From Symmetric Groups to Giant Gravitons

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Goal of the talk

Compute the large N spectrum of anomalous dimensions at one loop, of a class of operators whose classical dimension is of order N.

Interpret the results using AdS/CFT.

Take note: To capture the large N limit, we need to sum much more than just the planar diagrams.

$\mathcal{N} = 4$ SYM theory

We consider $\mathcal{N} = 4$ SYM theory on $R \times S^3$.

Consider the complex combinations $X = \phi_1 + i\phi_2,\ Y = \phi_3 + i\phi_4$, $Z = \phi_5 + i\phi_6$, built from the s-wave components of the 6 adjoint scalars.

The free two point function is

$$\langle Z^i{}_j (Z^\dagger)^k{}_l \rangle = \delta^i_l \delta^k_j$$

Consider operators built using n Z fields and m Y fields. Both n and m are of order N. Can restrict to subspaces of definite n and m.

$m = 0$, $n \neq 0$ - $\frac{1}{2}$-BPS sector

Correspond to the lowest weight states of half-BPS reps which satisfy $E = J_{56}$, where the J_{ij} are generators of the $SO(6)$ $\mathcal{R}$ symmetry.

$Z = X_5 + iX_6$ satisfies the $E = J_{56}$ condition. The independent gauge-invariant BPS states correspond to traces and products of traces.

$n = 1 : \ \mathrm{Tr}(Z)$
$n = 2 : \ \mathrm{Tr}(Z^2);\ (\mathrm{Tr}Z)^2$
$n = 3 : \ \mathrm{Tr}(Z^3);\ \mathrm{Tr}(Z^2)\mathrm{Tr}(Z);\ (\mathrm{Tr}Z)^3$

Large N factorization of correlators implies that distinct multi-trace structures are orthogonal in the large N limit.

$$\langle \frac{\mathrm{Tr}(Z^J)}{\sqrt{JN^J}} \frac{\mathrm{Tr}(Z^{\dagger\, J})}{\sqrt{JN^J}} \rangle = 1$$

$$\langle \frac{\mathrm{Tr}(Z^{J_1})}{\sqrt{J_1 N^{J_1}}} \frac{\mathrm{Tr}(Z^{J_2})}{\sqrt{J_2 N^{J_2}}} \frac{\mathrm{Tr}(Z^{\dagger\, J_3})}{\sqrt{J_3 N^{J_3}}} \rangle = \frac{\sqrt{J_1 J_2 J_3}}{N} \delta_{J_1+J_2;J_3}$$

Orthogonality breaks down at $J_i \sim N^{\frac{1}{3}}$

(Balasubramanian, Berkooz, Naqvi, Strassler, hep-th/0107119)

Multitraces

At $n = 2$ we have two gauge invariant operators

$$\mathrm{Tr} Z \mathrm{Tr} Z = Z^{i_1}_{i_1} Z^{i_2}_{i_2} \qquad \mathrm{Tr}(Z^2) = Z^{i_1}{}_{i_2} Z^{i_2}_{i_1}$$

They differ in how the lower labels are permuted with respect to the upper labels.

More generally

$$\mathrm{Tr}(\sigma Z^{\otimes n}) = Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} \cdots Z^{i_n}_{i_{\sigma(n)}}$$

gives a convenient description for talking about the complete set of multitrace operators. Permutations in the same conjugacy class determine the same operator.

Schur Polynomials

$$\chi_R(Z) = \frac{1}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)\mathrm{Tr}(\sigma Z^{\otimes n})$$

R specifies an irrep of S_n. $\chi_R(\sigma)$ is the character of σ in irrep R.

$$\chi_{\yng(3)} = \frac{1}{6}\left[\mathrm{Tr}(Z)^3 + 3\mathrm{Tr}(Z)\mathrm{Tr}(Z^2) + 2\mathrm{Tr}(Z^3)\right]$$

$$\chi_{\yng(2,1)} = \frac{1}{6}\left[2\mathrm{Tr}(Z)^3 - 2\mathrm{Tr}(Z^3)\right]$$

$$\chi_{\yng(1,1,1)} = \frac{1}{6}\left[\mathrm{Tr}(Z)^3 - 3\mathrm{Tr}(Z)\mathrm{Tr}(Z^2) + 2\mathrm{Tr}(Z^3)\right]$$

(Corley, Jevicki, Ramgoolam, hep-th/0111222)

