

Non-Abelian Self-Dual Tensor Field Theories From Twistor Space

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Outline

- Introduction And Motivation
- Chiral Fields In $6d$ And Their Twistorial Interpretation
- Non-Abelian Extensions And Supersymmetry
- Conclusions And Outlook

Introduction And Motivation

Major (Classical) Twistor Applications

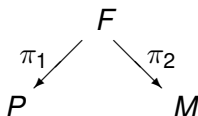
- Free fields in $4d \leftrightarrow$ cohomology groups via the Penrose transform
- $4d$ Yang–Mills (YM) instantons \leftrightarrow holomorphic vector bundles via the Penrose–Ward transform
- $4d$ gravitational instantons \leftrightarrow holomorphic contact structures via the Ward construction
- Full $4d$ YM \leftrightarrow holomorphic vector bundles that can be extended in a certain fashion \rightarrow full $\mathcal{N} = 3, 4$ SYM theory
- Applications to theories in $d < 4$ via dimensional reduction
- ...

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Differentially constrained data \leftrightarrow Algebraic data

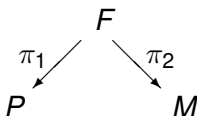
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- P twistor space

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where M , F and P are complex manifolds:

- M space-time
 - F correspondence space
 - P twistor space
- Then we have a **correspondence** between P and M , i.e. between **points** in one space and **subspaces** of the other:

$$\begin{array}{lcl} \pi_1(\pi_2^{-1}(x)) \hookrightarrow P & \leftrightarrow & x \in M \\ p \in P & \leftrightarrow & \pi_2(\pi_1^{-1}(p)) \hookrightarrow M \end{array}$$

Twistor Correspondence: $P \xleftarrow{\pi_1} F \xrightarrow{\pi_2} M$

- Using this correspondence, we can **transfer** data given on P to data on M and vice versa (e.g. vector bundles, sheaf cohomology groups, contact forms, ...)

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- Take some analytic object **Ob_P on P** and transform it to an object **Ob_M on M** ; this in turn is constrained by some **PDEs** as $\pi_1^* Ob_P$ has to be constant up the fibres of $\pi_1 : F \rightarrow P$

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- Under suitable topological conditions, the maps

$$Ob_P \mapsto Ob_M \quad \text{and} \quad Ob_M \mapsto Ob_P$$

define a **bijection** between $[Ob_P]$ and $[Ob_M]$ (the objects in question will only be defined up to equivalence)

Two Prime Examples

Penrose Transform

Consider 4d flat space $M = \mathbb{C}^4$ with $TM \cong S \otimes \tilde{S}$:

$$\begin{array}{ccc} F = \mathbb{P}(\tilde{S}^\vee) = \mathbb{C}^4 \times \mathbb{P}^1 & & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ P = \mathbb{P}^3 \setminus \mathbb{P}^1 & & M = \mathbb{C}^4 \end{array}$$

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Then

$$H^1(P, \mathcal{O}_P(-2h-2)) \cong \left\{ \begin{array}{l} \text{zero-rest-mass fields} \\ \text{of helicity } h \text{ on } M \end{array} \right\}$$

Penrose–Ward Transform

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Then there is a **1-1 correspondence** between:

- Holomorphic vector bundles E_P on P holomorphically trivial on any $\pi_1(\pi_2^{-1}(x)) = \mathbb{P}_x^1 \hookrightarrow P$ for $x \in M$
- Holomorphic vector bundles E_M on M equipped with a connection ∇ flat on each $\pi_2(\pi_1^{-1}(p)) = \mathbb{C}_p^2 \hookrightarrow M$ for $p \in P$, which implies $\nabla^2 = F = F^+$, i.e. self-dual YM

Question: In light of the success of twistor geometry in $d \leq 4$, can we apply similar ideas to theories in higher dimensions?

