Polylogarithms and Double Scissors Congruence Groups

for Gandalf and the Arithmetic Study Group

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Abstract

Polylogarithms are a class of special functions which have applications throughout the mathematics and physics worlds. I will begin by introducing the basis properties of polylogarithms and some reasons for interest in them, such as their functional equations and the role they play in Zagier's polylogarithm conjecture. From here I will turn to Aomoto polylogarithms, a more general class of functions and explain how they motivate a geometric view of polylogarithms as configurations of hyperplanes in $\mathbb{P}^n(\mathbb{C})$. This approach has been used by Goncharov to establish Zagier's conjecture for $n = 3$.

1 Definitions

Let's begin with the definition of the order p polylogarithm:

Definition 1 (p-th Polylogarithm)**.** For $p \in \mathbb{Z}_{>0}$:

$$\operatorname{Li}_p(z) \coloneqq \sum_{n=1}^{\infty} \frac{z^n}{n^p}, \quad \text{for } |z| < 1 \,.$$

Notice that $\operatorname{Li}_1(z) = -\log(1-z)$ is just the usual logarithm.

Since $\frac{\mathrm{d}}{\mathrm{d}t} \operatorname{Li}_p(z) = \frac{1}{z} \operatorname{Li}_{p-1}(z)$, we can analytically continue $\operatorname{Li}_p(z)$ to a multivalued holomorphic function on $\mathbb{C} \setminus \{0, 1\}$, via:

$$\operatorname{Li}_p(z) = \int_0^z \operatorname{Li}_{p-1}(t) \frac{\mathrm{d}t}{t}$$

Extra: It may also be worth introducing the related *multiple* polylogarithm functions. These are defined as:

Definition 2. For $p_1, \ldots, p_k \in \mathbb{Z}_{>0}$, the multiple polylogarithm $\operatorname{Li}_{p_1,\ldots,p_k}(x_1, \ldots, x_k)$ is defined by the series:

$$\operatorname{Li}_{p_1,\ldots,p_k}(x_1, \ldots, x_k) = \sum_{n_1 < \cdots < n_k} \frac{x_1^{n_1} \cdots x_k^{n_k}}{n_1^{p_1} \cdots n_k^{p_k}}$$

These functions arise quite naturally when considering products of polylogarithms, just by multiplying out the infinite sries. For example:

$$\begin{aligned}
\mathrm{Li}_a(x)\,\mathrm{Li}_b(y) &= \sum_{n=1}^{\infty} \frac{x^n}{n^a} \sum_{m=1}^{\infty} \frac{x^m}{m^a} \\
&= \sum_{n=1,m=1}^{\infty} \frac{x^n y^m}{n^a m^b} \\
&= \sum_{n<m} \frac{x^n y^m}{n^a m^b} + \sum_{n>m} \frac{x^n y^m}{n^a m^b} + \sum_{n=m} \frac{x^n y^m}{n^a m^b} \\
&= \mathrm{Li}_{a,b}(x,y) + \mathrm{Li}_{b,a}(y,x) + \mathrm{Li}_{a+b}(xy)
\end{aligned}$$

Why might we be interested in these functions? Aside from the fact that they have interesting mathematical properties on their own, these special functions crop up in a variety of places throughout mathematics and physics:

For example, in physics:

- As closed form solutions to Fermi-Dirac, and Bose-Einstein distributions
- Conformal Field Theory and Quantum Electrodynamics
- In the computation of Feynman diagram integrals
- And the computation of scattering amplitudes

On the maths side:

- Dilogarithms appear in the computation of volumes of hyperbolic tetrahedra (manifolds)
- Algebraic K-theory
- Cohomology of $\mathrm{GL}_n(\mathbb{C})$
- Low dimensional topology in Vassiliev-Kontsevich knot invariants
- In connection with the values of L-functions, as part of Zagier's polylogarithm conjecture

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I'd like to talk about the part they play in Zagier's polylogarithm conjecture in a little more detail, since double scissors congruence groups and the geometric point of view of polylogarithms has been used by Goncharov to prove the case $n = 2, 3$.

