Non-Abelian Self-Dual Tensor Field Theories From Twistor Space

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Outline

- Introduction And Motivation
- Chiral Fields In 6*d* And Their Twistorial Interpretation
- Non-Abelian Extensions And Supersymmetry
- Conclusions And Outlook

Introduction And Motivation

Major (Classical) Twistor Applications

- Free fields in 4*d* $\leftrightarrow$ cohomology groups via the Penrose transform
- 4*d* Yang–Mills (YM) instantons $\leftrightarrow$ holomorphic vector bundles via the Penrose–Ward transform
- 4*d* gravitational instantons $\leftrightarrow$ holomorphic contact structures via the Ward construction
- Full 4*d* YM $\leftrightarrow$ holomorphic vector bundles that can be extended in a certain fashion $\rightarrow$ full $\mathcal{N} = 3, 4$ SYM theory
- Applications to theories in $d < 4$ via dimensional reduction
- ...

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Differentially constrained data $\leftrightarrow$ Algebraic data

- The central objects are double fibrations of the form

$$P \xleftarrow{\;\pi_1\;} F \xrightarrow{\;\pi_2\;} M$$

where M, F and P are complex manifolds:

  - M space-time
  - F correspondence space
  - P twistor space

Twistor Basics

- The central objects are double fibrations of the form

$$\begin{array}{ccc} & F & \\ {\scriptstyle\pi_1}\swarrow & & \searrow{\scriptstyle\pi_2} \\ P & & M \end{array}$$

  where M, F and P are complex manifolds:
  - M space-time
  - F correspondence space
  - P twistor space

- Then we have a correspondence between P and M, i.e. between points in one space and subspaces of the other:

$$\begin{array}{ccc} \pi_1(\pi_2^{-1}(x)) \hookrightarrow P & \leftrightarrow & x \in M \\ p \in P & \leftrightarrow & \pi_2(\pi_1^{-1}(p)) \hookrightarrow M \end{array}$$

Twistor Correspondence: $P \xleftarrow{\pi_1} F \xrightarrow{\pi_2} M$

- Using this correspondence, we can transfer data given on P to data on M and vice versa (e.g. vector bundles, sheaf cohomology groups, contact forms, ...)

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- Take some analytic object Ob_P on P and transform it to an object Ob_M on M; this in turn is constrained by some PDEs as $\pi_1^*\mathrm{Ob}_P$ has to be constant up the fibres of $\pi_1 : F \to P$

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- Take some analytic object Ob_P on P and transform it to an object Ob_M on M; this in turn is constrained by some PDEs as $\pi_1^*\mathrm{Ob}_P$ has to be constant up the fibres of $\pi_1 : F \to P$
- Under suitable topological conditions, the maps

$$\mathrm{Ob}_P \mapsto \mathrm{Ob}_M \qquad \text{and} \qquad \mathrm{Ob}_M \mapsto \mathrm{Ob}_P$$

define a bijection between $[\mathrm{Ob}_P]$ and $[\mathrm{Ob}_M]$ (the objects in question will only be defined up to equivalence)

Two Prime Examples

Penrose Transform

Consider $4d$ flat space $M = \mathbb{C}^4$ with $TM \cong S \otimes \tilde{S}$:

$$\begin{array}{ccc} & F = \mathbb{P}(\tilde{S}^\vee) = \mathbb{C}^4 \times \mathbb{P}^1 & \\ & \pi_1 \swarrow \quad \searrow \pi_2 & \\ P = \mathbb{P}^3 \setminus \mathbb{P}^1 & & M = \mathbb{C}^4 \end{array}$$

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Then

$$H^1(P, \mathcal{O}_P(-2h-2)) \cong \left\{ \begin{array}{c} \text{zero-rest-mass fields} \\ \text{of helicity } h \text{ on } M \end{array} \right\}$$