Schur Polynomials

Number of Schur polynomials agrees with finite N counting.

$$\langle \chi_R(Z) \chi_S(Z)^\dagger \rangle = f_R \delta_{RS}$$

$$\mathrm{Tr}(\sigma Z^{\otimes n}) = \sum_R \chi_R(\sigma) \chi_R(Z)$$

Including Y: $m \neq 0$

How much of the $\frac{1}{2}$-BPS story can be generalized?

$$\mathrm{Tr}(\sigma Z^{\otimes n} \otimes Y^{\otimes m}) = Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} \cdots Z^{i_n}_{i_{\sigma(n)}} Y^{i_{n+1}}_{i_{\sigma(n+1)}} Y^{i_{n+2}}_{i_{\sigma(n+2)}} \cdots Y^{i_{m+n}}_{i_{\sigma(m+n)}}$$

again gives a convenient description for talking about the complete set of multitrace operators.

Permutations related by

$$\gamma \sigma_1 \gamma^{-1} = \sigma_2 \qquad \sigma_1, \sigma_2 \in S_{n+m} \quad \gamma \in S_n \times S_m$$

determine the same operator.

Restricted Schur Polynomials

$$\chi_{R,(r,s)\alpha\beta}(Z,Y) = \frac{1}{n!m!} \sum_{\sigma \in S_{n+m}} \mathrm{Tr}_{(r,s)\alpha\beta}(\Gamma^R(\sigma)) \mathrm{Tr}(\sigma Z^{\otimes n} \otimes Y^{\otimes m})$$

R is an irrep of S_{n+m}. We can subduce the $S_n \times S_m$ irrep (r,s) from R. α, β keep track of which copy we subduce.

$$\chi_{\square\square,(\square,\square)} = \mathrm{Tr}(Z)\mathrm{Tr}(Y) + \mathrm{Tr}(ZY)$$
$$\chi_{\substack{\square\\\square},(\square,\square)} = \mathrm{Tr}(Z)\mathrm{Tr}(Y) - \mathrm{Tr}(ZY)$$

(Berenstein, Balasubramanian, Feng, Huang, hep-th/0411205; Bhattacharyya, Collins, dMK, arXiv:0801.2061)

Restricted schurs

Number of restricted Schur polynomials agrees with finite N counting.

$$\langle \chi_{R,(r,s)\mu\nu}(Z,Y)\chi_{S,(t,u)\alpha\beta}(Z,Y)^\dagger\rangle = N(R,r,s)\delta_{RS}\delta_{rt}\delta_{su}\delta_{\mu\alpha}\delta_{\nu\beta}$$

$$\mathrm{Tr}(\sigma Z^{\otimes n} Y^{\otimes m}) = \sum_R \chi_{R,(r,s)\alpha\beta}(\sigma)\chi_{R,(r,s)\beta\alpha}(Z,Y)$$

(Collins, arXiv:0810.4217; Bhattacharyya, Collins, dMK, arXiv:0801.2061; Bhattacharyya, dMK, Stephanou, arXiv:0805.3025)

Other bases

For operators built from Z and Z^* using the Brauer algebra. (Kimura, Ramgoolam, arXiv:0709.2158)

A basis will have good global quantum numbers. (Brown, Heslop, Ramgoolam, arXiv:0711.0176, arXiv:0806.1911)

A restricted Schur basis for fermions. (dMK, Diaz, Nokwara, arXiv:1212.5935)

A restricted Schur basis for ABJM. (dMK, Mohammed, Murugan, Prinsloo, arXiv:1202.4925)

A basis for quiver gauge theories. (Pasukonis, Ramgoolam, arXiv:1301.1980)

Dilatation Operator

$$D = -g_{YM}^2 \mathrm{Tr}\left(\left[Z, Y\right]\left[\frac{d}{dZ}, \frac{d}{dY}\right]\right)$$

(Beisert, Kristjansen, Staudacher, hep-th/0303060)

$$D\chi_{R,(r,s)ji} = \sum_{S,(t,u)kl} M_{R,(r,s)ji;S,(t,u)kl}\chi_{S,(t,u)lk}$$

$$M_{R,(r,s)jk;T,(t,u)lq} = -g_{YM}^2 \sum_{R'} N_{R,R',r,s,T,t,u}$$

$$\times \mathrm{Tr}\left(\left[\Gamma^R(1, m+1), P_{R,(r,s)jk}\right] I_{R'\,T'}\left[\Gamma^T(1, m+1), P_{T,(t,u)ql}\right] I_{T'\,R'}\right).$$

$$\mathrm{Tr}_{(r,s)\alpha\beta}(*) = \mathrm{Tr}_R(P_{R,(r,s)\alpha\beta}*)$$

(De Comarmond, dMK, Jefferies, arXiv:1012.3884)

A Decoupled Sector

Focus on operators $\chi_{R,(r,s)\alpha\beta}$ for which R and r have p rows and s has $\leq p$ rows.

