Seiberg-Witten Theories on Ellipsoids

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Introduction

AGT relation (2009) : a correspondence between

2D CFTs (Liouville / Toda) ⟷ **4D N=2 SUSY (SW) theories**

-- coupling $b=1$

-- general coupling?

-- round 4-sphere

-- deformed 4-spheres?

cf: Liouville CFT

$$\mathcal{L} = \partial\phi\bar{\partial}\phi + e^{2b\phi}$$

$$c = 1 + 6Q^2;\ \ Q \equiv b + \frac{1}{b}$$

Claim: SW theories on ellipsoid backgrounds,

$$\frac{x_0^2}{r^2} + \frac{x_1^2 + x_2^2}{\ell^2} + \frac{x_3^2 + x_4^2}{\tilde{\ell}^2} = 1$$

(with some additional background fields,)

reproduce Liouville/Toda correlators for general coupling $b \equiv \sqrt{\ell/\tilde{\ell}}$.

Plan:

* 4D N=2 Killing spinor equation
* SW theory on curved space
* SUSY on ellipsoids
* partition function

1. 4D N=2 Killing Spinor Equation

Killing Spinors (KS)

. . . characterize rigid SUSY on curved backgrounds.

[Example] Killing spinors on n-sphere satisfy

[main equation]

$$D_m\epsilon \equiv \left(\partial_m + \frac{1}{4}\omega_m^{ab}\Gamma^{ab}\right)\epsilon \;=\; \Gamma_m\tilde{\epsilon} \qquad (1)$$

[auxiliary equation]

automatic ⇒

$$(\Gamma^m D_m)^2\epsilon = -\frac{n^2}{4\ell^2}\epsilon \qquad (2)$$

On less-symmetric spheres, (1) have no solutions.

We need to generalize the KS equation.

[Example] **3D Ellipsoids**

$$\frac{x_1^2 + x_2^2}{\ell^2} + \frac{x_3^2 + x_4^2}{\tilde{\ell}^2} = 1 \,.$$

When $\ell = \tilde{\ell}$, there is a pair of KSs $\epsilon_\pm$ satisfying

$$\Big(\partial_m + \frac{1}{4}\omega_m^{ab}\Gamma^{ab}\Big)\epsilon_\pm = -\frac{i}{2\ell}\Gamma_m\epsilon_\pm$$

After squashing they satisfy

$$\Big(\partial_m + \frac{1}{4}\omega_m^{ab}\Gamma^{ab} \mp iV_m\Big)\epsilon_\pm = -\frac{i}{2f}\Gamma_m\epsilon_\pm \,.$$

suitably chosen background fields

They were used to formulate 3D N=2 theories on ellipsoids. (Hama-KH-Lee '11)

For **SW theories on 4D ellipsoids**,

we look for KSs satisfying pseudo-reality

$$\xi \equiv (\xi_{\alpha A}, \bar{\xi}_{\dot{\alpha} A})$$

$$(\xi_{\alpha A})^* = \epsilon^{\alpha\beta}\epsilon^{AB}\xi_{\beta B}$$

$$(\bar{\xi}_{\dot{\alpha} A})^* = \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{AB}\bar{\xi}_{\dot{\beta} B}$$

Indices

$\alpha = 1, 2$: chiral spinor

$\dot{\alpha} = 1, 2$: anti-chiral spinor

$A = 1, 2$: N=2 SUSY

Expectation: SUSY on 4D ellipsoids requires turning on

$SU(2)_R$ gauge field

$$D_m \xi_A \equiv \partial_m \xi_A + \frac{1}{4}\omega_m^{ab}\sigma^{ab}\xi_A + i\xi_B V_m{}^B{}_A$$

$$D_m \bar{\xi}_A \equiv \partial_m \bar{\xi}_A + \frac{1}{4}\omega_m^{ab}\bar{\sigma}^{ab}\bar{\xi}_A + i\bar{\xi}_B V_m{}^B{}_A$$

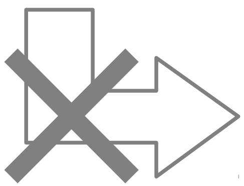
It turned out we need more background fields.

[main equation]

$$D_m \xi_A + T^{kl} \sigma_{kl} \sigma_m \bar{\xi}_A = -i \sigma_m \bar{\xi}'_A,$$
$$D_m \bar{\xi}_A + \bar{T}^{kl} \bar{\sigma}_{kl} \bar{\sigma}_m \xi_A = -i \bar{\sigma}_m \xi'_A,$$

T^{kl} : ASD 2-form

$\bar{T}^{kl}$: SD 2-form



does not look automatic

[auxiliary equation]

$$\sigma^m \bar{\sigma}^n D_m D_n \xi_A + 4 D_l T_{mn} \cdot \sigma^{mn} \sigma^l \bar{\xi}_A = M \xi_A,$$
$$\bar{\sigma}^m \sigma^n D_m D_n \bar{\xi}_A + 4 D_l \bar{T}_{mn} \cdot \bar{\sigma}^{mn} \bar{\sigma}^l \xi_A = M \bar{\xi}_A,$$

M : scalar

-- In 4D N=2 SUGRA, these background fields appear as auxiliary fields in gravity multiplet.

