Moduli space of dyonic instantons in 5d SYM

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Dyonic Instantons – Outline

We investigate the properties of dyonic instantons in 5D $\mathcal{N} = 2$ SYM using the moduli space approximation. This is clearly much simpler than analysing the full field theory. Most of the talk will concern 2 dyonic instantons with SU(2) gauge group. In this talk I will:

- Briefly outline what Instantons and Dyonic Instantons are
- Present the results for the Metric and Potential on Moduli Space
- Discuss some properties of the moduli space, including geodesic submanifolds
- Present some examples of moduli space dynamics

Instantons in 4+1D SYM

Instantons in 5D SYM are $\frac{1}{2}$-BPS particles. In the 4 spatial directions they are self-dual solutions with topological charge. In string theory they correspond to D0-branes in the $\frac{1}{4}$-BPS D0-D4 system.

- D0-branes carry momentum on M-Theory S^1 – instantons are KK modes of 6D field theory.
- Can find moduli space of solutions with fixed energy for given topological charge.
- Field theory kinetic energy induces metric on moduli space.
- Instanton size is a modulus – possible evolution to zero size singularity and generic instability to spreading out.

Dyonic Instantons in 4+1D SYM

Also find $\frac{1}{4}$-BPS dyonic instantons. In string theory these are D0-F1 bound states or supertubes between D4-branes.

- D0-F1 lifts to M2 with momentum, SDS on M5.
- Same moduli space and metric as for instantons.
- On Coulomb branch BPS bound includes electric charge contribution $\rightarrow$ Potential on moduli space.
- Potential increases with size, so does not suffer unbounded size instability.
- Single dyonic instanton has conserved charge L, and for $L \neq 0$ cannot evolve to zero size singularity.

Dyonic Instantons in 4+1D SYM

Consider the bosonic sector of 5D SYM with only one of the five scalars fields non-vanishing.

$$S = -\int \mathrm{d}^5 x \mathrm{Tr}\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\phi D^\mu\phi\right)$$

Static finite energy solutions must approach pure gauge at spatial infinity

$$A_i \to g^{-1}\partial_i g$$

and so have a conserved topological charge

$$k = \frac{-1}{8\pi^2}\int_{\mathbb{R}^4} \mathrm{Tr}(F \wedge F)$$

given by the winding number of

$$g : S^3_\infty \to SU(2) \sim S^3$$

Dyonic Instantons in 4+1D SYM

We can derive the BPS conditions from the Hamiltonian

$$\begin{aligned} H &= \frac{1}{2}\int \mathrm{d}^4x\mathrm{Tr}\left(F_{i0}F_{i0} + D_i\phi D_i\phi + \frac{1}{2}F_{ij}F_{ij} + D_0\phi D_0\phi\right) \\ &= \frac{1}{2}\int \mathrm{d}^4x\mathrm{Tr}\left([F_{i0}\pm D_i\phi]^2 + \frac{1}{4}[F_{ij}\pm * F_{ij}]^2 + [D_0\phi]^2\right) \\ &\quad \pm 2\pi^2 k \mp Q_E \\ &\geq 2\pi^2|k| + |Q_E| \end{aligned}$$

where the electric charge is

$$Q_E = \int \mathrm{d}^4x\mathrm{Tr}\left(F_{i0}D_i\phi\right)$$

Dyonic Instantons in 4+1D SYM

So for fixed integer (instanton number/charge) $k > 0$ the BPS conditions are:

$$F_{ij} = *F_{ij} \to A_i(z;x)\ , \quad \text{Moduli } z$$

$$F_{i0} = \mp D_i\phi\ , \quad D_0\phi = 0 \text{ and eom } D^i D_i\phi = 0 \to A_0 = \mp\phi(z;x)$$

For each self-dual solution $A_i(z)$ (independent of scalar VEV), find unique solution $\phi(z)$ (dependent on scalar VEV).

Now moduli space approximation

- Each timeslice is a BPS configuration given by moduli $z^r(t)$
- $A_i(z;x) \to A_i(z(t);x)$
- $A_0 = \mp\phi \to A_0 = \mp\phi + \epsilon_r \dot{z}^r$

Metric on Moduli Space – $\phi = 0$

If we define (zero modes) $\delta_r A_i \equiv \partial_r A_i - D_i \epsilon_r$ we see that

$$D_i\left(\delta_r A_j\right) - D_j\left(\delta_r A_i\right) = \epsilon_{ijkl} D_k(\delta_r A_l) \quad \text{(linearised } F = *F\text{)}$$

and since $F_{i0} = -\dot{z}^r \delta_r A_i$, Gauss' Law

$$0 = D^i F_{i0} = -\dot{z}^r D^i \delta_r A_i$$

is satisfied if we choose ϵ_r so that

$$D^i(\delta_r A_i) = 0$$

The SYM action now becomes

$$S = \frac{1}{2}\int \mathrm{d}^5 x \,\mathrm{Tr}(F_{i0}F_{i0}) = \frac{1}{2}\int \mathrm{d}t g_{rs}\dot{z}^r \dot{z}^s$$

so moduli space dynamics is geodesic motion with metric

$$g_{rs}(z) = \langle \delta_r A_i, \delta_s A_i \rangle = \int \mathrm{d}^4 x \,\mathrm{Tr}\left(\delta_r A_i \delta_s A_i\right)$$

Metric on Moduli Space – $\phi \neq 0$

Now we write

$$A_0 = \phi \to A_0 = \phi + \dot{z}^r \epsilon_r$$

but Gauss' Law is

$$0 = D_i F_{i0} + [D_0\phi, \phi] = [D_0\phi, \phi] = \mathcal{O}(|\dot{z}||q|^2)$$

where $|q|$ is the magnitude of the scalar VEV.