At large N the dilatation operator does not mix operators with different values of p.

Sectors with long rows / columns decouple.

(dMK, Mashile, Park, arXiv:1004.1108)

The way forwards

Even though we have reduced problem of mixing dramatically by giving decoupled subsectors of operators that don't mix, we are still not able to diagonalize the one loop dilatation operator.

We will introduce another approximation - the displaced corners approximation - defined precisely below.

It is not used when evaluating the action of D - but rather simplifies the construction of $P_{R,(r,s)\alpha\beta}$.

How do we justify this? Compute some exact answers (for $m = O(1)$) and compare to results obtained in the displaced corners approximation.

So now: describe the exact answers

$m = 3$ sector

$$\chi_A(b_0, b_1) = \chi_{\ydiagram{2,2,2,2,2,1,1,1,1,1}\ \ydiagram{2,2,2,2,2,1,1}\ \ydiagram{1,1,1}}(Z, Y) \qquad \chi_B(b_0, b_1) = \chi_{\ydiagram{2,2,2,2,2,1,1,1,1,1}\ \ydiagram{2,2,2,2,1,1,1,1}\ \ydiagram{1,1,1}}(Z, Y)$$

$$\chi_C(b_0, b_1) = \chi_{\ydiagram{2,2,2,2,2,1,1,1,1,1}\ \ydiagram{2,2,2,2,1,1,1,1}\ \ydiagram{2,1}}(Z, Y) \qquad \chi_D(b_0, b_1) = \chi_{\ydiagram{2,2,2,2,2,1,1,1,1,1}\ \ydiagram{2,2,2,1,1,1,1,1,1}\ \ydiagram{1,1,1}}(Z, Y)$$

$$\chi_E(b_0, b_1) = \chi_{\ydiagram{2,2,2,2,2,1,1,1,1,1}\ \ydiagram{2,2,2,1,1,1,1,1,1}\ \ydiagram{2,1}}(Z, Y) \qquad \chi_F(b_0, b_1) = \chi_{\ydiagram{2,2,2,2,2,1,1,1,1,1}\ \ydiagram{2,2,1,1,1,1,1,1,1,1}\ \ydiagram{1,1,1}}(Z, Y)$$

Explicit expression for

We find that $D\chi_{*,(b_0,b_1)}$ is given by a VCF.

Explicit expression for

We find that $D\chi_{*,(b_0,b_1)}$ is given by a VCF.

VCF = Very Complicated Formula

One term

$$\hat{D}O_B(b_0,b_1) = \sqrt{(N-b_0-b_1-1)(N-b_0)}\left[-\frac{4}{3}\sqrt{\frac{(b_1+2)(b_1-1)}{b_1\ (b_1+1)}}\frac{(b_1-2)(b_1+3)}{b_1\ (b_1+1)}O_B(b_0+1,b_1-2)\right.$$

$$+\frac{2}{3}\ \frac{b_1+3}{b1}\sqrt{\frac{(b_1+2)(b_1-1)}{(b_1+1)\ b_1}}\sqrt{2}O_C(b_0+1,b_1-2)-\frac{32}{3}\ \frac{{b_1}^2+2\,b_1-3}{b_1(b_1+1)(b_1+2)^2}\sqrt{\frac{b_1+2}{b_1}}O_D(b_0,b_1)$$

$$\left.-\frac{2\sqrt{2}}{3}\sqrt{\frac{b_1+2}{b_1}}\frac{(b_1+3)(3b_1-2)}{b_1(b_1+2)(b_1+1)}O_E(b_0,b_1)+8\sqrt{\frac{(b_1+3)\,b_1}{(b_1+2)\,(b_1+1)}}\frac{1}{(b_1+1)\,(b_1+2)}O_F(b_0-1,b_1+2)\right]$$