Penrose–Ward Transform

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$$\boxed{\pi_1(\pi_2^{-1}(x)) = \mathbb{P}^1_x} \hookrightarrow P = \mathbb{P}^3 \setminus \mathbb{P}^1 \xleftarrow{\pi_1} F \xrightarrow{\pi_2} M = \mathbb{C}^4 \hookleftarrow \boxed{\mathbb{C}^2_p = \pi_2(\pi_1^{-1}(p))}$$

Then there is a 1-1 correspondence between:

- Holomorphic vector bundles E_P on P holomorphically trivial on any $\pi_1(\pi_2^{-1}(x)) = \mathbb{P}^1_x \hookrightarrow P$ for $x \in M$
- Holomorphic vector bundles E_M on M equipped with a connection ∇ flat on each $\pi_2(\pi_1^{-1}(p)) = \mathbb{C}^2_p \hookrightarrow M$ for $p \in P$, which implies $\nabla^2 = F = F^+$, i.e. self-dual YM

Question: In light of the success of twistor geometry in $d \leq 4$, can we apply similar ideas to theories in higher dimensions?

Chiral Fields In 6*d* And Their Twistorial Interpretation

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- Then choose coordinates $x^{AB} = -x^{BA}$ with $\partial_{AB} = -\partial_{BA}$, where $A, B, \ldots = 1, \ldots, 4$ and the metric is $\frac{1}{2}\varepsilon_{ABCD}$.
- Null-momentum p_{AB} is given by

$$\tfrac{1}{2} p_{AB} p_{CD} \varepsilon^{ABCD} = p_{AB} p^{AB} = 0,$$

so that

$$p_{AB} = k_{Aa} k_{Bb} \varepsilon^{ab} , \qquad p^{AB} = \tilde{k}^{A\dot{a}} \tilde{k}^{B\dot{b}} \varepsilon_{\dot{a}\dot{b}} ,$$

where $a, \dot{a}, \ldots$ are $SL(2,\mathbb{C}) \times \widetilde{SL(2,\mathbb{C})}$ little group indices.

Chiral Fields

- Interested in fields that transform trivially under $\widetilde{SL(2,\mathbb{C})} \to$ chiral fields with $(\mathbf{2h+1},\mathbf{1})$ and $h \in \frac{1}{2}\mathbb{N}_0$ which we call spin

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- The $\mathcal{N}=(2,0)$ tensor multiplet consists of a self-dual 3-form $H = \mathrm{d}B$ in the $(\mathbf{3},\mathbf{1})$ representation, four Weyl fermions ψ^I in the $(\mathbf{2},\mathbf{1})$ and five scalars ϕ^{IJ} in the $(\mathbf{1},\mathbf{1})$:

$$\partial^{AC} H_{CB} \;=\; \partial^{AC}\psi_C \;=\; \Box\phi \;=\; 0\;,$$

where

$$\left\{\begin{array}{rcl} H &=& \mathrm{d}B \\ H &=& *H \end{array}\right\} \leftrightarrow \left\{\begin{array}{rcl} (H_{AB}, H^{AB}) &=& (\partial_{C(A}B_{B)}{}^C, \partial^{C(A}B_C{}^{B)}) \\ H^{AB} &=& 0 \end{array}\right\}$$

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- The corresponding plane waves are

$$H_{AB\,ab} \;=\; k_{A(a}k_{Bb)}\,\mathrm{e}^{\mathrm{i}\,x\cdot p}\;,\quad \psi_{Aa} \;=\; k_{Aa}\,\mathrm{e}^{\mathrm{i}\,x\cdot p}\;,\quad \phi \;=\; \mathrm{e}^{\mathrm{i}\,x\cdot p}\;.$$

