

Multiple Zeta Values

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Outline

- 1 Motivation
- 2 Definitions
- 3 Algebraic Properties
- 4 Motivic Multiple Zeta Values
- 5 Symmetric Insertion

Riemann Zeta Function

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Riemann zeta function is

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$$\zeta(2k) = \frac{(-1)^{k+1} B_{2k} (2\pi)^{2k}}{2(2k)!}$$

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 \end{aligned}$$

Multiple Zeta Values

Definition

The **multiple zeta value** (MZV) $\zeta(a_1, a_2, \dots, a_k)$ is defined by

$$\zeta(a_1, a_2, \dots, a_k) := \sum_{0 < n_1 < n_2 < \dots < n_k} \frac{1}{n_1^{a_1} n_2^{a_2} \dots n_k^{a_k}}.$$

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- For convergence $a_k > 1$
- Sum $a_1 + \dots + a_k$ is the **weight**

Integral Representation

Can write MZV as a Chen iterated integral

$$\begin{aligned}\zeta(a_1, \dots, a_k) &= \int_{[0,1]} \frac{dx}{1-x} \left(\frac{dx}{x}\right)^{a_1-1} \cdots \frac{dx}{1-x} \left(\frac{dx}{x}\right)^{a_k-1} \\ &=: I(0; 10^{a_1-1} \cdots 10^{a_k-1}; 1)\end{aligned}$$

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- Encode $\zeta(a_1, \dots, a_k)$ as binary word $10^{a_1-1} \dots 10^{a_k-1}$
- Also write $\zeta(yx^{a_1-1} \dots yx^{a_k-1})$

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MZVs satisfy many relations

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- Can we describe all relations?
- Are they weight graded?

Duality

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Substitute $x \mapsto 1 - x$ in integral representation

$$\begin{aligned} & \zeta(a_1, \dots, a_k) \\ &= I(0; 10^{a_1-1} \dots 10^{a_k-1}; 1) \\ &= I(0; 1^{a_k-1}0 \dots 1^{a_1-1}0; 1) \end{aligned}$$

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Example

$$\zeta(1, 2) = I(0; 110; 1) = I(0; 100; 1) = \zeta(3)$$

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Definition

The **shuffle product** \sqcup is defined by recursively by:

- For any words w_1, w_2 , and letters $a, b \in \{x, y\}$

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Example

$$\begin{aligned} \zeta(2)\zeta(2) &= \zeta(yx)\zeta(yx) = \zeta(yx \sqcup yx) \\ &= \zeta(2 \cdot xyxy + 4 \cdot yyxx) = 2\zeta(2, 2) + 4\zeta(1, 3) \end{aligned}$$

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$$\begin{aligned} \zeta(a)\zeta(b) &= \zeta(yx^{a-1})\zeta(yx^{b-1}) = \zeta(yx^{a-1} * yx^{b-1}) \\ &= \zeta(yx^{a-1}yx^{b-1} + yx^{b-1}yx^{a-1} + yx^{a+b-1}) \\ &= \zeta(a, b) + \zeta(b, a) + \zeta(a + b) \end{aligned}$$

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$$\begin{aligned} \zeta(2)\zeta(2) &= \zeta(yx)\zeta(yx) = \zeta(yx * yx) \\ &= \zeta(2 \cdot yxyx + yxxx) = 2\zeta(2, 2) + \zeta(2 + 2) \end{aligned}$$

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Compare these two products

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$$\zeta(w_1 \sqcup w_2) - \zeta(w_1 * w_2) = 0$$

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$$\text{So } 4\zeta(1, 3) = \zeta(4)$$

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- Does this give all relations?

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- Motivic MZV relations imply classical MZV relations

'Transcendental' Galois Theory

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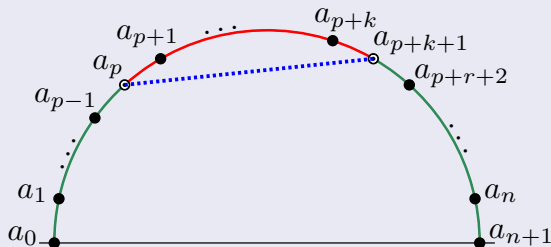
$$D_k I^m(w) := \sum_{\substack{S \text{ subword} \\ \text{of length } k+2}} I^{\mathcal{L}}(S) \otimes I^m(w/\text{int } S)$$

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Mnemonic



■ $w = a_0 a_1 \cdots a_n a_{n+1}$

■ $S = a_p \cdots a_{p+k+1}, \quad w/\text{int } S = a_0 \cdots a_{p-1} a_{p+k+1} \cdots a_{n+1}$

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Theorem (Brown, 2011)

If Z^m has weight N and $D_k Z^m = 0$ for $3 \leq k \leq N$ odd, then

$$Z^m \in \zeta^m(N)\mathbb{Q}$$

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n repetitions of 2 n repetitions of 10

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- So $\zeta(2, \dots, 2) \in \zeta(2n)\mathbb{Q} = \pi^{2n}\mathbb{Q}$

Cyclic Insertion

Conjecture (BBBL)

Given a_0, a_1, \dots, a_{2n} with sum m we get

$$\sum_{\substack{\text{cyclic shifts} \\ \text{of } a_i}} \zeta(2^{a_0}, 1, 2^{a_1}, 3, \dots, 1, 2^{a_{2n-1}}, 3, 2^{a_{2n}}) \stackrel{?}{=} \frac{\pi^{4n+2m}}{(4m+2n+1)!}$$

Examples

$$\zeta(2, 1, 3) + \zeta(1, 2, 3) + \zeta(1, 3, 2) \stackrel{?}{=} \frac{\pi^6}{7!}$$

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$$\begin{aligned} &\zeta(2, 2, 1, 3, 2, 1, 3) + \zeta(1, 2, 2, 3, 1, 2, 3) + \zeta(1, 3, 2, 2, 1, 3, 2) \\ &+ \zeta(2, 1, 3, 1, 2, 2, 3) + \zeta(2, 1, 3, 1, 3, 2, 2) \stackrel{?}{=} \frac{\pi^{14}}{15!} \end{aligned}$$

Symmetric Insertion

Theorem

Given a_0, a_1, \dots, a_{2n} with sum m we get

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Sketch of Proof

- Lift to motivic MZVs
- Write the iterated integral nicely as

$$I^m((01)^{a_0+1} (10)^{a_1+1} \dots (01)^{a_{2n+1}+1})$$

- Compute D_{2k+1}
- Pairwise cancel using symmetry



Summary

- Definition of multiple zeta values
- Irrationality and transcendence?
- Relations between MZVs
 - Shuffle and stuffle product
 - Double shuffle generates all?
- Motivic MZVs
 - Combinatorial tools
 - Transcendental Galois theory
- Symmetric insertion