De Wit, van Holten, van-Proeyen, '80

2. SW Theories on Curved Space

Vector Multiplet

$$(A_m,\ \phi,\ \bar{\phi},\ \lambda_{\alpha A},\ \bar{\lambda}_{\dot{\alpha} A},\ D_{AB})$$

[SUSY]

$$\mathbf{Q}A_m = i\xi^A \sigma_m \bar{\lambda}_A - i\bar{\xi}^A \bar{\sigma}_m \lambda_A,$$

$$\mathbf{Q}\phi = -i\xi^A \lambda_A,$$

$$\mathbf{Q}\bar{\phi} = +i\bar{\xi}^A \bar{\lambda}_A,$$

$$\cdots\cdot$$

Under the assumption

1. fields and KS are pseudoreal,
2. Q preserves the pseudoreality,

SUSY rule & action can be found by just dialing real coefficients.

Note: pseudoreality $\Rightarrow$ $\phi^\dagger = \phi,\ \bar{\phi}^\dagger = \bar{\phi}$

Vector Multiplet

$$(A_m,\ \phi,\ \bar{\phi},\ \lambda_{\alpha A},\ \bar{\lambda}_{\dot{\alpha} A},\ D_{AB})$$

[Action]

$$\mathcal{L}_{\rm YM} = {\rm Tr}\Big[\frac{1}{2}F_{mn}^2 + 16F_{mn}(\bar{\phi}T^{mn} + \phi\bar{T}^{mn}) + 64\bar{\phi}^2 T_{mn}^2 + 64\phi^2\bar{T}_{mn}^2$$
$$-4D_m\bar{\phi}D^m\phi + 2M\bar{\phi}\phi + 4[\phi,\bar{\phi}]^2$$
$$-\frac{1}{2}D^{AB}D_{AB} + ({\rm fermions})\Big]$$

Note: $\phi^\dagger = \phi,\ \ \bar{\phi}^\dagger = \bar{\phi}$ ⇒ $\mathcal{L}_{\rm YM}$ unbounded!

We need to rotate the integration contour for some fields by 90 degrees.

Hypermultiplet

$$q_{AI},\ \psi_{\alpha I},\ \bar{\psi}_{\dot{\alpha} I},\ F_{AI}$$

$I = 1, \cdots, 2r$: repr. index of gauge symmetry.

Pseudoreality:

$(q_{AI})^{\dagger} = \epsilon^{AB}\Omega^{IJ}q_{BJ}$ $\quad\quad$ Ω^{IJ} : *Sp(r)* invariant tensor

Off-shell SUSY:

For 4D N=2 hypermultiplets,

one cannot realize all the 8 SUSYs off-shell at once,

BUT <u>any one of them</u> can be realized off-shell.

[Example] Free hypermultiplets on flat space

[action]

$$\mathcal{L}_{\rm mat} = \partial_m q^{AI} \partial^m q_{AI} - i\bar{\psi}^I \bar{\sigma}^m \partial_m \psi_I$$

[SUSY]

$$\mathbf{Q} q_{AI} = -i\xi_A \psi_I + i\bar{\xi}_A \bar{\psi}_I,$$

$$\mathbf{Q} \psi_I = 2\sigma^m \bar{\xi}_A \partial_m q_I^A$$

$$\mathbf{Q} \bar{\psi}_I = 2\bar{\sigma}^m \xi_A \partial_m q_I^A$$

$\mathbf{Q}^2(\text{field}) = 2i\bar{\xi}^A \bar{\sigma}^m \xi_A \cdot \partial_m(\text{field})$ on all the fields **up to EOM.**

[Example] Free hypermultiplets on flat space

[action]

$$\mathcal{L}_{\rm mat} = \partial_m q^{AI}\partial^m q_{AI} - i\bar{\psi}^I\bar{\sigma}^m\partial_m\psi_I - F^{AI}F_{AI}$$

[SUSY]

$$\mathbf{Q}q_{AI} = -i\xi_A\psi_I + i\bar{\xi}_A\bar{\psi}_I,$$

$$\mathbf{Q}\psi_I = 2\sigma^m\bar{\xi}_A\partial_m q_I^A + 2\check{\xi}_A F_I^A,$$

$$\mathbf{Q}\bar{\psi}_I = 2\bar{\sigma}^m\xi_A\partial_m q_I^A + 2\bar{\check{\xi}}_A F_I^A,$$

$$\mathbf{Q}F_{AI} = i\check{\xi}_A\sigma^m\partial_m\bar{\psi}_I - i\bar{\check{\xi}}_A\bar{\sigma}^m\partial_m\psi_I$$

$$\mathbf{Q}^2(\text{field}) = 2i\bar{\xi}^A\bar{\sigma}^m\xi_A\cdot\partial_m(\text{field})$$ on all the fields **off-shell**

provided "$\check{\xi}$ is orthogonal to ξ".