Twistor Space For Chiral Fields

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- One can show that

$$P \;\cong\; T^\vee \mathbb{P}^3 \otimes \mathcal{O}_{\mathbb{P}^3}(2) \;\hookrightarrow\; \mathcal{O}_{\mathbb{P}^3}(1) \otimes \mathbb{C}^4 \;\cong\; \mathbb{P}^7 \setminus \mathbb{P}^3$$

so we may use coordinates (z^A, λ_A) with $z^A \lambda_A = 0$ and thus

$$\begin{array}{ccc} & F & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ P & & M \end{array}$$

with π_2 being the trivial projection and

$$\pi_1 \;:\; (x^{AB}, \lambda_A) \;\mapsto\; (z^A, \lambda_A) \;=\; (x^{AB}\lambda_B, \lambda_A)$$

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$$
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 & \pi_1 \swarrow & & \searrow \pi_2 & \\
P & & & & M
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$$

Because of $z^A = x^{AB}\lambda_B$ we have a geometric correspondence:

$$
\begin{array}{ccc}
\pi_1(\pi_2^{-1}(x)) \cong \mathbb{P}^3_x \hookrightarrow P & \leftrightarrow & x \in M \\
p \in P & \leftrightarrow & \pi_2(\pi_1^{-1}(p)) \cong \mathbb{C}^3_z \hookrightarrow M
\end{array}
$$

where

$$
\mathbb{C}^3_p \;:\; x^{AB} \;=\; x_0^{AB} + \varepsilon^{ABCD}\mu_C\lambda_D
$$

which is a totally null 3-plane.

Penrose Transform: H^3

- Then for $h \in \frac{1}{2}\mathbb{N}_0$

$$H^3(P, \mathcal{O}_P(-2h-4)) \;\cong\; \left\{ \begin{array}{c} \text{chiral zero-rest-mass fields} \\ \text{of spin } h \text{ on } M \end{array} \right\}$$

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- This can be interpreted as a contour integral

$$\psi_{A_1\cdots A_{2h}}(x) \;=\; \oint_\gamma \Omega^{(3,0)}\, \lambda_{A_1}\cdots\lambda_{A_{2h}} f_{-2h-4}(x\cdot\lambda, \lambda)\ ,$$

where γ is topologically a 3-torus and

$$\Omega^{(3,0)} \;:=\; \tfrac{1}{4!}\varepsilon^{ABCD}\lambda_A \mathrm{d}\lambda_B \wedge \mathrm{d}\lambda_C \wedge \mathrm{d}\lambda_D$$

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What about $h < 0$?

Penrose–Ward Transform: H^2

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  by means of a Penrose–Ward transform.
- Note that in the case of interest for the self-dual 3-forms, we have $h = 1$ and thus $H^2(P, \mathcal{O}_P)$, which in turn is isomorphic to $H^2(P, \mathcal{O}^*_P)$. Hence, holomorphic bundle 1-gerbes on twistor space correspond to self-dual 3-form fields on space-time.

Twistor Action

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where $\Omega^{(4,0)}(z) := \frac{1}{4!}\varepsilon_{ABCD}\,\mathrm{d}\,z^A\wedge\mathrm{d}\,z^B\wedge\mathrm{d}\,z^C\wedge\mathrm{d}\,z^D$ and $\Omega^{(3,0)}(\lambda) := \frac{1}{4!}\varepsilon^{ABCD}\lambda_A\,\mathrm{d}\,\lambda_B\wedge\mathrm{d}\,\lambda_C\wedge\mathrm{d}\,\lambda_D$.

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- Then,

$$S \;=\; \int \Omega^{(6,0)}\wedge B^{(0,2)}_{2h-2}\wedge\bar\partial C^{(0,3)}_{-2h-4}\;.$$

Non-Abelian Extensions And Supersymmetry

Principal Bundles

- Let $M = \bigcup_a U_a$ be a manifold and G a Lie group. A principal bundle looks locally like $U_a \times G$. It can be described by transition functions $g_{ab} : U_a \cap U_b \to G$ with

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- Two transition functions $g_{ab} \sim g'_{ab}$ are said to be equivalent if $g'_{ab} = g_a g_{ab} g_b^{-1}$ for $g_a : U_a \to G$. A principal bundle is said to be trivial if $g_{ab} = g_a g_b^{-1} \sim 1$.