Metric on Moduli Space $\phi \neq 0$

Now for metric note, $F_{i0} = -\dot{y}^r \delta_r A_i$ after a change of coordinates $y^r = z^r - |q| K^r t$.

- $D_i \phi$ satisfies linearised self-dual equation (like gauge transformation)
- $D_i \phi$ satisfies gauge fixing condition $D_i \left(D_i \phi \right) = 0$ due to eom
- Therefore $D_i \phi$ is a linear combination of zero modes $D_i \phi = |q| K^r \delta_r A_i$
- K^r is Killing since $D_i \phi$ is symmetry – global gauge transformation on moduli space.

So we find the same metric as for instantons

$$\frac{1}{2} \int \mathrm{d}^5 x \mathrm{Tr}(F_{i0} F_{i0}) = \frac{1}{2} \int \mathrm{d}t g_{rs} \dot{y}^r \dot{y}^s$$

Potential on Moduli Space

The potential on moduli space is given by

$$V = \frac{1}{2}\int \mathrm{d}^4x \mathrm{Tr}(D_i\phi D_i\phi) = \frac{1}{2}|q|^2 g_{rs}K^r K^s$$

which is the form expected from supersymmetry, with K a moduli space Killing vector.

Noting that $\mathrm{Tr}(D_0\phi D_0\phi) = \mathcal{O}(|\dot{z}|^2|q|^2)$ we have the effective action

$$S = \int \mathrm{d}t \left(\frac{1}{2}g_{rs}\dot{y}^r\dot{y}^s - V\right) + \mathcal{O}(|\dot{z}|^2|q|^2)$$

ADHM Construction

Now, to actually find all self-dual configurations, and to then project onto the moduli space orthogonal to gauge transformations, sounds like a difficult problem! However, fortunately this can be reduced to a (still non-trivial) algebraic problem:

- ADHM Construction – Gives parametrisation of self-dual configurations
- Osborn projection – Method to project onto gauge-fixed moduli space within ADHM construction

This gives a method to calculate the metric, and it is also possible to find the potential in terms of the ADHM data.

ADHM – Constraints and Gauge Field

The ADHM data is encoded in a $(N+2k)\times 2k$ matrix $\Delta(x)$. The ADHM constraint is

$$\Delta^\dagger\Delta = f^{-1}(x)\otimes I_2$$

with f being an invertible $k\times k$ matrix.
Then find $(N+2k)\times N$ matrix $U(x)$ satisfying

$$\Delta^\dagger U = 0\ , \quad U^\dagger U = I_N$$

and this produces self-dual $\mathrm{U}(N)$ gauge fields

$$A_i = U^\dagger\partial_i U$$

ADHM – Gauge Invariance

Note that the construction is U(N) gauge invariant since U is only defined up to $U \to UV$ with $A_i \to V^\dagger A_i V + V^\dagger \partial_i V$.
Note that we can also transform

$$\Delta \to Q\Delta R \ , \quad U \to QU$$

with $Q^\dagger Q = I$. Osborn showed that an appropriate choice of Q and R, depending on the moduli, results in a moduli space with tangent vectors orthogonal to gauge transformations. I.e. it implements a projection so that $\partial_r A_i$ are the zero modes $\delta_r A_i$.

In general the moduli space has dimension $4kN$. We will focus on $N = 2$ and $k = 1, 2$.

ADHM – Scalar Field

It is also possible to solve for the scalar field, using an ansatz

$$\phi = iU^\dagger \begin{pmatrix} q & 0 \\ 0 & P \end{pmatrix} U$$

where the scalar VEV is iq, and the real anti-symmetric $2k \times 2k$ matrix P can be found from the eom for ϕ.

ADHM – 2 SU(2) Instantons

Note that we can represent 4-vectors such as x^i as quaternions. When $N = 2$ we can also use (imaginary) quaternions for SU(2), and Δ can be expressed as a $(k+1) \times k$ quaternion matrix. In this notation we have

$$\Delta(x) = \begin{pmatrix} v_1 & v_2 \\ \tilde{\rho}+\tau & \sigma \\ \sigma & \tilde{\rho}-\tau \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ x & 0 \\ 0 & x \end{pmatrix}$$

- $v_I \to$ instanton sizes $\rho_I = |v_I|$ and embeddings in SU(2)
- $\tilde{\rho} \to$ Centre of Mass – Decouples so ignore
- $\tau \to$ Instanton relative position (for large $|\tau|$)
- $\sigma = \frac{\tau}{4|\tau|^2}(\overline{v}_2 v_1 - \overline{v}_1 v_2)$

Outline of Moduli Space properties

I will not present further details of the calculation of the moduli space metric and potential. Instead we will see the results and features of the moduli space and its dynamics.