“Orthogonality”

$$\xi_A \check{\xi}_B - \bar{\xi}_A \bar{\check{\xi}}_B = 0,$$

$$\xi^A \xi_A + \bar{\check{\xi}}^A \bar{\check{\xi}}_A = 0,$$

$$\bar{\xi}^A \bar{\xi}_A + \check{\xi}^A \check{\xi}_A = 0,$$

$$\xi^A \sigma^m \bar{\xi}_A + \check{\xi}^A \sigma^m \bar{\check{\xi}}_A = 0.$$

For any given ξ, the choice of $\check{\xi}$ is unique up to local SU(2) rotations.

$\check{\xi}_A,\ \bar{\check{\xi}}_A,\ F_A$: doublets under $SU(2)_{\check{\mathrm{R}}}$

(summary) Actions

$$\mathcal{L}_{\rm YM} = \mathrm{Tr}\Big[\frac{1}{2}F_{mn}F^{mn} + 16F_{mn}(\bar\phi T^{mn} + \phi\bar T^{mn}) + 64\bar\phi^2 T^2_{mn} + 64\phi^2\bar T^2_{mn}$$

$$-4D_m\phi D^m\bar\phi + 2M\bar\phi\phi - 2i\lambda^A\sigma^m D_m\bar\lambda_A - 2\lambda^A[\bar\phi,\lambda_A] + 2\bar\lambda^A[\phi,\bar\lambda_A]$$

$$+4[\phi,\bar\phi]^2 - \frac{1}{2}D^{AB}D_{AB}\Big]$$

$$\mathcal{L}_{\rm mat} = \frac{1}{2}D_m q^A D^m q_A - q^A\{\phi,\bar\phi\}q_A + \frac{i}{2}q^A D_{AB}q^B + \frac{1}{8}(R+M)q^A q_A$$

$$-\frac{i}{2}\bar\psi\bar\sigma^m D_m\psi - \frac{1}{2}\psi\phi\psi + \frac{1}{2}\bar\psi\bar\phi\bar\psi + \frac{i}{2}\psi\sigma^{kl}T_{kl}\psi - \frac{i}{2}\bar\psi\bar\sigma^{kl}\bar T_{kl}\bar\psi$$

$$-q^A\lambda_A\psi + \bar\psi\bar\lambda_A q^A - \frac{1}{2}F^A F_A$$

$$\mathcal{L}_{\rm FI} \equiv w^{AB} D_{AB} - M(\phi + \bar{\phi})$$

$$-64\phi T^{kl} T_{kl} - 64\bar{\phi}\bar{T}^{kl}\bar{T}_{kl} - 8F^{kl}(T_{kl} + \bar{T}_{kl})\,.$$

A bilinear of Killing spinor

3. SUSY on Ellipsoids

Strategy :

1. choose a nice KS on round 4-sphere: $(\xi_A, \bar{\xi}_A)$
2. introduce squashing (deform the metric),