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- Let $\mathfrak{g} := \mathrm{Lie}G$, a connection is locally given by a $\mathfrak{g}$-valued connection 1-form A_a with curvature 2-form $F_a := \mathrm{d}\, A_a + A_a \wedge A_a$ so that on overlaps $U_a \cap U_b$

$$A_a = g_{ab}^{-1} A_b g_{ab} + g_{ab}^{-1} \mathrm{d}\, g_{ab} , \quad F_a = g_{ab}^{-1} F_b g_{ab}$$

and gauge transformations

$$A_a \mapsto \tilde{A}_a := g_a^{-1} A_a g_a + g_a^{-1} \mathrm{d}\, g_a , \quad F_a \mapsto \tilde{F}_a := g_a^{-1} F_a g_a .$$

Question: How can one generalise this to incorporate gauge potentials of higher form-degree?

Principal 2-Bundles: Part I

- Let $M = \bigcup_a U_a$ be a manifold and (G, H) a pair of Lie groups together with an automorphism action $\triangleright$ of G on H and a group homomorphism $t : H \to G$ such that

$$t(g \triangleright h) \;=\; gt(h)g^{-1} \;, \quad t(h_1) \triangleright h_2 \;=\; h_1 h_2 h_1^{-1}$$

called the equivariance and Peiffer conditions. This is known as a Lie crossed module.

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- A principal 2-bundle looks locally like $U_a \times (G, H)$. It can be described by transition functions $g_{ab} : U_a \cap U_b \to G$ and $h_{abc} : U_a \cap U_b \cap U_c \to H$ with

$$t(h_{abc}) g_{ab} g_{bc} \;=\; g_{ac} \;, \quad h_{acd} h_{abc} \;=\; h_{abd}(g_{ab} \triangleright h_{bcd})$$

Principal 2-Bundles: Part II

- Two transition functions $(g_{ab}, h_{abc}) \sim (g'_{ab}, h'_{abc})$ are said to be equivalent if

$$g_a g'_{ab} \;=\; t(h_{ab}) g_{ab} g_b \;, \quad h_{ac} h_{abc} \;=\; (g_a \rhd h'_{abc}) h_{ab} (g_{ab} \rhd h_{bc})$$

and trivial $(g_{ab}, h_{abc}) \sim (1, 1)$ if

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- Let $(\mathfrak{g}, \mathfrak{h}) := (\mathrm{Lie}\, G, \mathrm{Lie}\, H)$, a connection is locally given by a $\mathfrak{g}$-valued 1-form A_a and an $\mathfrak{h}$-valued 2-form B_a which are patched together by means of

$$\begin{aligned} A_a &= g_{ab}^{-1} A_b g_{ab} + g_{ab}^{-1} \mathrm{d}\, g_{ab} + t(\Lambda_{ab}) , \\ B_a &= g_{ab}^{-1} \rhd B_b + A_a \rhd \Lambda_{ab} + \mathrm{d}\, \Lambda_{ab} - \Lambda_{ab} \wedge \Lambda_{ab} , \end{aligned}$$

with

$$h_{abc}^{-1} (A_a \rhd h_{abc}) + h_{abc}^{-1} \mathrm{d}\, h_{abc} + g_{ab} \rhd \Lambda_{bc} + \Lambda_{ab} = h_{abc}^{-1} \Lambda_{ac} h_{abc} .$$