$$+\sqrt{(N-b_0-b_1-2)(N-b_0+1)}\left[\frac{2}{3}\sqrt{\frac{(b_1+4)(b_1+1)}{(b_1+2)(b_1+3)}}\frac{\sqrt{2}b_1}{(b_1+3)}O_C(b_0-1,b_1+2)\right.$$

$$-\frac{4}{3}\sqrt{\frac{(b_1+4)(b_1+1)}{(b_1+3)(b_1+2)}}\frac{(b_1+5)\,b_1}{(b_1+3)(b_1+2)}O_B(b_0-1,b_1+2)$$

$$\left.+4\sqrt{\frac{b_1+4}{b_1+2}}\frac{b_1}{(b_1+3)(b_1+2)}O_A(b_0,b_1)\right]+(N-b_0-b_1-1)\left[-4\sqrt{\frac{b_1-1}{b_1+1}}\frac{(b_1+3)}{(b_1+1)\,b_1}O_A(b_0+1,b_1-2)\right.$$

$$+\frac{4}{3}\ \frac{(b_1+3)(b_1^3+5b_1^2+8b_1-12)}{(b_1+1)b_1(b_1+2)^2}O_B(b_0,b_1)-\frac{2\sqrt{2}}{3}\ \frac{(b_1^2+2b_1-4)(b_1+3)}{(b_1+1)(b_1+2)^2}O_C(b_0,b_1)$$

$$\left.-\frac{8}{3}\sqrt{\frac{b_1+3}{b_1+1}}\frac{(b_1+4)\,b_1}{(b_1+2)^2\,(b_1+1)}O_D(b_0-1,b_1+2)+\frac{4}{3}\sqrt{2}\sqrt{\frac{b_1+3}{b_1+1}}\frac{b_1}{(b_1+2)^2}O_E(b_0-1,b_1+2)\right]$$

$$+(N-b_0+1)\left[\frac{4}{3}\ \frac{(b_1+4)\,{b_1}^2}{(b_1+3)\,(b_1+2)^2}O_B(b_0,b_1)+\frac{8}{3}\ \frac{\sqrt{(b_1+1)(b_1+3)}b_1\ (b_1+4)}{(b_1+3)^2\,(b_1+2)^2}O_D(b_0-1,b_1+2)\right.$$

$$\left.-\frac{2}{3}\ \frac{\sqrt{2}\,(b_1+4)\,b_1}{(b_1+2)^2}O_C(b_0,b_1)+\frac{2}{3}\ \frac{\sqrt{2}\sqrt{(b_1+1)(b_1+3)}b_1\ (b_1+4)}{(b_1+3)^2\,(b_1+2)^2}O_E(b_0-1,b_1+2)\right]$$

Results

$m = 2$; 4 operators mix; $\omega = 0$ $(\times 3)$ $\quad \omega = 8g_{YM}^2$ $(\times 1)$

$m = 3$; 6 operators mix; $\omega = 0$ $(\times 4)$ $\quad \omega = 8g_{YM}^2$ $(\times 2)$

$m = 4$; 9 operators mix; $\omega = 0$ $(\times 5)$ $\quad \omega = 8g_{YM}^2$ $(\times 3)$
$\omega = 16g_{YM}^2$ $(\times 1)$

$m = 5$ 12 operators mix; $\omega = 0$ $(\times 6)$ $\quad \omega = 8g_{YM}^2$ $(\times 4)$
$\omega = 16g_{YM}^2$ $(\times 2)$

These results are all recovered in the displaced corners approximation.

The Displaced Corners Approximation

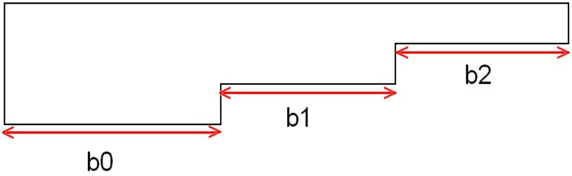


Figure: Example of a three row Young diagram.

In the displaced corners approximation we assume that b_0, b_1, b_2 are all of order N.

This limit simplifies the action of the symmetric group which is responsible for a new $U(p)$ symmetry. (dMK, Dessein, Giataganas, Mathwin, arXiv:1108.2761)

A New Symmetry

$\chi_{R,(r,s)\mu\nu}$

$$R = \ydiagram{15,10,5}$$

$$r = \ydiagram{12,9,3}$$

$$R = \begin{ytableau} {} & {} & {} & {} & {} & {} & {} & {} & {} & {} & {} & {} & * & * & * \\ {} & {} & {} & {} & {} & {} & {} & {} & {} & * \\ {} & {} & {} & * & * \end{ytableau}$$

$$\vec{m} = (3, 1, 2)$$

New symmetry leads to a further conservation law - the dilatation operator does not mix operators with different $\vec{m}$.