   while requiring $(\xi_A, \bar{\xi}_A)$ to remain KS

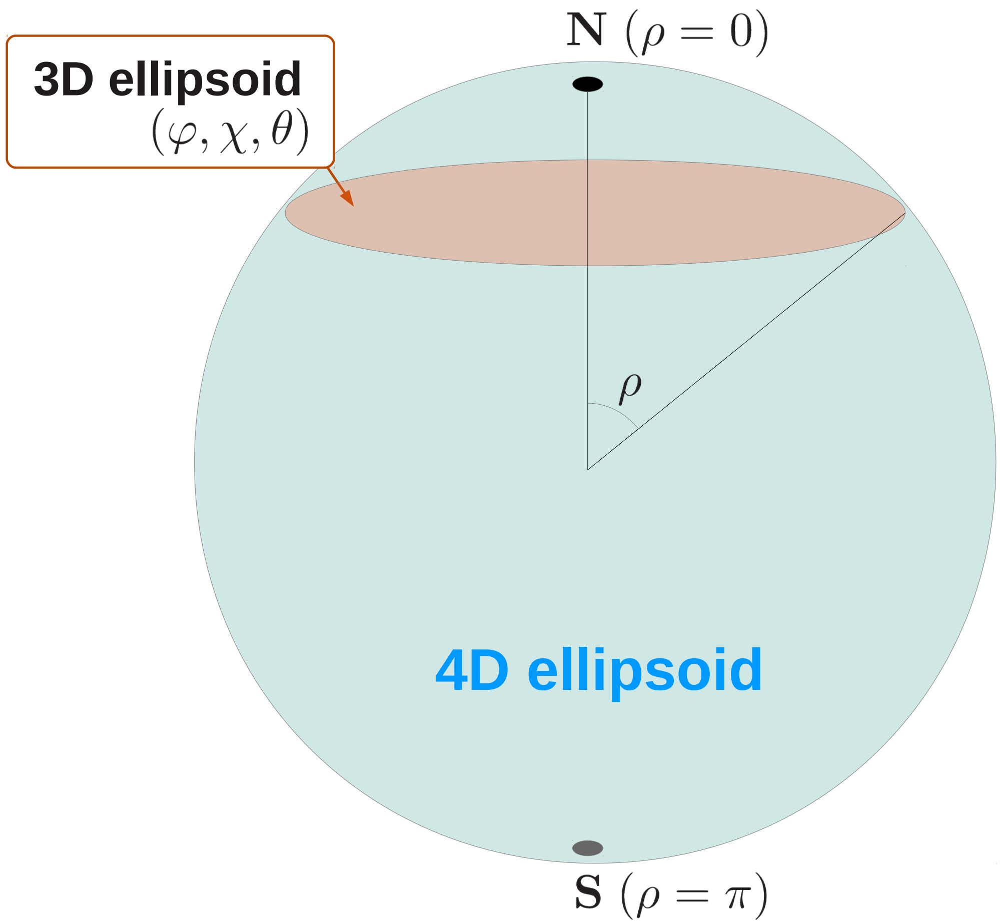
⇨ Determine the background fields

$$(T_{kl},\ \bar{T}_{kl},\ V_m{}^A{}_B,\ M)$$

Polar Coordinates

$$\frac{x_0^2}{r^2} + \frac{x_1^2 + x_2^2}{\ell^2} + \frac{x_3^2 + x_4^2}{\tilde{\ell}^2} = 1$$

$$x_0 = r \cos\rho,$$
$$x_1 = \ell \sin\rho \cos\theta \cos\varphi,$$
$$x_2 = \ell \sin\rho \cos\theta \sin\varphi,$$
$$x_3 = \tilde{\ell} \sin\rho \sin\theta \cos\chi,$$
$$x_4 = \tilde{\ell} \sin\rho \sin\theta \sin\chi,$$



φ : rotation angle about (x_1, x_2)-plane

χ (x_3, x_4)-plane

Round 4-Sphere

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \sin^2\rho \cdot \mathrm{d}s^2_{S^3} \;=\; E^a E^a,$$

$E^a = \sin\rho \cdot e^a \;\; (a = 1, 2, 3),$ (e^a: vielbein on 3-sphere)

$E^4 = \mathrm{d}\rho$

We see separation of variables in KS equation.

A nice solution satisfying pseudoreality etc. is

$$\xi_{\alpha A}\Big|_{A=1} = \sin\tfrac{\rho}{2} \cdot \epsilon_+, \qquad \bar{\xi}^{\dot{\alpha}}_{A}\Big|_{A=1} = \cos\tfrac{\rho}{2} \cdot i\epsilon_+,$$

$$\xi_{\alpha A}\Big|_{A=2} = \sin\tfrac{\rho}{2} \cdot \epsilon_-, \qquad \bar{\xi}^{\dot{\alpha}}_{A}\Big|_{A=2} = \cos\tfrac{\rho}{2} \cdot (-i\epsilon_-).$$

($\epsilon_\pm$: a pair of KSs on round 3-sphere)

4D Ellipsoids

$$\frac{x_0^2}{r^2} + \frac{x_1^2 + x_2^2}{\ell^2} + \frac{x_3^2 + x_4^2}{\tilde{\ell}^2} = 1$$

$$\begin{aligned} \mathrm{d}s^2 &= E^a E^a, \\ E^1 &= \sin\rho\, e^1 \\ E^2 &= \sin\rho\, e^2 \\ E^3 &= \sin\rho\, e^3 + h(\theta)\,\mathrm{d}\rho \\ E^4 &= g(\rho,\theta)\mathrm{d}\rho \end{aligned}$$

e^a : vielbein on 3D ellipsoid

$$\begin{aligned} \mathrm{d}s^2 = \ & \ell^2 \cos^2\theta \mathrm{d}\varphi^2 \\ & + \tilde{\ell}^2 \sin^2\theta \mathrm{d}\chi^2 \\ & + (\ell^2 \sin^2\theta + \tilde{\ell}^2 \cos^2\theta)\mathrm{d}\theta^2 \end{aligned}$$