Chiral Fields In $6d$ And Their Twistorial Interpretation

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- Null-momentum p_{AB} is given by

$$\frac{1}{2}p_{AB}p_{CD}\varepsilon^{ABCD} = p_{AB}p^{AB} = 0,$$

so that

$$p_{AB} = k_{Aa}k_{Bb}\varepsilon^{ab}, \quad p^{AB} = \tilde{k}^{A\dot{a}}\tilde{k}^{B\dot{b}}\varepsilon_{\dot{a}\dot{b}},$$

where a, \dot{a}, \dots are $SL(2, \mathbb{C}) \times \widetilde{SL(2, \mathbb{C})}$ little group indices.

Chiral Fields

- Interested in fields that transform **trivially** under $\widetilde{SL}(2, \mathbb{C}) \rightarrow$ chiral fields with $(\mathbf{2h} + \mathbf{1}, \mathbf{1})$ and $h \in \frac{1}{2}\mathbb{N}_0$ which we call spin

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- The $\mathcal{N} = (2, 0)$ tensor multiplet consists of a self-dual 3-form $H = dB$ in the $(\mathbf{3}, \mathbf{1})$ representation, four Weyl fermions ψ^I in the $(\mathbf{2}, \mathbf{1})$ and five scalars ϕ^J in the $(\mathbf{1}, \mathbf{1})$:

$$\partial^{AC} H_{CB} = \partial^{AC} \psi_C = \square \phi = 0,$$

where

$$\left\{ \begin{array}{l} H = dB \\ H = *H \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} (H_{AB}, H^{AB}) = (\partial_{C(A} B_{B)}{}^C, \partial^{C(A} B_C{}^{B)}) \\ H^{AB} = 0 \end{array} \right\}$$

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- The corresponding plane waves are

$$H_{ABab} = k_{A(a} k_{Bb)} e^{ix \cdot p}, \quad \psi_{Aa} = k_{Aa} e^{ix \cdot p}, \quad \phi = e^{ix \cdot p}.$$

Twistor Space For Chiral Fields

- Starting from space-time M with coordinates x^{AB} , define the correspondence space F to be $F := \mathbb{P}(S^\vee) \cong \mathbb{C}^6 \times \mathbb{P}^3$ with coordinates (x^{AB}, λ_A) .

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- One can show that

$$P \cong T^\vee \mathbb{P}^3 \otimes \mathcal{O}_{\mathbb{P}^3}(2) \hookrightarrow \mathcal{O}_{\mathbb{P}^3}(1) \otimes \mathbb{C}^4 \cong \mathbb{P}^7 \setminus \mathbb{P}^3$$

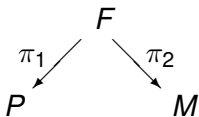
so we may use coordinates (z^A, λ_A) with $z^A \lambda_A = 0$ and thus

$$\begin{array}{ccc} & F & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ P & & M \end{array}$$

with π_2 being the trivial projection and

$$\pi_1 : (x^{AB}, \lambda_A) \mapsto (z^A, \lambda_A) = (x^{AB} \lambda_B, \lambda_A)$$

Twistor Space For Chiral Fields



Because of $z^A = x^{AB}\lambda_B$ we have a geometric correspondence:

$$\begin{array}{ccc} \pi_1(\pi_2^{-1}(x)) \cong \mathbb{P}_x^3 \hookrightarrow P & \leftrightarrow & x \in M \\ p \in P & \leftrightarrow & \pi_2(\pi_1^{-1}(p)) \cong \mathbb{C}_z^3 \hookrightarrow M \end{array}$$

where

$$\mathbb{C}_p^3 : x^{AB} = x_0^{AB} + \varepsilon^{ABCD} \mu_C \lambda_D$$

which is a **totally null 3-plane**.

- Then for $h \in \frac{1}{2}\mathbb{N}_0$

$$H^3(P, \mathcal{O}_P(-2h-4)) \cong \left\{ \begin{array}{l} \text{chiral zero-rest-mass fields} \\ \text{of spin } h \text{ on } M \end{array} \right\}$$

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- This can be interpreted as a contour integral

$$\psi_{A_1 \dots A_{2h}}(x) = \oint_{\gamma} \Omega^{(3,0)} \lambda_{A_1} \cdots \lambda_{A_{2h}} f_{-2h-4}(x \cdot \lambda, \lambda),$$

where γ is topologically a 3-torus and

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What about $h < 0$?