- Brief review of single instanton
- Metric and potential for 2 instantons
- Geodesic submanifolds – Moduli Subspaces
- Some dynamical properties of 2 instantons (in moduli space approximation)

Instanton number 1

For a single dyonic instanton the moduli space is 4-dimensional after removing the trivial centre of mass coordinates. The 4 moduli are parametrised by a quaternion v describing the instanton size $\rho = |v|$ and gauge orientation. The effective action is

$$S = 8\pi^2 \int \mathrm{d}t \left(|\dot{v}|^2 - |q|^2|v|^2\right)$$

As the action is independent of the gauge orientation parameters, there are conserved quantities. In detail, if this motion in $SU(2)$ is parametrised by angle θ, we have

- $S \sim \int \mathrm{d}t \left(\dot{\rho}^2 + \rho^2\dot{\theta}^2 - |q|^2\rho^2\right)$
- $L = \rho^2\dot{\theta}$ is conserved
- EOM $\rightarrow \ddot{\rho} - L^2\rho^{-3} + |q|^2\rho = 0$

Dyonic Instanton Oscillations

The general solution describes the instanton size oscillating with "amplitude" A,

$$\rho = \sqrt{A \sin(2|q|t) + \sqrt{\frac{L^2}{|q|^2} + A^2}}$$

- The case $A = 0$ with $\dot{\theta} = \pm|q|$ describes a static dyonic instanton with fixed size.
- In general, provided $L \neq 0$ the size oscillates in such a way that the instanton can be arbitrarily large, but will never evolve to zero size.

Instanton number 2

For $k = 2$ it is possible to find the metric and potential. Previously this was known for large separation up to terms of order $|\tau|^{-2}$.

- $ds^2 = 8\pi^2 \left(|dv_1|^2 + |dv_2|^2 + |d\tau|^2 + |d\sigma|^2 - N_A^{-1} dk^2\right)$
- $V = 8\pi^2 |q|^2 \left(|v_1|^2 + |v_2|^2 - N_A^{-1} |\overline{v}_2 \hat{q} v_1 - \overline{v}_1 \hat{q} v_2|^2\right)$

where

$$\hat{q} = q/|q| \; , \quad N_A = |v_1|^2 + |v_2|^2 + 4(|\tau|^2 + |\sigma|^2)$$

$$dk = \overline{v}_1 dv_2 - \overline{v}_2 dv_1 + 2(\overline{\tau} d\sigma - \overline{\sigma} d\tau)$$

$$\sigma = \frac{\tau}{4|\tau|^2} (\overline{v}_2 v_1 - \overline{v}_1 v_2)$$

- Note $\hat{q}$ dependence in last term of potential.

Symmetries

Now there are two types of symmetry to consider.

- Remaining symmetries of the ADHM data $\rightarrow$ Identifications on moduli space
- Symmetries of the metric and potential $\rightarrow$ conserved charges and fixed points $\rightarrow$ geodesic submanifolds

The ADHM identifications include orbifold singularities for zero size instantons and also describe right-angled scattering – see later.

Moduli space singularities

Some discrete symmetries in the ADHM data include:

- $(v_1, v_2, \tau) \to (-v_1, v_2, \tau)$
- $(v_1, v_2, \tau) \to (v_1, -v_2, \tau)$

The $\mathbb{Z}_2$ symmetries have fixed points at $v_1 = 0$ and $v_2 = 0$, so the moduli space has orbifold singularities where either instanton has zero size.

Half-Dimension Geodesic Submanifold – $\mathbb{C} \subset \mathbb{H}$

Imaginary unit quaternionic involution by p, $p^2 = -1$

$$v_1 \to p v_1 \overline{p}\ , \quad v_2 \to p v_2 \overline{p}\ , \quad \tau \to p \tau \overline{p}$$

This requires p parallel to q (in scalar VEV) to be symmetry of potential.

Get fixed points when v_1, v_2, τ are in complex subspace spanned by $\{1, q\}$. This geodesic submanifold has half the dimension of the (relative) moduli space, i.e. dimension 6, so is much simpler to explore. However it seems to capture all qualitative features of the full moduli space dynamics.

- τ – Relative position in complex plane
- v_1, v_2 – Instanton sizes $|v_1|, |v_2|$ and phase in unbroken U(1)

Vortices and σ-model lumps

- The reduction from 4 to 2 spatial dimensions is in fact the Hanany-Tong construction of the moduli space of vortices. This gives a qualitative description of vortex dynamics.
- Actually since we have zero non-commutativity parameter, we are in the strong coupling limit of the 3D Yang-Mills-Higgs system, so this is the moduli space of charge 2 O(3) σ-model lumps.
- There is a further geodesic submanifold (of dimension 4) where we fix the relative instanton sizes and gauge orientations, and this is qualitatively similar to the charge 2 Q-lump moduli space of a deformed O(3) σ-model, with the deformation introducing the potential.