We solve the KS equation, with our nice solution inserted,

in favor of the background fields $(T_{kl},\ \bar{T}_{kl},\ V_m{}^A{}_B,\ M)$

Result (Hama-KH '12)

A family of ellipsoid background was found,

for which the background fields $(T_{kl},\ \bar{T}_{kl},\ V_m{}^A{}_B,\ M)$

depend on 3 arbitrary functions

$$c_1(\rho,\theta),\ c_2(\rho,\theta),\ c_3(\rho,\theta).$$

The auxiliary fields

$$i\mathbf{T} \equiv \sigma_{kl} T^{kl}, \quad i\bar{\mathbf{T}} \equiv \bar{\sigma}_{kl} \bar{T}^{kl}$$

$$\mathbf{T} = \frac{1}{4}\Big(\frac{1}{f} - \frac{1}{g}\Big)\boldsymbol{\tau}^1_\theta + \frac{h}{4fg}\boldsymbol{\tau}^2_\theta + \tan\frac{\rho}{2}\left(+c_1\boldsymbol{\tau}^1_\theta + c_2\boldsymbol{\tau}^2_\theta + c_3\boldsymbol{\tau}^3\right),$$

$$\bar{\mathbf{T}} = \frac{1}{4}\Big(\frac{1}{f} - \frac{1}{g}\Big)\boldsymbol{\tau}^1_\theta - \frac{h}{4fg}\boldsymbol{\tau}^2_\theta + \cot\frac{\rho}{2}\left(-c_1\boldsymbol{\tau}^1_\theta + c_2\boldsymbol{\tau}^2_\theta + c_3\boldsymbol{\tau}^3\right),$$

$$\boldsymbol{\tau}^1_\theta \equiv \boldsymbol{\tau}^1 \cos\theta + \boldsymbol{\tau}^2 \sin\theta, \quad \boldsymbol{\tau}^2_\theta \equiv \boldsymbol{\tau}^2 \cos\theta - \boldsymbol{\tau}^1 \sin\theta$$

$$M = \frac{1}{f^2} - \frac{1}{g^2} + \frac{h^2}{f^2g^2} - \frac{4}{fg}$$

$$+8\Big(\frac{1}{g}\partial_\rho - \frac{h}{gf\sin\rho}\partial_\theta + \frac{\ell^2\tilde{\ell}^2\cos\rho}{gf^4\sin\rho} + \frac{\cos\rho(\ell^2+\tilde{\ell}^2-f^2)}{gf^2\sin\rho} - \frac{\cos\rho}{f\sin\rho}\Big)c_1$$

$$+8\Big(\frac{1}{f\sin\rho}\partial_\theta + \frac{h\ell^2\tilde{\ell}^2\cos\rho}{g^2f^4\sin\rho} + \frac{2\cot 2\theta}{f\sin\rho} - \frac{h\cos\rho}{fg\sin\rho}\Big)c_2 - 16(c_1^2 + c_2^2 + c_3^2).$$

The Square of SUSY

$$\mathbf{Q}^2 = i\mathcal{L}_v + \mathrm{Gauge}(\Phi)$$
$$+ \mathrm{Lorentz} + \mathrm{R}_{\mathrm{SU}(2)}(\Theta^A{}_B) + \check{\mathrm{R}}_{\mathrm{SU}(2)}(\check{\Theta}^A{}_B) + \mathrm{Scale} + \mathrm{R}_{\mathrm{U}(1)}$$

non-zero and important, but gauge-dependent

These are absent for our <u>nice</u> Killing spinor.

Isometry (rotation)

$$v \equiv 2\bar{\xi}^A \bar{\sigma}^m \xi_A \cdot \partial_m = \frac{1}{\ell}\partial_\varphi + \frac{1}{\tilde{\ell}}\partial_\chi,$$

⟺ Omega background

Field-dependent gauge rotation

$$\Phi \equiv -2i\phi\bar{\xi}^A\bar{\xi}_A + 2i\bar{\phi}\xi^A\xi_A - iv^n A_n.$$

⟺ Topologically twisted gauge theory

Topological Twist Revisited

Topological twist identifies SU(2)R with the Lorentz SU(2) for anti-chiral spinors.

$$\xi_{\alpha A} \equiv 0, \quad \bar{\xi}^{\dot{\alpha}}_{A} \equiv \delta^{\dot{\alpha}}_{A} \ (\text{constant})$$

satisfies our KS equation if $T_{kl} = \bar{T}_{kl} = 0$ and

$$\frac{1}{4}\Omega^{ab}_{m}(\bar{\sigma}^{ab})^{\dot{\alpha}}{}_{\dot{\beta}}\delta^{\dot{\beta}}{}_{A} + i\delta^{\dot{\alpha}}{}_{B}V_{m}{}^{B}{}_{A} \ = \ 0.$$