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- Note that in the case of interest for the self-dual 3-forms, we have $h = 1$ and thus $H^2(P, \mathcal{O}_P)$, which in turn is isomorphic to $H^2(P, \mathcal{O}_P^*)$. Hence, **holomorphic bundle 1-gerbes** on twistor space correspond to self-dual 3-form fields on space-time.

Twistor Action

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- Then,

$$S = \int \Omega^{(6,0)} \wedge B_{2h-2}^{(0,2)} \wedge \bar{\partial} C_{-2h-4}^{(0,3)}.$$

Non-Abelian Extensions And Supersymmetry

Principal Bundles

- Let $M = \bigcup_a U_a$ be a manifold and G a Lie group. A principal bundle looks locally like $U_a \times G$. It can be described by transition functions $g_{ab} : U_a \cap U_b \rightarrow G$ with

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- Two transition functions $g_{ab} \sim g'_{ab}$ are said to be **equivalent** if $g'_{ab} = g_a g_{ab} g_b^{-1}$ for $g_a : U_a \rightarrow G$. A principal bundle is said to be **trivial** if $g_{ab} = g_a g_b^{-1} \sim 1$.

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- Let $\mathfrak{g} := \text{Lie}G$, a connection is locally given by a \mathfrak{g} -valued **connection 1-form** A_a with **curvature 2-form** $F_a := dA_a + A_a \wedge A_a$ so that on overlaps $U_a \cap U_b$

$$A_a = g_{ab}^{-1} A_b g_{ab} + g_{ab}^{-1} dg_{ab}, \quad F_a = g_{ab}^{-1} F_b g_{ab}$$

and **gauge transformations**

$$A_a \mapsto \tilde{A}_a := g_a^{-1} A_a g_a + g_a^{-1} dg_a, \quad F_a \mapsto \tilde{F}_a := g_a^{-1} F_a g_a.$$

Question: How can one generalise this to incorporate gauge potentials of higher form-degree?

Principal 2-Bundles: Part I

- Let $M = \bigcup_a U_a$ be a manifold and (G, H) a pair of Lie groups together with an automorphism action \triangleright of G on H and a group homomorphism $t : H \rightarrow G$ such that

$$t(g \triangleright h) = gt(h)g^{-1}, \quad t(h_1) \triangleright h_2 = h_1 h_2 h_1^{-1}$$

called the **equivariance** and **Peiffer** conditions. This is known as a **Lie crossed module**.

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- A principal 2-bundle looks locally like $U_a \times (G, H)$. It can be described by transition functions $g_{ab} : U_a \cap U_b \rightarrow G$ and $h_{abc} : U_a \cap U_b \cap U_c \rightarrow H$ with

$$t(h_{abc})g_{ab}g_{bc} = g_{ac}, \quad h_{acd}h_{abc} = h_{abd}(g_{ab} \triangleright h_{bcd})$$

Principal 2-Bundles: Part II

- Two transition functions $(g_{ab}, h_{abc}) \sim (g'_{ab}, h'_{abc})$ are said to be **equivalent** if

$$g_a g'_{ab} = t(h_{ab}) g_{ab} g_b, \quad h_{ac} h_{abc} = (g_a \triangleright h'_{abc}) h_{ab} (g_{ab} \triangleright h_{bc})$$

and **trivial** $(g_{ab}, h_{abc}) \sim (1, 1)$ if

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- Let $(g, \mathfrak{h}) := (\text{Lie}G, \text{Lie}H)$, a connection is locally given by a g -valued **1-form** A_a and an \mathfrak{h} -valued **2-form** B_a which are patched together by means of

$$A_a = g_{ab}^{-1} A_b g_{ab} + g_{ab}^{-1} d g_{ab} + t(\Lambda_{ab}),$$

$$B_a = g_{ab}^{-1} \triangleright B_b + A_a \triangleright \Lambda_{ab} + d \Lambda_{ab} - \Lambda_{ab} \wedge \Lambda_{ab},$$

with

$$h_{abc}^{-1} (A_a \triangleright h_{abc}) + h_{abc}^{-1} d h_{abc} + g_{ab} \triangleright \Lambda_{bc} + \Lambda_{ab} = h_{abc}^{-1} \Lambda_{ac} h_{abc}.$$