D in the Displaced Corners Approximation

$$DO_{R,(r,s)\mu_1\mu_2} = -g_{YM}^2 \sum_{u\nu_1\nu_2} \sum_{i<j} M^{(ij)}_{s\mu_1\mu_2;u\nu_1\nu_2} \Delta_{ij} O_{R,(r,u)\nu_1\nu_2}$$

Δ_{ij} acts only on the Young diagrams R, r and $M^{(ij)}_{s\mu_1\mu_2;u\nu_1\nu_2}$ acts only on the labels $s\mu_1\mu_2$.

Action of Δ_{ij}

$$\Delta_{12} O(b_0, b_1, b_2) = (2N + 2b_0 + 2b_1 + b_2) O(b_0, b_1, b_2)$$
$$-\sqrt{(N + b_0 + b_1)(N + b_0 + b_1 + b_2)} (O(b_0, b_1 - 1, b_2 + 2) + O(b_0, b_1 + 1, b_2 - 2))$$

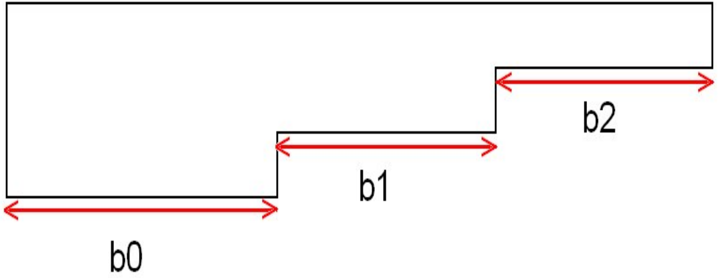


Figure: Example of labeling for a three row Young diagram.

Δ_{ij} eigenproblem

The Δ_{ij} operator can be expressed as an element of the $u(N)$ Lie algebra.

Fundamental representation of $u(N)$ represents the elements of the Lie algebra as $N \times N$ matrices $(E_{kl})_{ab} = \delta_{ak}\delta_{bl}$. Introduce

$$Q_{ij} = \frac{E_{ii} - E_{jj}}{2}, \qquad Q_{ij}^{+} = E_{ij}, \qquad Q_{ij}^{-} = E_{ji}\,,$$

$$\left[Q_{ij}, Q_{ij}^{+}\right] = Q_{ij}^{+}, \qquad \left[Q_{ij}, Q_{ij}^{-}\right] = -Q_{ij}^{-}, \qquad \left[Q_{ij}^{+}, Q_{ij}^{-}\right] = 2Q_{ij}\,.$$

$$Q_{ij}^{+}|\lambda, \Lambda\rangle = c_{+}|\lambda + 1, \Lambda\rangle\,, \qquad c_{+} = \sqrt{(\Lambda + \lambda + 1)(\Lambda - \lambda)}\,,$$

and

$$Q_{ij}^{-}|\lambda, \Lambda\rangle = c_{-}|\lambda - 1, \Lambda\rangle\,, \qquad c_{-} = \sqrt{(\Lambda + \lambda)(\Lambda - \lambda + 1)}\,.$$

Δ_{ij} eigenproblem

$$\Delta_{12} O(b_0, b_1, b_2) = (2N + 2b_0 + 2b_1 + b_2) O(b_0, b_1, b_2)$$
$$-\sqrt{(N + b_0 + b_1)(N + b_0 + b_1 + b_2)}(O(b_0, b_1 - 1, b_2 + 2) + O(b_0, b_1 + 1, b_2 - 2))$$

$$Q_{ij}^{\pm}|\lambda, \Lambda\rangle = c_{\pm}|\lambda + 1, \Lambda\rangle\,, \qquad c_+ = \sqrt{(\Lambda + \lambda + 1)(\Lambda - \lambda)}\,,$$
$$c_- = \sqrt{(\Lambda + \lambda)(\Lambda - \lambda + 1)}\,.$$

$$\Delta_{ij} = -\frac{1}{2}(E_{ii} + E_{jj}) + Q_{ij}^- + Q_{ij}^+\,.$$
$$c_- = \sqrt{(N + b_0 + b_1)(N + b_0 + b_1 + b_2 + 1)},$$
$$c_+ = \sqrt{(N + b_0 + b_1 + 1)(N + b_0 + b_1 + b_2)}$$
$$\Lambda = pN + n, \qquad \lambda = \frac{1}{2} b_i\,.$$