Outline of Dynamics on Moduli Space

We will now present some details of the dynamics on moduli space. Specific features we will consider are

- Right-angled scattering
- Evolution to zero size singularity
- More general scattering
- Orbiting instantons

Right-angled scattering

There is a discrete symmetry of the ADHM data interchanging τ and σ

$$\begin{pmatrix} v_1 & v_2 \\ \tau & \sigma \\ \sigma & -\tau \end{pmatrix} \to \begin{pmatrix} (v_1+v_2)/\sqrt{2} & (v_1-v_2)/\sqrt{2} \\ \sigma & \tau \\ \tau & -\sigma \end{pmatrix}$$

So if τ is the relative position of the instantons for large $|\tau|$, the same should be true for σ. Indeed the eigenvalues of the block are $\pm\sqrt{\tau^2+\sigma^2}$.

Recalling that

$$\sigma = \frac{\tau}{4|\tau|^2}(\overline{v}_2 v_1 - \overline{v}_1 v_2)$$

we see that this is consistent since τ large and σ large are exclusive.

Further, if we imagine an evolution with decreasing real τ we see that we end up with increasing imaginary σ. In general this corresponds to 90° scattering in a plane determined by τ, v_1 and v_2.

In actual dyonic instanton dynamics the scattering can be more complicated, but this picture of the evolution seems to be a good approximation in some circumstances.

We can illustrate this right-angled "scattering" in an example with

$$v_1 = 1\,, \quad v_2 = i$$

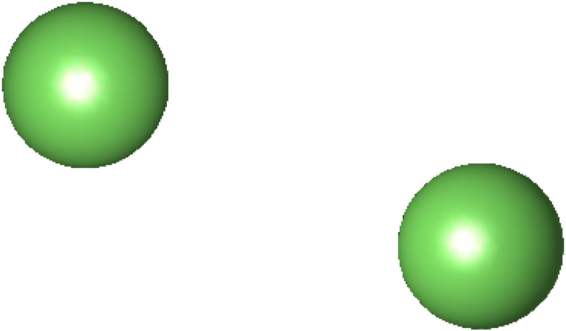
and evolving real

$$\tau = 3, \frac{9}{10}, \frac{1}{\sqrt{2}}, \frac{1}{6}$$

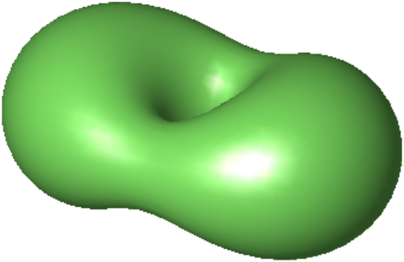
corresponding to

$$\sigma = \frac{-i}{6}, \frac{-5i}{9}, \frac{-i}{\sqrt{2}}, -3i$$

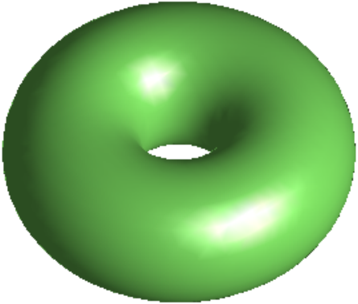
We plot the instanton charge isosurface in (x_1, x_2, x_3).
Note that the instantons both have width $|v_1| = |v_2| = 1$.



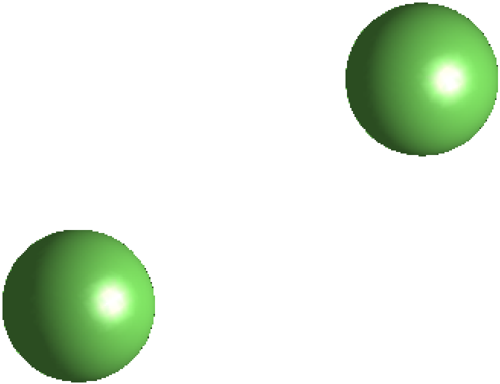
$$\tau = 3\,, \quad \sigma = \frac{-i}{6}\,, \quad \sqrt{\tau^2 + \sigma^2} \approx \tau = 3$$



$$\tau = \frac{9}{10}\,, \quad \sigma = \frac{-5i}{9}\,, \quad \sqrt{\tau^2 + \sigma^2} \approx 0.71 < 1$$



$$\tau = \frac{1}{\sqrt{2}}\,, \quad \sigma = \frac{-i}{\sqrt{2}}\,, \quad \sqrt{\tau^2 + \sigma^2} = 0$$

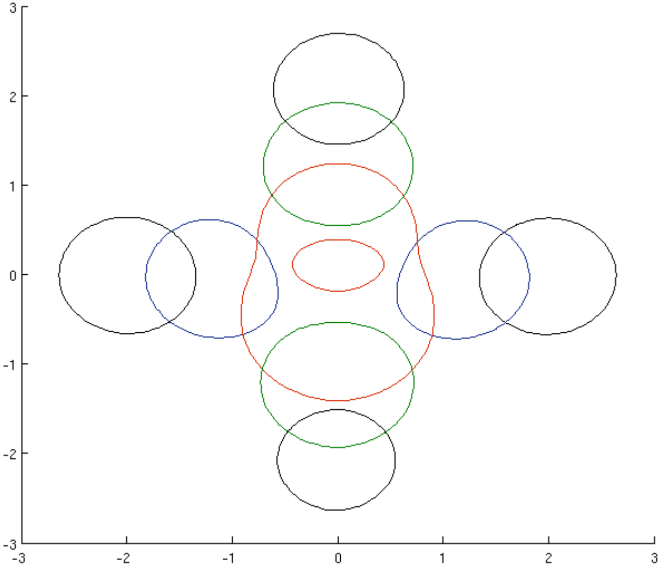


$$\tau = \frac{1}{6}\,, \quad \sigma = -3i\,, \quad \sqrt{\tau^2 + \sigma^2} \approx \sigma = -3i$$

Actual Scattering Example

The following plot shows the actual evolution in the moduli space approximation. The coloured contours represent the two instanton shapes and positions in the complex plane (within the 6-dimensional moduli subspace.)