Omega-Background Revisited

$$\boxed{\begin{aligned} \bar{\xi}^{\dot{\alpha}}_{A} &= \frac{1}{\sqrt{2}}\delta^{\dot{\alpha}}{}_{A}, \\ \xi_{\alpha A} &= -\frac{1}{2\sqrt{2}}\left(\frac{1}{\ell}(x_1\sigma_2 - x_2\sigma_1)_{\alpha A} + \frac{1}{\tilde{\ell}}(x_3\sigma_4 - x_4\sigma_3)_{\alpha A}\right) \end{aligned}} \qquad (3)$$

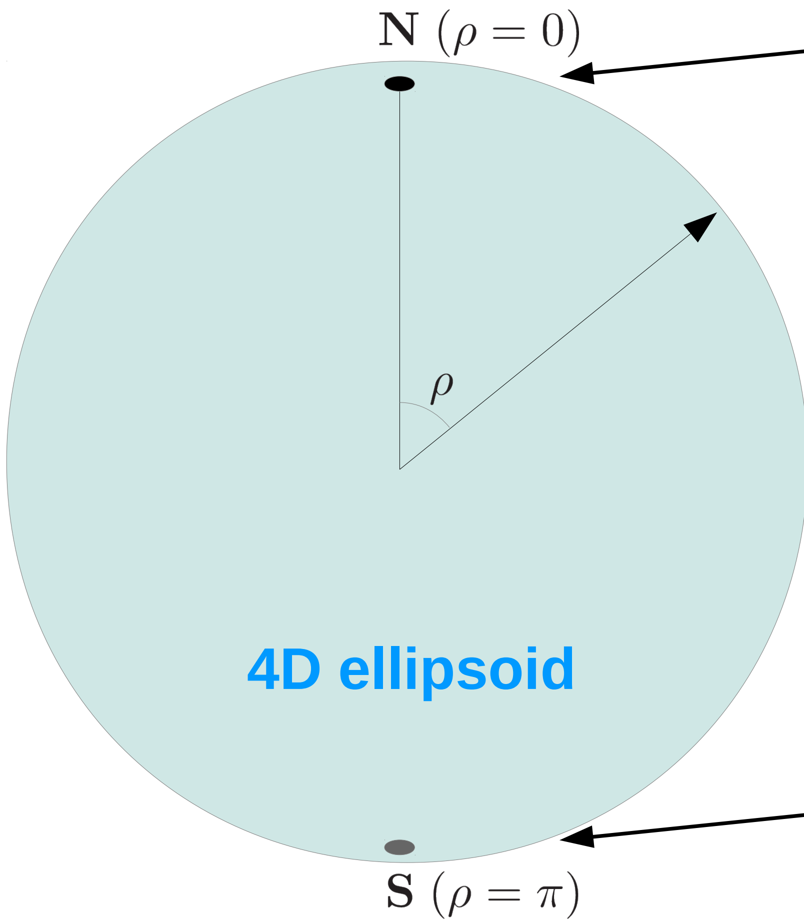
* Note : $\quad 2\bar{\xi}^{A}\bar{\sigma}^{m}\xi_{A}\partial_{m} = \frac{1}{\ell}(x_1\partial_2 - x_2\partial_1) + \frac{1}{\tilde{\ell}}(x_3\partial_4 - x_4\partial_3)\,.$

(3) satisfies our KS equation if $\bar{T}_{kl} = V_{m}{}^{A}{}_{B} = 0$ and

$$\frac{1}{2}T_{kl}\mathrm{d}x^{k}\mathrm{d}x^{l} = \frac{1}{16}\left(\frac{1}{\tilde{\ell}} - \frac{1}{\ell}\right)(\mathrm{d}x_1\mathrm{d}x_2 - \mathrm{d}x_3\mathrm{d}x_4)$$

* Our <u>nice</u> Killing spinor coincides with (3) up to Lorentz rotation
  near the north pole.