Δ_{ij} eigenproblem at large N

$$\Delta_{ij} \to \left(\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} - \frac{x_i x_j}{4} \right)$$

$$x_i = \frac{r_i - r_0}{\sqrt{N + b_0}}$$

(dMK, Kemp, Smith, arXiv:1111.1058)

D in the Displaced Corners Approximation

$$DO_{R,(r,s)\mu_1\mu_2} = -g_{YM}^2 \sum_{u\nu_1\nu_2} \sum_{i<j} M^{(ij)}_{s\mu_1\mu_2;u\nu_1\nu_2} \Delta_{ij} O_{R,(r,u)\nu_1\nu_2}$$

Δ_{ij} acts only on the Young diagrams R, r and $M^{(ij)}_{s\mu_1\mu_2;u\nu_1\nu_2}$ acts only on the labels $s\mu_1\mu_2$.

Y Eigenproblem

Example: (from Young diagrams with 4 rows and 8 labeled boxes)

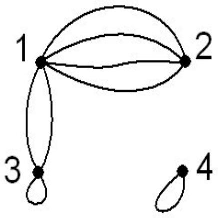
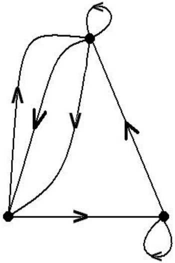


Figure: Example of a pictorial labeling.

$$DO(b_0, b_1, b_2, b_3) = -g_{YM}^2 (4\Delta_{12} + 2\Delta_{13}) O(b_0, b_1, b_2, b_3)$$

Enumerate graphs with double coset



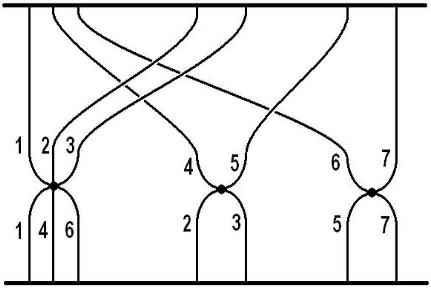


Figure: The graph determines an element of $H \setminus S_{m_1+m_2+m_3}/H$ where $H = S_{m_1} \times S_{m_2} \times S_{m_3}$.

Count Graphs

The cardinality of the double coset is

$$\sum_{s \vdash m} (M^s_{1_H})^2$$

where $M^s_{1_H}$ is the number of times the one dimensional irrep of H is subduced by irrep s of S_m. This equals the number of restricted Schur polynomials that can be defined.

(dMK, Ramgoolam, arXiv:1204.2153)

Fourier transform applied to the double coset

Complete set of functions on the double coset

$$C^{s;\mu\nu}(\sigma) = \sum_{ij} \sqrt{d_s}\Gamma^s_{ij}(\sigma) B_{j\mu} B_{i\nu}$$

$$\frac{1}{|H|}\sum_{\gamma\in H} \Gamma^s_{ik}(\sigma) = \sum_{\mu} B_{i\mu} B_{k\mu}$$

$$O_{R,r}(\sigma) = \sum_{s,\mu,\nu} C^{s;\mu\nu}(\sigma)\chi_{R,(r,s)\mu\nu}(Z,Y)$$

$$DO_{R,r}(\sigma) = -g^2_{YM}\sum_{i<j} n_{ij}(\sigma)\Delta_{ij} O_{R,r}(\sigma)$$

(dMK, Ramgoolam, arXiv:1204.2153)

Summary

Families of operators with a definite scaling dimension are labelled by a graph.

The action of the dilatation operator on each family reduces to a set of decoupled harmonic oscillators.

$$O_{\vec{n}}(\sigma) = \sum_{s,\mu,\nu} \sum_{ij} \sqrt{d_s} \sum_{r} \Gamma^{s}_{ij}(\sigma) B_{j\mu} B_{i\nu} \psi_{HO,\vec{n}}(r) \chi_{R,(r,s)\mu\nu}(Z,Y)$$

Thanks for your attention!