Recall the Dedekind zeta function of a number field F, defined as:

$$\zeta_F(s) = \prod_{\mathfrak{p} \neq (0) \subset \mathbb{Z}_F} \frac{1}{1 - \mathrm{N}(\mathfrak{p})^{-s}} \,.$$

This converges for $\mathrm{Re}(s) > 1$, and can be extended to a meromorphic function on $\mathbb{C}$ with a simple pole at $s = 1$. The analytic class number formula gives the residue at this pole as:

$$\lim_{s \to 1} (s-1)\zeta_F(s) = \frac{2^{r_1}(2\pi)^{r_2} h_F}{w_F \sqrt{|D_F|}} \, \mathrm{Reg}_F \,.$$

The important bit for this talk is Reg_F, the regulator. Roughly this is the volume for a fundamental domain for the lattice of units in logarithmic space, and so is a sum of $(r_1 + r_2 - 1)$-fold products of logarithms of elements of F / an $(r_1 + r_2 - 1)$-fold determinant of logarithms of elements of F. So '$\zeta_F(1)$' is related to order 1-polylogarithms.

Zagier's polylogarithm conjecture seek to generalise this as follows:

Conjecture 3 (Zagier)**.** *There exists $y_1, \dots, y_{r(n+1)} \in \mathbb{Q}[F \setminus \{0,1\}]$ such that:*

$$\zeta_F(n) = \pi^{nr(n)} D_F^{-1/2} \det \left[\mathcal{L}_n(\sigma_i(y_j))\right]_{1 \leq i,j \leq r(n+1)}$$

where $\sigma_1, \dots, \sigma_{r_1}$ are the real embeddings, and $\sigma_{1+r_1}, \dots, \sigma_{1+r_1+r_2}$ are each one of the pairs of complex embeddings. And $r(n) = r_2$ if n odd, $r(n) = r_1 + r_2$ if n even.

Extra: [Here $\mathcal{L}_n(z)$ is Bloch-Wigner-Ramakrishnan modification of the polylogarithm, which is explicitly given by Zagier as:

$$\mathcal{L}_n(x) := \mathrm{Re}_p \left(\sum_{j=0}^{p} \frac{2^j B_j}{j!} (\log|z|)^j \operatorname{Li}_{p-j}(z) \right)$$

where B_j is the j-th Bernoulli number, $\operatorname{Li}_0(z) = -1/2$ and Re_p means Re for p odd and Im for p even.]

For example, we have:

Example 4. Consider the number field $F = \mathbb{Q}(\sqrt{-5})$. Then:

$$\zeta_F(2) = \frac{\pi^2}{30\sqrt{20}} \left(4D(2+\sqrt{-5}) + 3D\left(\frac{2+\sqrt{-5}}{4}\right) + 20D\left(\frac{2+\sqrt{-5}}{3}\right) \right)$$

where here D is the Bloch-Wigner dilogarithm (another modification of $\operatorname{Li}(x)$, although it is essentially $\mathcal{L}_2(z)$ from above), defined as:

$$D(z) := \operatorname{Im}(\operatorname{Li}_2(z)) + \arg(1-z)\log|z|$$

This has been proven for $n = 2, 3$ by Goncharov using a geometric point of view, and there are some partial results leading to the $n = 4$ case. Zagier himself proved a slightly weaker version of his conjecture for $n = 2$.

Extra: [Slightly weaker in that the arguments to the dilogs aren't necessarily in F, but in F'/F some extension of F.

The idea is to look at $\operatorname{vol}(\mathbb{H}^3/\Gamma)$, where $\Gamma = \mathrm{SL}_2(\mathbb{Z}_F)$. The volume of this quotient is given by Humbert as $|D|^{3/2} \zeta_F(2)/4\pi^2$.

But on the other hand, this space can be triangulated into hyperbolic polyhedra with vertices in $\mathbb{P}^1(F)$, meaning this volume can be written in terms of dilogarithms, as I mentioned when talking about applications of polylogarithms.]