Summarizing,



Omega-deformation
of topologically twisted theory

$$T_{kl} \sim \frac{1}{\tilde{\ell}} - \frac{1}{\ell}, \ \bar{T}_{kl} \sim 0.$$

$$\xi_A \sim \sin\frac{\rho}{2}$$

$$\bar{\xi}_A \sim \cos\frac{\rho}{2}$$

Omega-deformation
of anti-topologically twisted theory

$$T_{kl} \sim 0, \ \bar{T}_{kl} \sim \frac{1}{\tilde{\ell}} - \frac{1}{\ell}.$$

Omega-deformation parameter : $\epsilon_1 = \frac{1}{\ell}, \ \epsilon_2 = \frac{1}{\tilde{\ell}}.$

4. Partition Function

Localization technique

- saddle points
- gauge fixing
- 1-loop determinant

[Pestun '07]

Saddle Points

= Coulomb branch moduli space (coordinate: a_0)

$$A_m = 0,\ \phi = \bar{\phi} = -\frac{i}{2}a_0,\ D_{AB} = -ia_0 \cdot w_{AB}$$

$$q_A = F_A = 0$$

(background field)

Classical value of SYM and FI action:

$$\frac{1}{g^2}S_{\rm YM}\Big|_{\rm saddle\ pt.} = \frac{8\pi^2}{g^2}\ell\tilde{\ell}{\rm Tr}(a_0^2)$$

$$\zeta S_{\rm FI}\Big|_{\rm saddle\ pt.} = -16i\pi^2\ell\tilde{\ell}\zeta a_0\,.$$

* independent of the arbitrary functions c_1, c_2, c_3

Gauge Fixing

* Introduce constant field a_0, ghosts $c, \bar{c}, B$
* Define BRST symmetry so that $\mathbf{Q}_B^2[X] = \mathrm{Gauge}(a_0)[X]$

$$\mathbf{Q}_B c = icc + a_0.$$

* Determine SUSY transformation of ghosts so that

$$\widehat{\mathbf{Q}}^2[X] \equiv (\mathbf{Q} + \mathbf{Q}_B)^2[X] = \left\{ i\mathcal{L}_v + \mathrm{Gauge}(a_0) + (\cdots) \right\}[X]$$

$\mathrm{R}_{\mathrm{SU}(2)}, \check{\mathrm{R}}_{\mathrm{SU}(2)}$, Lorentz

$$(\mathbf{Q} + \mathbf{Q}_B)^2[X] = \left\{ i\mathcal{L}_v + \mathrm{Gauge}(\Phi) + (\cdots) + \mathrm{Gauge}(a_0) \right\}[X]$$
$$+\mathbf{Q}[\mathrm{Gauge}(c)X] + \mathrm{Gauge}(c)[\mathbf{Q}X]$$
$$= \mathrm{Gauge}(\mathbf{Q}c)[X]$$

$$\mathbf{Q}c = -\Phi = 2i\phi\bar{\xi}^A\bar{\xi}_A - 2i\bar{\phi}\xi^A\xi_A + iv^n A_n$$

Change of Variables

For vector multiplet, (10+10)

$$A_m\,,\ \phi\,,\ \bar{\phi}\,,\ \lambda_{\alpha A}\,,\ \bar{\lambda}_{\dot{\alpha} A}\,,\ D_{AB}\,;\ c\,,\ \bar{c}\,,\ B$$

4, 1, 1, 4, 4, 3, 1, 1, 1

$$\vec{X} \equiv (A_m\,,\ \phi - \bar{\phi}) \qquad \vec{\Xi} \equiv (2\bar{\xi}_{(A}\bar{\lambda}_{B)} - 2\xi_{(A}\lambda_{B)}\,,\ c\,,\ \bar{c})$$

5, 5

$$\widehat{\mathbf{Q}}\vec{X} \qquad \widehat{\mathbf{Q}}\vec{\Xi} \equiv (D_{AB} + \cdots\,,\ -\Phi + \cdots\,,\ B)$$

5, 5

Deformation of Lagrangian: $\mathcal{L}_{\text{YM}} + t\,\widehat{\mathbf{Q}}\mathcal{V}$

$$\mathcal{V}\Big|_{\text{quad.}} = (\widehat{\mathbf{Q}}X, \Xi)\begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix}\begin{pmatrix} X \\ \widehat{\mathbf{Q}}\Xi \end{pmatrix}$$

$$\widehat{\mathbf{Q}}\mathcal{V}\Big|_{\text{quad.}} = (X, \widehat{\mathbf{Q}}\Xi)\begin{pmatrix} -\mathbf{H} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix}\begin{pmatrix} X \\ \widehat{\mathbf{Q}}\Xi \end{pmatrix} - (\widehat{\mathbf{Q}}X, \Xi)\begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \mathbf{H} \end{pmatrix}\begin{pmatrix} \widehat{\mathbf{Q}}X \\ \Xi \end{pmatrix}$$

$$\mathbf{H} \equiv \widehat{\mathbf{Q}}^2$$

Determinant & Index

$$(Z_{\text{1-loop}})^2 \sim \frac{\det \mathbf{H}\big|_{\Xi}}{\det \mathbf{H}\big|_{X}} = \frac{\det \mathbf{H}\big|_{\mathrm{Coker}D_{10}}}{\det \mathbf{H}\big|_{\mathrm{Ker}D_{10}}} \qquad \Big([D_{10}, \mathbf{H}] = 0\Big)$$