Note that the setup is not exactly symmetric (and there is a small impact parameter.) However, we clearly see the formation of a single charge two lump followed by separation of two instantons, with right-angled scattering.


3
2
1
0
-1
-2
-3
-3
-2
-1
0
1
2
3

Exchange of angular momentum

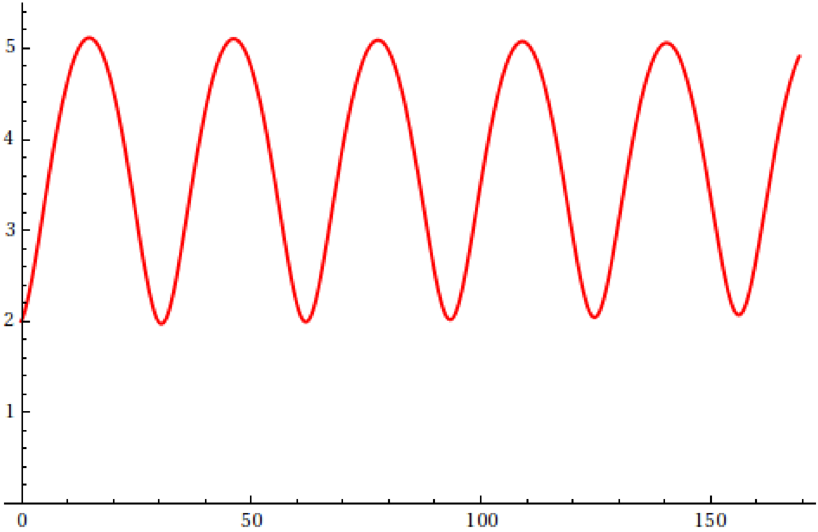
Now, for a single instanton, a non-zero conserved angular momentum ensured it could not reach zero size. However, for two instantons we still only have one conserved charge

$$L = \rho_1^2 \dot{\theta}_1 + \rho_2^2 \dot{\theta}_2 + \mathcal{O}\left(\frac{1}{|\tau|^2}\right) \approx L_1 + L_2$$

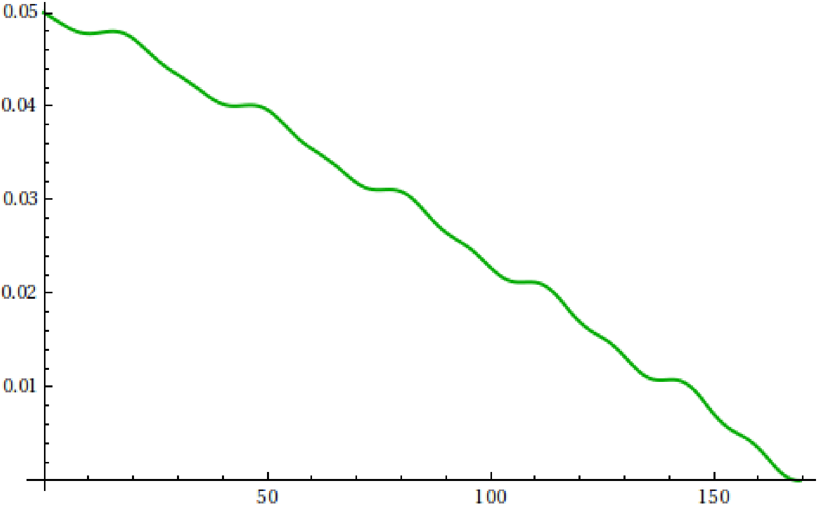
The instantons interact so, even for large separations, there is only approximate conservation of individual angular momentum.
It is a possibility that the exchange might always be from larger to smaller, but this is not generally true. We can find examples where say $L_1 \gg L_2 > 0$ but L_2 decreases.

However, recall from single instanton that its size oscillates. Having zero angular momentum just means that its minimum width reaches zero. Since L_2 will just pass through zero we will only reach zero size if the oscillation phase is fine tuned. This is possible with the result that to reach the zero size singularity we must fine tune:

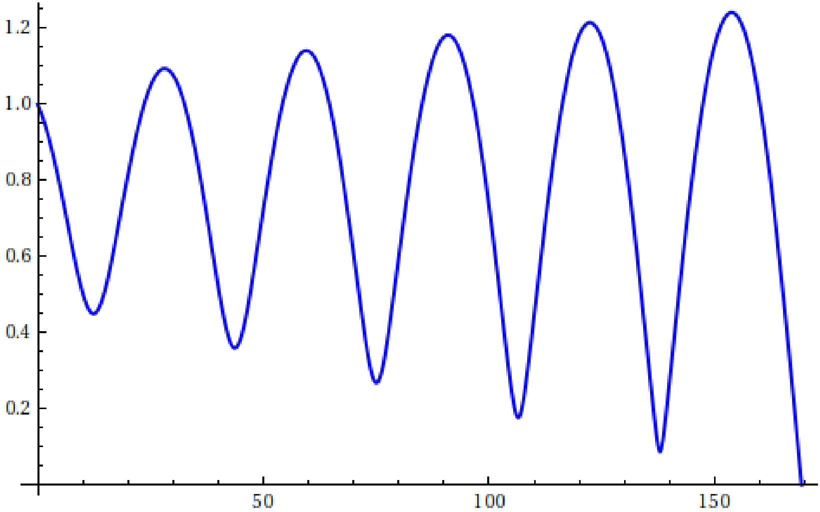
- 1 parameter in the 6-dimensional submanifold
- 3 parameters in the full moduli space



Oscillating width of Instanton 1, ρ_1, with L_1 approximately constant.



Angular momentum of instanton 2, $L_2 = \rho_2^2 \dot{\theta}_2$.



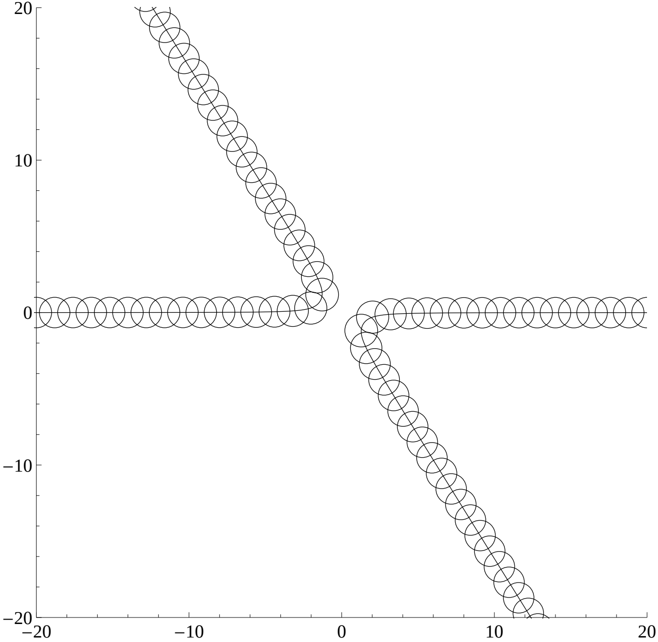
Oscillating width of Instanton 2, ρ_2.

More Instanton Scattering

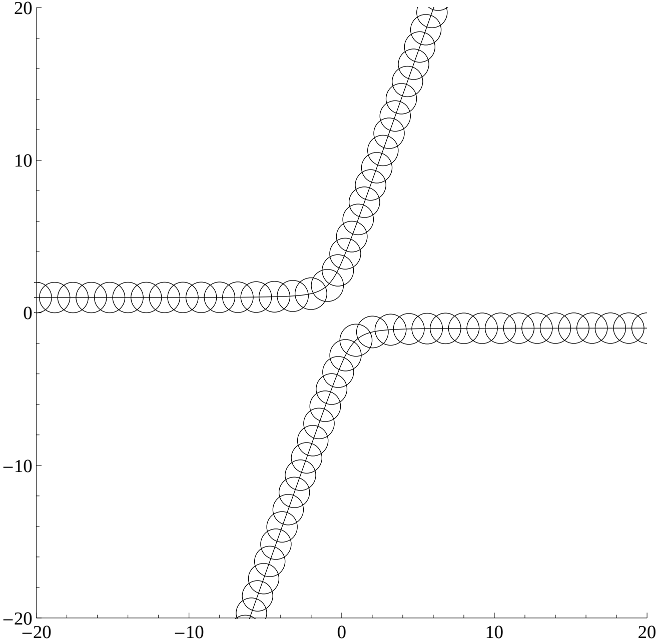
We will now consider some examples of dyonic instanton scattering, and some general analysis of the scattering angle. We will work within the 6-dimensional geodesic submanifold. Some features we will see are:

- Even for head-on scattering, scattering angle varies from 90° as the VEV is turned on.
- Sign of impact parameter is relevant.

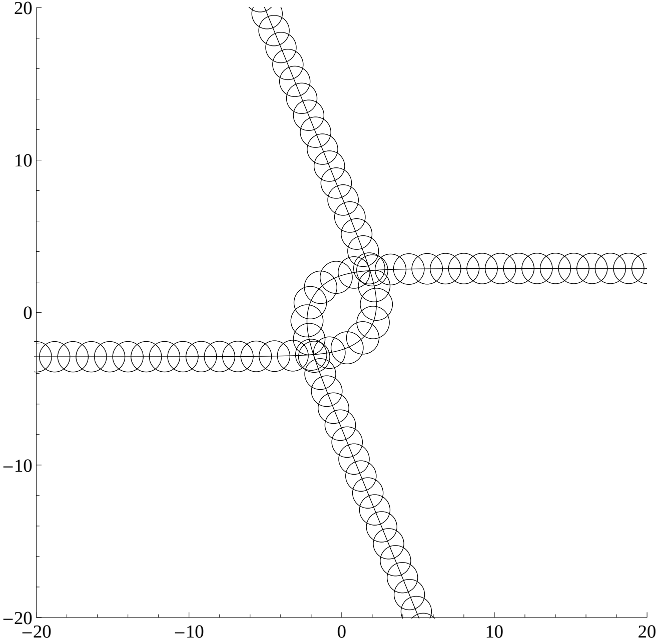
In the following diagrams, the circles indicate the size, but not necessarily the shape of the 2 instantons. The initial instanton widths are 1, initial separation is 50, and the initial velocity is 0.03.