2 Algebraic Properties

No talk about polylogarithms is complete without saying something about their functional equations. Understanding the functional equations is one of the mean avenues of exploration of these functions.

The first such example comes the first property you learn about logarithms:

$$\log(xy) = \log(x) + \log(y)\,,$$

which can be rewritten as a functional equation for $\mathrm{Li}_1(x)$.

We have some 'trivial' functional equations which hold for all weight polylogarithms, such as the duplication formula:

$$\mathrm{Li}_p(z^2) = 2^{p-1}[\mathrm{Li}_p(z) + \mathrm{Li}_p(-z)]\,,$$

which can be prove just by looking at the power series expansion of both sides for $|z| < 1$.

There is also an inversion formula, which in the case $p = 2$ for simplicity reads:

$$\mathrm{Li}_2(1/z) = -\mathrm{Li}_2(z) - \pi^2/6 - \tfrac{1}{2}\log^2(-z)$$

A less trivial/more interesting example is the main functional equation for the dilogarithm:

Theorem 5 (Five Term Relation for the Dilogarithm)**.** *The following holds:*

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(y) + \mathrm{Li}_2\left(\tfrac{1-x}{1-xy}\right) + \mathrm{Li}_2(1-xy) + \mathrm{Li}_2\left(\tfrac{1-y}{1-xy}\right) =$$
$$\tfrac{\pi^2}{6} - \log(x)\log(1-x) - \log(y)\log(1-y) + \log\left(\tfrac{1-x}{1-xy}\right)\log\left(\tfrac{1-y}{1-xy}\right)$$

Proof. A fairly straightforward proof of this is just to differentiate to show this is constant. □

Extra: Writing the arguments as cross ratios leads to a very symmetric form where z_i are in $\mathbb{P}^1(\mathbb{C})$:

$$\sum_i (-1)^i D(\mathrm{r}(z_0, \ldots, \widehat{z_i}, \ldots, z_5)) = 0$$

One expects to have such non-trivial functional equations for all order polylogarithms, but so far we know functional equations only up to the 8-log.

Extra: [A functional equation for the trilogarithm discoevered by Goncharov reads:

$$\mathcal{L}_3(-xyz) + \sum_{\text{cyclic } xyz} \Big\{\mathcal{L}_3(zx - x + 1) + \mathcal{L}_3(\tfrac{zx-x+1}{zx}) - \mathcal{L}_3(\tfrac{zx-x+1}{z}) +$$
$$\mathcal{L}_3(\tfrac{x(yz-z+1)}{-(zx-x+1)}) + \mathcal{L}_3(z) + \mathcal{L}_3(\tfrac{yz-z+1}{y(zx-x+1)}) - \mathcal{L}_3(\tfrac{yz-z+1}{yz(zx-x+1)})\Big\} = 3\,\mathcal{L}_3(1)\,.]$$

Other areas of interest include special values of polylogarithms and ladders (where powers of one value are related to each other).

Extra: [We also have special values for the dilogarithm. Only these few are known. Compare this with that happens for many other functions where either they have infinitely many special values which can be easily described, or none.

$$\mathrm{Li}_2(0) = 0, \quad \mathrm{Li}_2(1) = \tfrac{\pi^2}{6}, \quad \mathrm{Li}_2(-1) = -\tfrac{\pi^2}{12}, \quad \mathrm{Li}^2\left(\tfrac{1}{2}\right) = \tfrac{\pi^2}{12} - \tfrac{1}{2}\log^2(2),$$
$$\mathrm{Li}_2\left(\tfrac{3-\sqrt{5}}{2}\right) = \tfrac{\pi^2}{15} - \log^2\left(\tfrac{1+\sqrt{5}}{2}\right), \quad \mathrm{Li}_2\left(\tfrac{-1+\sqrt{5}}{2}\right) = \tfrac{\pi^2}{10} - \log^2\left(\tfrac{1+\sqrt{5}}{2}\right),$$
$$\mathrm{Li}_2\left(\tfrac{1-\sqrt{5}}{2}\right) = -\tfrac{\pi^2}{15} + \tfrac{1}{2}\log^2\left(\tfrac{1+\sqrt{5}}{2}\right), \quad \mathrm{Li}_2\left(\tfrac{-1-\sqrt{5}}{2}\right) = -\tfrac{\pi^2}{10} + \tfrac{1}{2}\log^2\left(\tfrac{1+\sqrt{5}}{2}\right)\,.$$

