c_i \pi(i)}{x} + o(x^{-1});
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Recurrence classification

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Suppose that \((B_p)\) holds for some \(p > 2\) and conditions \((Q_\infty)\) and \((M_L)\) hold. The following sufficient conditions apply.

- If \(\sum_{i \in S} (2c_i - s_i^2) \pi(i) > 0\), then \(X_n\) is transient.
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- If \(\sum_{i \in S} (2c_i + s_i^2) \pi(i) < 0\), then \(X_n\) is positive-recurrent.
The missing details

The proof relied on the following:

- Random variable $\tau$ has exponential tails.
- Control of $X_k - X_0$ for $k \leq \tau$.
- $\lim_{x \to \infty} E_{x,0} \sum_{k=0}^{\tau-1} 1\{\eta_k = i\} = \frac{\pi(i)}{\pi(0)}$

All these follow from a coupling of $(X_n, \eta_n)$ with $(\eta_n^*)$ the Markov chain on $S$ with transition matrix $(q(i,j))$. 
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All these follow from a coupling of $(X_n, \eta_n)$ with $(\eta_n^*)$ the Markov chain on $S$ with transition matrix $(q(i,j))$.

E.g. if $\tau^* = \min\{n > 0 : \eta_n^* = 0\}$, then conditional on $\eta_n$ and $\eta_n^*$ remaining coupled up to time $m$ we have $\tau \leq m$ if and only if $\tau^* \leq m$. 
Example: persistent random walk on $\mathbb{Z}_+$

Nearest-neighbour random walk $(X_n)$ on $\mathbb{Z}_+$ where the distribution of $X_{n+1}$ depends on the current position $X_n$ and the current direction $X_n - X_{n-1}$. Setting $\eta_n = X_n - X_{n-1}$, we can model this as a Markov chain $(X_n, \eta_n)$ on $\mathbb{Z}_+ \times S$, where $S = \{+1, -1\}$. For $c > 0$ the walk has a marginal preference to continue in the positive direction, and a marginal aversion to continuing in the negative direction. (For large $x$ the local behaviour is approx like SRW on $\mathbb{Z}_+$.)
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Nonzero transition probabilities are $p(x, i, x + j, j) = q_x(i, j)$ with

$$q_x(i, j) = \begin{cases} \frac{1}{2} + \frac{ic}{2x} + o(x^{-1}) & \text{if } j = i \\ \frac{1}{2} - \frac{ic}{2x} + o(x^{-1}) & \text{if } j \neq i \end{cases}$$

For $c > 0$ the walk has a marginal preference to continue in the positive direction, and a marginal aversion to continuing in the negative direction. (For large $x$ the local behaviour is approx like SRW on $\mathbb{Z}_+$.)
Persistent random walk on $\mathbb{Z}_+$

We calculate the moments

$$\mu_1(x, i) = \frac{c}{x} + o(x^{-1}) \text{ and } \mu_2(x, i) = 1 \text{ for } i \in S.$$ 

Hence, our results tell us that

- $X_n$ is transient if $c > 1/2$,
- $X_n$ is null-recurrent if $-1/2 < c < 1/2$,
- $X_n$ is positive-recurrent if $c < -1/2$. 

![Durham University logo]