One can calculate the determinant from the index,

$$\begin{aligned}\mathrm{Ind}(D_{10}) &= \mathrm{Tr}_X e^{-i\tau\mathbf{H}} - \mathrm{Tr}_{\Xi} e^{-i\tau\mathbf{H}} \\ &= \mathrm{Tr}_{\mathrm{Ker}D_{10}} e^{-i\tau\mathbf{H}} - \mathrm{Tr}_{\mathrm{Coker}D_{10}} e^{-i\tau\mathbf{H}}\end{aligned}$$

-- Index depends only on the terms in D_{10} of highest order in derivatives.

-- Index localizes onto fixes points of the Killing vector v (=N,S)

-- Needs a regularization since D_{10} is not elliptic.

Localization

$$\mathrm{Tr}e^{-i\tau\mathbf{H}} \sim \int \mathrm{d}^4x\delta^4(x-x') \qquad (x' \equiv e^{\tau\mathcal{L}_v}x)$$
$$= \det(1-\partial x'/\partial x)^{-1}$$
$$= |(1-q_1)(1-q_2)|^{-2} \quad (q_1 \equiv e^{i\tau/\ell},\, q_2 \equiv e^{i\tau/\tilde{\ell}})$$

* Near the N,S-poles,

$$D_{10}\Big|_{\text{near }\mathbf{N}(\mathbf{S})} \,\tilde{:}\, A \longmapsto \left\{(1\pm *)\mathrm{d}A\,,\ \mathrm{d}^*A\right\}$$

1-form [4] **SD(ASD) 2-form [3]** **0-form [1]**

[Atiyah-Bott]

$$\mathrm{ind}(D_{10}) = \frac{(q_1+\bar{q}_1+q_2+\bar{q}_2)-(1+q_1q_2+\bar{q}_1\bar{q}_2)-1}{|(1-q_1)(1-q_2)|^2} + \left(\text{south pole}\right)$$
$$= \left[-\frac{1+q_1q_2}{(1-q_1)(1-q_2)}\right] + \left[-\frac{1+q_1q_2}{(1-q_1)(1-q_2)}\right]$$

Ellipsoid Partition Function:

$$Z = \int_{\text{Cartan}} \mathrm{d}\hat{a}_0 e^{-2\pi \mathrm{Im}\tau \mathrm{Tr}(\hat{a}_0^2)} \cdot Z_{\text{1-loop}} \cdot |Z_{\text{Nek}}|^2$$

For gauge group G and hyper rep. R, the 1-loop part reads

$$Z_{\text{1-loop}} = \prod_{\alpha \in \Delta_+} \Upsilon(i\hat{a}_0 \cdot \alpha)\Upsilon(-i\hat{a}_0 \cdot \alpha) \prod_{\rho \in R} \Upsilon(i\hat{a}_0 \cdot \rho + \tfrac{Q}{2})^{-1}$$

$$\Upsilon(x) = \prod_{m,n \geq 0} (mb + nb^{-1} + x)(mb + nb^{-1} + Q - x)$$

$$Q = b + b^{-1}, \; b \equiv \sqrt{\ell/\tilde{\ell}}.$$

It correctly reproduces the Liouville DOZZ factor

for general central charge.

A Quick check of AGT

EX) SU(2) N_f=4 SQCD

$$Z_{\text{1-loop}} = \frac{\Upsilon(Q+2ip)\Upsilon(Q-2ip)}{\Upsilon(\cdots)\Upsilon(\cdots)\Upsilon(\cdots)\Upsilon(\cdots)\Upsilon(\cdots)\Upsilon(\cdots)\Upsilon(\cdots)\Upsilon(\cdots)}$$

$$= C(p_1, p_2, p)C(p_3, p_4, -p)$$

(up to factors for external legs)

cf) Liouville 3-point structure constant

$$C(p_1, p_2, p_3) = \frac{\text{const}\cdot\Upsilon(Q+2ip_1)\Upsilon(Q+2ip_2)\Upsilon(Q+2ip_3)}{\Upsilon(\frac{Q}{2}+ip_{1+2+3})\Upsilon(\frac{Q}{2}+ip_{1+2-3})\Upsilon(\frac{Q}{2}+ip_{1-2+3})\Upsilon(\frac{Q}{2}+ip_{1-2-3})}.$$

Conclusion

Motivated by AGT correspondence, we found

-- Generalized KS equation for 4D N=2 SUSY

-- SUSY 4D ellipsoid background

-- Ellipsoid partition function which reproduces
   Liouville/Toda correlators for general b