Polylogarithm ladders have deep connections to K-theory. Examples of ladders: if

$\rho = (\sqrt{5}-1)/2$, then:

$$\operatorname{Li}_2(\rho^6) = 4\operatorname{Li}_2(\rho^3) + 3\operatorname{Li}_2(\rho^2) - 6\operatorname{Li}_2(\rho) + \tfrac{7}{30}\pi^2$$
$$\operatorname{Li}_2(\rho^{12}) = 2\operatorname{Li}_2(\rho^6) + 3\operatorname{Li}_2(\rho^4) + 4\operatorname{Li}_2(\rho^3) - 6\operatorname{Li}_2(\rho^2) + \tfrac{1}{10}\pi^2\,.]$$

3 Aomoto Polylogarithms

I'd now like to introduce a more geometrical point of view of polylogarithms, and this is done by the Aomoto polylogarithms.

Let L and M be a pair of simplices in n-dimensional projective space $\mathbb{P}^n(\mathbb{C})$ over $\mathbb{C}$. Such a simplex is a collection of $n+1$ hyperplanes $L = (L_0, \ldots, L_n)$. (A face of L is a non-empty intersection of hyperplanes from L. A pair of simplices is admisslbe if they have no common face of the same dimension.)

We can take one of these simplices as a region Δ_M to integrate over. (Take a n-cycle representing a generator of $H(\mathbb{P}^n(C), M)$.) The other defines us a differential form as follows. Let the equation of L_i be $f_i = 0$ in homogeneous coordinates. then:

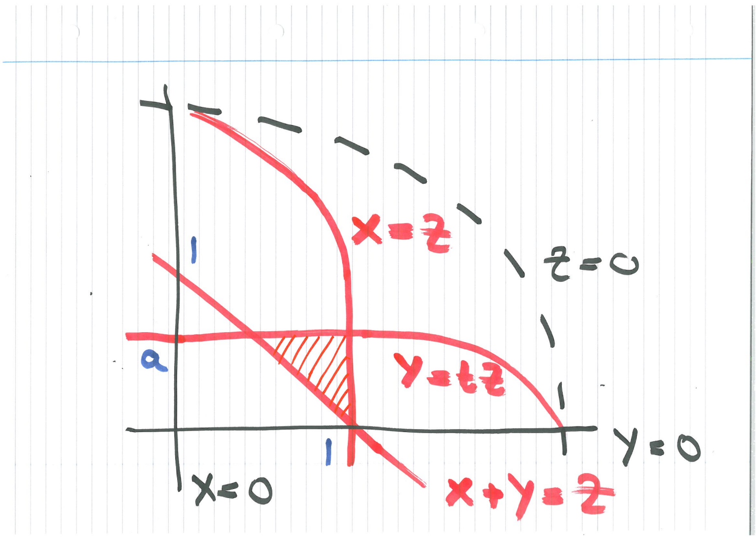
$$\omega_L = \mathrm{d}\log(f_1/f_0) \wedge \cdots \wedge \mathrm{d}\log(f_n/f_0)$$

Integrating ω_L over Δ_M gives us the weight n Aomoto polylogarithm:

$$\Lambda(L, M) := \int_{\Delta_M} \omega_L$$

Example 6 (Dilogarithm)**.** Aomoto polylogarithms genuinely do generalise classical polylogarithms. If we take $L = (Z = 0, X = 0, Y = 0)$ and $M = (X + Y = Z, X = Z, Y = tZ)$, then $a(L, M) = \operatorname{Li}_2(t)$, the dilogarithm. In this case we get:

$$\omega_L = \mathrm{d}\log(X/Z) \wedge \mathrm{d}\log(Y/Z) = \frac{\mathrm{d}x}{x} \wedge \frac{\mathrm{d}y}{y}$$