Principal 2-Bundles: Part III

- The curvatures are $F_a := \mathrm{d}\, A_a + A_a \wedge A_a$ and $H_a := \nabla B_a := \mathrm{d}\, B_a + A_a \triangleright B_a$ and we have the following gauge transformations:

$$
\begin{aligned}
A_a &\mapsto \tilde{A}_a := g_a^{-1} A_a g_a + g_a^{-1}\, \mathrm{d}\, g_a + t(\Lambda_a)\ ,\\
B_a &\mapsto \tilde{B}_a := g_a^{-1} \triangleright B_a + \tilde{A}_a \triangleright \Lambda_a + \mathrm{d}\, \Lambda_a - \Lambda_a \wedge \Lambda_a\ ,\\
F_a &\mapsto \tilde{F}_a := g_a^{-1} F_a g_a + t(\mathrm{d}\, \Lambda_a - \Lambda_a \wedge \Lambda_a) + \\
&\qquad\qquad + t(\Lambda_a) \wedge \tilde{A}_a + \tilde{A}_a \wedge t(\Lambda_a)\ ,\\
H_a &\mapsto \tilde{H}_a := g_a^{-1} \triangleright H_a + (\tilde{F}_a - t(\tilde{B}_a)) \triangleright \Lambda_a\ .
\end{aligned}
$$

- Thus, provided $F_a = t(B_a)$, the 3-form curvature transforms covariantly. This is called the fake curvature constraint.

Non-Abelian Self-Dual Tensor Field Equations

- Let us consider the following set of non-Abelian self-dual tensor equations

$$H \;=\; \mathrm{d}\,B + A \triangleright B\,, \quad H \;=\; \star_6 H\,, \quad F \;=\; \mathrm{d}\,A + A \wedge A \;=\; t(B)$$

on space-time $M \cong \mathbb{C}^6$. In spinor notation, this reads as

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on space-time $M \cong \mathbb{C}^6$. In spinor notation, this reads as

$$H^{AB} = \nabla^{C(A} B_C{}^{B)} \;=\; 0\,, \quad F_A{}^B \;=\; t(B_A{}^B)$$

- Can we use twistor theory to derive these equations including the just-mentioned gauge transformations from algebraic data on twistor space?

Penrose–Ward Transform: $P \overset{\pi_1}{\leftarrow} F \overset{\pi_2}{\rightarrow} M$

Theorem: There is a bijection between

(i) equivalence classes of (topologically trivial) holomorphic principal 2-bundles over the twistor space P that are holomorphically trivial when restricted to any complex projective 3-space $\pi_1(\pi_2^{-1}(x)) \hookrightarrow P$ for $x \in M$ and

(ii) gauge equivalence classes of (complex holomorphic) solutions to the non-Abelian self-dual tensor field equations on space-time M.

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Remark: The proof makes use of cohomology theory and Riemann–Hilbert problems on twistor space. The non-uniquess of the RH problems is the origin of the gauge transformations on space-time.

Question: What about supersymmetry?

Supertwistor Space: Part I

- Consider $\mathcal{N} = (n, 0)$ superspace $M = \mathbb{C}^{6|8n}$ with coordinates (x^{AB}, η_I^A) with $I, J, \ldots = 1, \ldots, 2n$. The derivatives

$$P_{AB} := \frac{\partial}{\partial x^{AB}} , \quad D_A^I := \frac{\partial}{\partial \eta_I^A} - 2\Omega^{IJ}\eta_J^B \frac{\partial}{\partial x^{AB}}$$

obey

$$\{D_A^I, D_B^J\} = -4\Omega^{IJ} P_{AB} .$$

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- Define the correspondence space F to be $F := \mathbb{C}^{4|8n} \times \mathbb{P}^3$ with coordinates $(x^{AB}, \eta_I^A, \lambda_A)$.
- Introduce a rank-$3|6n$ distribution $\langle V^A, V^{I\,AB}\rangle \hookrightarrow TF$ by $V^A := \lambda_B \partial^{AB}$ and $V^{I\,AB} = \frac{1}{2}\varepsilon^{ABCD}\lambda_C D_D^I$ which is integrable. Hence, we have foliation $P := F/\langle V^A, V^{I\,AB}\rangle$.