Head-on scattering, $|q| = 0.1$, scattering angle $\approx 122^\circ$.



Impact parameter $= -2$, $|q| = 0.1$, scattering angle reduced.



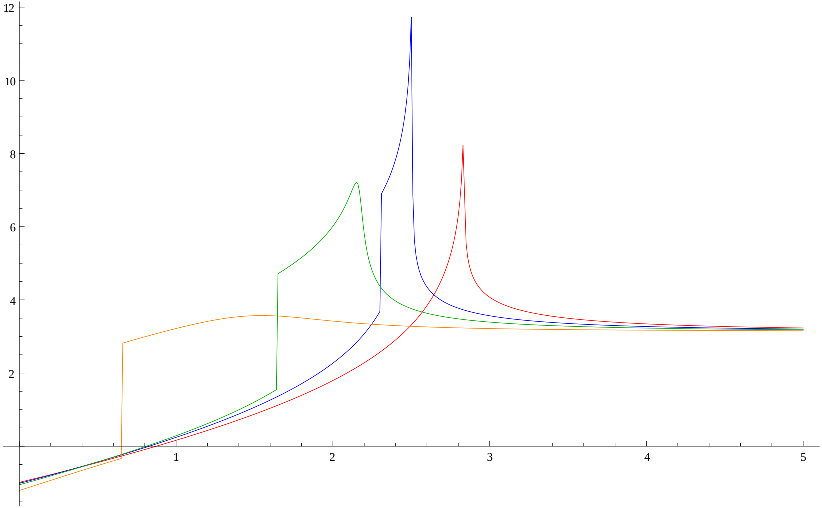
Impact parameter $\approx +6$, $|q| = 0.1$, instantons wind around.

Scattering Angle v Impact Parameter

The following plot shows the scattering angle in terms of χ – the angle in the plane made by the outgoing trajectory of the instanton from the right. We can see that there is a complicated dependence on the impact parameter and scalar VEV. Some obvious features are:

- Angle jumps by π – Instantons coincide during scattering so lose identity.
- Spikes in angle – Fine-tuned conditions allow instantons to orbit when close before finally separating.

Scattering Angle v Impact Parameter

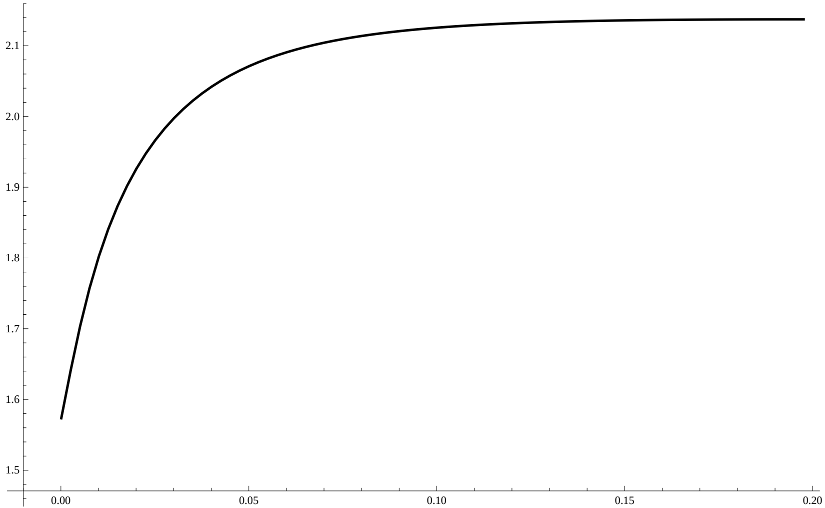


Final angle χ v Positive Impact Parameter, for
$|q| = 0.02$, $|q| = 0.05$, $|q| = 0.07$ and $|q| = 0.1$.

Head-on scattering angle v Scalar VEV

The follow plot shows how the scattering angle depends on the scalar VEV. In particular we see the interpolation from right-angled scattering for $|q| = 0$ to $\approx 122^\circ$ for $|q| = 0.2$.

Head-on scattering angle v Scalar VEV



Scattering Angle v Scalar VEV scale $|q|$.