- The curvatures are $F_a := dA_a + A_a \wedge A_a$ and $H_a := \nabla B_a := dB_a + A_a \triangleright B_a$ and we have the following **gauge transformations**:

$$A_a \mapsto \tilde{A}_a := g_a^{-1} A_a g_a + g_a^{-1} dg_a + t(\Lambda_a),$$

$$B_a \mapsto \tilde{B}_a := g_a^{-1} \triangleright B_a + \tilde{A}_a \triangleright \Lambda_a + d\Lambda_a - \Lambda_a \wedge \Lambda_a,$$

$$F_a \mapsto \tilde{F}_a := g_a^{-1} F_a g_a + t(d\Lambda_a - \Lambda_a \wedge \Lambda_a) + t(\Lambda_a) \wedge \tilde{A}_a + \tilde{A}_a \wedge t(\Lambda_a),$$

$$H_a \mapsto \tilde{H}_a := g_a^{-1} \triangleright H_a + (\tilde{F}_a - t(\tilde{B}_a)) \triangleright \Lambda_a.$$

- Thus, provided $F_a = t(B_a)$, the 3-form curvature transforms covariantly. This is called the **fake curvature constraint**.

Non-Abelian Self-Dual Tensor Field Equations

- Let us consider the following set of **non-Abelian self-dual tensor equations**

$$H = dB + A \triangleright B, \quad H = \star_6 H, \quad F = dA + A \wedge A = t(B)$$

on space-time $M \cong \mathbb{C}^6$. In spinor notation, this reads as

$$H^{AB} = \nabla^{C(A} B_C{}^{B)} = 0, \quad F_A{}^B = t(B_A{}^B)$$

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- Can we use twistor theory to derive these equations including the just-mentioned gauge transformations from algebraic data on twistor space?

Theorem: There is a bijection between

- (i) equivalence classes of (topologically trivial) holomorphic principal 2-bundles over the twistor space P that are holomorphically trivial when restricted to any complex projective 3-space $\pi_1(\pi_2^{-1}(x)) \hookrightarrow P$ for $x \in M$ and
- (ii) gauge equivalence classes of (complex holomorphic) solutions to the non-Abelian self-dual tensor field equations on space-time M .

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- (i) equivalence classes of (topologically trivial) holomorphic principal 2-bundles over the twistor space P that are holomorphically trivial when restricted to any complex projective 3-space $\pi_1(\pi_2^{-1}(x)) \hookrightarrow P$ for $x \in M$ and
- (ii) gauge equivalence classes of (complex holomorphic) solutions to the non-Abelian self-dual tensor field equations on space-time M .

Remark: The proof makes use of cohomology theory and Riemann–Hilbert problems on twistor space. The non-uniqueness of the RH problems is the origin of the gauge transformations on space-time.

Question: What about supersymmetry?

Supertwistor Space: Part I

- Consider $\mathcal{N} = (n, 0)$ superspace $M = \mathbb{C}^{6|8n}$ with coordinates (x^{AB}, η_I^A) with $I, J, \dots = 1, \dots, 2n$. The derivatives

$$P_{AB} := \frac{\partial}{\partial x^{AB}}, \quad D_A^I := \frac{\partial}{\partial \eta_I^A} - 2\Omega^{IJ} \eta_J^B \frac{\partial}{\partial x^{AB}}$$

obey

$$\{D_A^I, D_B^J\} = -4\Omega^{IJ} P_{AB}.$$

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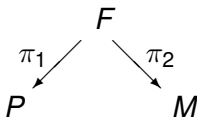
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- Define the correspondence space F to be $F := \mathbb{C}^{4|8n} \times \mathbb{P}^3$ with coordinates $(x^{AB}, \eta_I^A, \lambda_A)$.
- Introduce a **rank-3|6n** distribution $\langle V^A, V^{IAB} \rangle \hookrightarrow TF$ by $V^A := \lambda_B \partial^{AB}$ and $V^{IAB} = \frac{1}{2} \varepsilon^{ABCD} \lambda_C D_D^I$ which is integrable. Hence, we have foliation $P := F / \langle V^A, V^{IAB} \rangle$.