Supertwistor Space: Part II

- On P, we may use coordinates (z^A, η_I, λ_A) with $z^A\lambda_A = \Omega^{IJ}\eta_I\eta_J$ and thus

$$\begin{array}{ccc} & F & \\ {\scriptstyle \pi_1} \swarrow & & \searrow {\scriptstyle \pi_2} \\ P & & M \end{array}$$

  with π_2 being the trivial projection and

$$\pi_1 : (x^{AB}, \eta_I^A, \lambda_A) \;\mapsto\; (z^A, \eta_I, \lambda_A) \;=$$
$$= ((x^{AB} + \Omega^{IJ}\eta_I^A\eta_J^B)\lambda_B, \eta_I^A\lambda_A, \lambda_A)$$

- A point $x \in M$ corresponds to a complex projective 3-space in P, while a point $p \in P$ corresponds to a totally null $3|6n$-superplane with

$$x^{AB} \;=\; x_0^{AB} + \varepsilon^{ABCD}\mu_C\lambda_D + 2\Omega^{IJ}\varepsilon^{CDE[A}\lambda_C\theta_{IDE}\eta_{0}{}_J^{B]}\;,$$
$$\eta_I^A \;=\; \eta_{0}{}_I^A + \varepsilon^{ABCD}\lambda_B\theta_{ICD}\;.$$

Theorem: There is a bijection between

(i) equivalence classes of (topologically trivial) holomorphic principal 2-bundles over the supertwistor space P that are holomorphically trivial when restricted to any complex projective 3-space $\pi_1(\pi_2^{-1}(x)) \hookrightarrow P$ for $x \in M$ and

(ii) gauge equivalence classes of (complex holomorphic) solutions to the constraint system

$$\begin{aligned} F_A{}^B &= t(B_A{}^B)\ , \quad F_{AB}{}^I_C = t(B_{AB}{}^I_C)\ , \quad F^{IJ}_{AB} = t(B^{IJ}_{AB})\ , \\ H^{AB} &= 0\ , \\ H_A{}^{BI}{}_C &= \delta^B_C \psi^I_A - \tfrac{1}{4}\delta^B_A \psi^I_C\ , \\ H_{AB}{}^{IJ}_{CD} &= \varepsilon_{ABCD}\phi^{IJ}\ , \\ H^{IJK}_{ABC} &= 0\ , \end{aligned}$$

on $\mathcal{N} = (n, 0)$ superspace.

Remarks

- We obtain the fields $(H_{AB}, \psi^I_A, \phi^{IJ})$ which transform on-shell under gauge transformations as

$$(H_{AB}, \psi^I_A, \phi^{IJ}) \;\mapsto\; g^{-1} \triangleright (H_{AB}, \psi^I_A, \phi^{IJ}).$$

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- For $n = 1$, $(H_{AB}, \psi^I_A, \phi^{IJ})$ constitutes an $\mathcal{N} = (1,0)$ tensor multiplet consisting of a self-dual 3-form, 2 spinors, and a scalar. For $n = 2$, we get the $\mathcal{N} = (2,0)$ tensor multiplet after imposing the constraint $\Omega_{IJ}\phi^{IJ} = 0$ (which is familiar from $\mathcal{N} = 4$ SYM in 4d)

Summary

- Discussed a twistor space suitable for 6*d* chiral fields.
- Found Penrose and Penrose–Ward transforms to encode spinor fields in terms of cohomology.
- Found novel (supersymmetric) non-Abelian tensor field equations based in principal 2-bundles.
- Also in 6*d*, twistor theory is a natural language for self-duality. Big advantage: e.o.m. plus gauge symmetry simply arise from algebraic data on twistor space.
- Not mention here but performed in our paper, everything can be dimensionally reduced to discuss the twistor theory of Abelian and non-Abelian self-dual strings.
- Non-chiral 6*d* theories such $\mathcal{N} = (1,1)$ SYM theory can also be discussed by means of twistors.

C Sämann, R Wimmer, M Wolf, JHEP 1205 (2012) 020

Thank You!