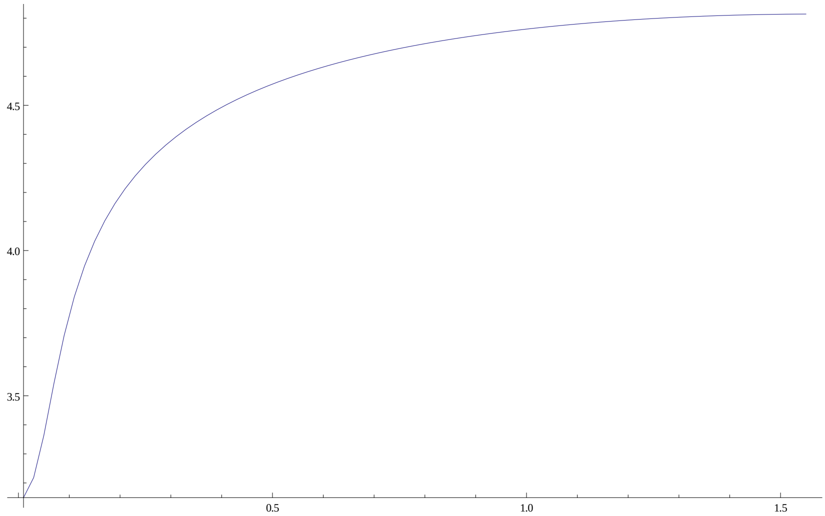
Scattering Angle v Relative Gauge Orientation

There are two different types of 4-dimensional geodesic submanifolds with fixed relative instanton properties (size and gauge orientation):

- $v_1 \propto v_2 \rightarrow$ Fixed ratio of sizes but no interactions in moduli space approximation
- $v_1 = iv_2 \rightarrow$ Equal sizes and strongest interactions (similar to Q-lumps)

We can illustrate the increasing interaction strength by plotting the scattering angle χ as a function of the relative gauge orientation. We see next an example for impact parameter -0.5.

Scattering Angle v Relative Gauge Orientation

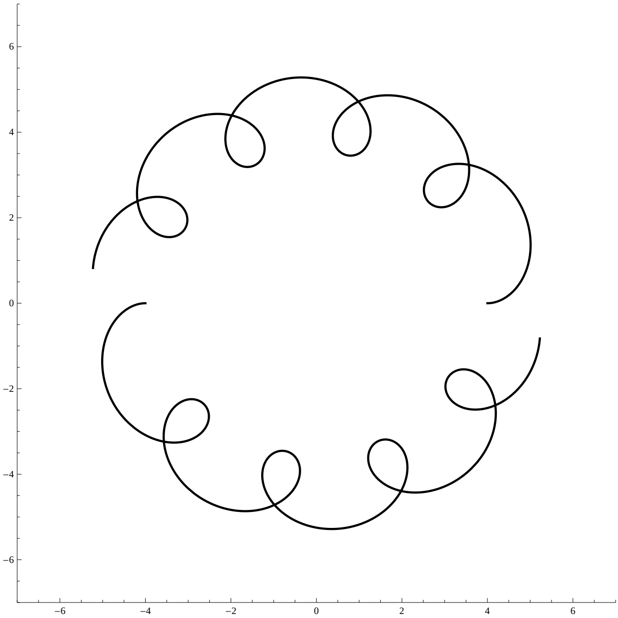


Angle χ v Relative Gauge angle

Orbiting Dyonic Instantons

It is possible to find examples where 2 dyonic instantons orbit each other, with the separation also oscillating.

One way to study this behaviour is to start with 2 separated instantons and gently push them apart. The instantons interact, but the force is not parallel to the direction of separation and motion. This can lead to an interesting orbiting motion.



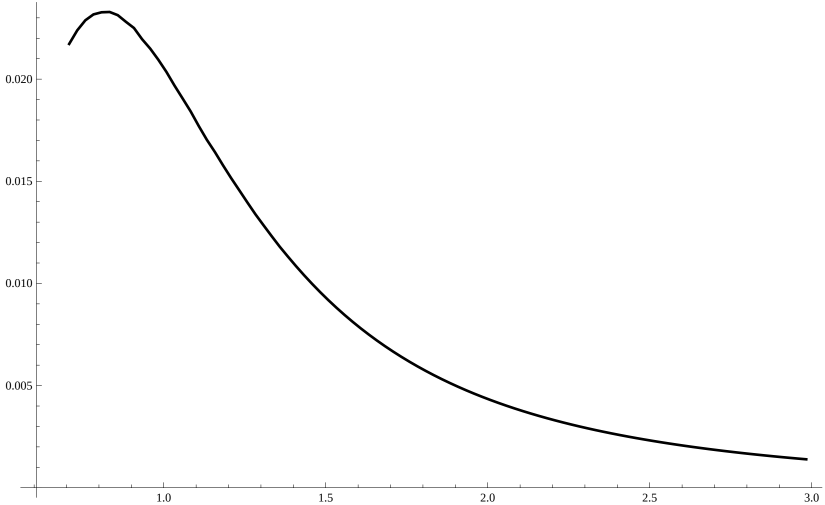
Example of dyonic instantons orbiting after initial gentle push apart.

How well bound are the instantons?

There are many questions about the stability of these orbits. A simple one is what happens as we increase the initial velocity?

- If the initial push is too hard, the instantons will just separate.
- We can plot this escape velocity as a function of the initial separation.

The result is that there is a fall off as expected with increasing separation, and some interesting behaviour for small separation.



Minimum initial escape velocity against initial separation $|\tau|$.

Outlook and Work in Progress

There are still many features to study, and many ways to generalise the system.

- Introduce non-commutativity parameter – resolve zero size singularity.
- Quantum mechanics on moduli space – bound states, transfer of angular momentum (and commutative limit)
- Extend to $k > 2$ – some motivation required!
- Extend to SU(3) – richer structure with more charges. SU(3)?