Supertwistor Space: Part II

- On P , we may use coordinates (z^A, η_I, λ_A) with $z^A \lambda_A = \Omega^{IJ} \eta_I \eta_J$ and thus



with π_2 being the trivial projection and

$$\begin{aligned} \pi_1 : (x^{AB}, \eta_I^A, \lambda_A) &\mapsto (z^A, \eta_I, \lambda_A) = \\ &= ((x^{AB} + \Omega^{IJ} \eta_I^A \eta_J^B) \lambda_B, \eta_I^A \lambda_A, \lambda_A) \end{aligned}$$

- A point $x \in M$ corresponds to a complex projective 3-space in P , while a point $p \in P$ corresponds to a **totally null 3|6n-superplane** with

$$\begin{aligned} x^{AB} &= x_0^{AB} + \varepsilon^{ABCD} \mu_C \lambda_D + 2\Omega^{IJ} \varepsilon^{CDE[A} \lambda_C \theta_{IDE} \eta_0^{B]} , \\ \eta_I^A &= \eta_{0I}^A + \varepsilon^{ABCD} \lambda_B \theta_{ICD} . \end{aligned}$$

Penrose–Ward Transform: $P \xleftarrow{\pi_1} F \xrightarrow{\pi_2} M$

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- (ii) gauge equivalence classes of (complex holomorphic) solutions to the constraint system

$$F_A^B = t(B_A^B), \quad F_{ABC}^I = t(B_{ABC}^I), \quad F_{AB}^{IJ} = t(B_{AB}^{IJ}),$$

$$H^{AB} = 0,$$

$$H_A^{BI}{}_C = \delta_C^B \psi_A^I - \frac{1}{4} \delta_A^B \psi_C^I,$$

$$H_{ABCD}^{IJ} = \varepsilon_{ABCD} \phi^{IJ},$$

$$H_{ABC}^{IJK} = 0,$$

on $\mathcal{N} = (n, 0)$ superspace.

- We obtain the fields $(H_{AB}, \psi_A^I, \phi^{IJ})$ which transform on-shell under gauge transformations as

$$(H_{AB}, \psi_A^I, \phi^{IJ}) \mapsto g^{-1} \triangleright (H_{AB}, \psi_A^I, \phi^{IJ}).$$

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- For $n = 1$, $(H_{AB}, \psi_A^I, \phi^{IJ})$ constitutes an $\mathcal{N} = (1, 0)$ tensor multiplet consisting of a self-dual 3-form, 2 spinors, and a scalar. For $n = 2$, we get the $\mathcal{N} = (2, 0)$ tensor multiplet after imposing the constraint $\Omega_{IJ} \phi^{IJ} = 0$ (which is familiar from $\mathcal{N} = 4$ SYM in $4d$)

Summary

- Discussed a twistor space suitable for $6d$ chiral fields.
- Found Penrose and Penrose–Ward transforms to encode spinor fields in terms of cohomology.
- Found novel (supersymmetric) non-Abelian tensor field equations based in principal 2-bundles.
- Also in $6d$, twistor theory is a natural language for self-duality. Big advantage: e.o.m. plus gauge symmetry simply arise from algebraic data on twistor space.
- Not mention here but performed in our paper, everything can be dimensionally reduced to discuss the twistor theory of Abelian and non-Abelian self-dual strings.
- Non-chiral $6d$ theories such $\mathcal{N} = (1, 1)$ SYM theory can also be discussed by means of twistors.

C Sämann, R Wimmer, M Wolf, JHEP 1205 (2012) 020

Thank You!