Extra: We the integrate this over the interior of the region Δ_M defined by $x+y=1$, $x=1$ and $y=t$. Doing this integral by first integrating wrt to x, an integrating term by term the power series for $\log(1-y)/y$, we get:

$$\begin{aligned}
\int_{\Delta_M} \frac{\mathrm{d}x}{x} \wedge \frac{\mathrm{d}y}{y} &= \int_0^t \left[\int_{1-y}^1 \frac{1}{xy}\,\mathrm{d}x\right]\mathrm{d}y \\
&= \int_0^t \frac{-\log(1-y)}{y}\,\mathrm{d}y \\
&= \int_0^t \left[\frac{1}{y}\sum_{n=1}^{\infty}\frac{y^n}{n}\right]\mathrm{d}y \\
&= \sum_{n=1}^{\infty}\frac{t^n}{n^2} \\
&=: \mathrm{Li}_2(t)
\end{aligned}$$

Properties of Aomoto Polylogarithms:

Non-degeneracy $\Lambda(L,M)=0$ if L or M is degenerate (lies in a hyperplane). If L, then we get a repeated term in the differential form, so it goes to 0. If M, then we're integrating over a 0-volume region, and get 0.

Skew Symmetry $\Lambda(\sigma L, M) = (-1)^{\operatorname{sgn}\sigma}\Lambda(L,M) = \Lambda(L,\sigma M)$, for any permutation $\sigma \in \S_{n+1}$. (Where σL means to permute the order of the hyperplanes in L by σ.) Applying to M will change the orientation of the simplex, applying to L will change the order of the differential forms.

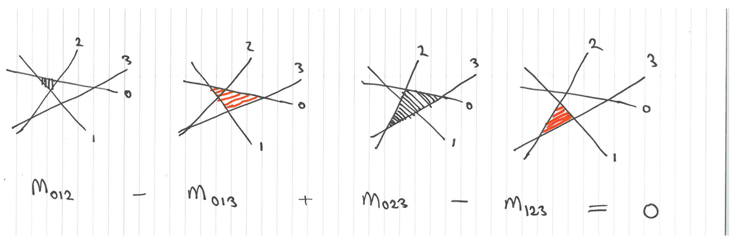
Additivity in L For any collection of hyperplanes $L_0,\dots,L_{n+1}$ the following holds:

$$\sum_{i=0}^{n+1}(-1)^i\Lambda(\widehat{L}^i, M) = 0$$

Additivity in M For any collection of hyperplanes $M_0,\dots,M_{n+1}$ the following holds:

$$\sum_{i=0}^{n+1}(-1)^i\Lambda(L, \widehat{M}^i) = 0$$

(Here $\widehat{L}^i$ means $L_0,\dots,\widehat{L_i},\dots,L_{n+1}$, missing out L_i.) Additivity in M comes from integrating over each region twice with opposite sign. For example when $n=2$:



Projective Invariance For any $g \in \mathrm{PGL}_{n+1}(\mathbb{C})$:

$$\Lambda(gL, gM) = \Lambda(L, M)$$

Which is just changing variables.

Extra: A reasonable question is how exactly Aomoto and classical relate. Every classical polylogarithm is an Aomoto polylogarithm integrating the coordinate simplex differential forms over a special choice of integration simplex. The other way is not so clear cut. For $n = 1, 2, 3$ they are the same thing: every Aomoto can be expressed as classical. However for $n \geq 4$ things differ. One way of seeing the results for $n = 1, 2, 3$ comes from just cutting up a general configuration and showing is can be written in terms of classical. But for $n = 4$ there are obstructions which prevent this.

4 Double Scissors Congruence Groups

It's on these properties above that we model the Double Scissors congruence groups. Conjecturally this should capture all the functional equations of the Aomoto polylogarithm. We incorporate the above properties into the definition:

Definition 7 (Double Scissors Congruence Group)**.** For n a positive integer, and F a field, define the abelian group $A_n(F)$ as follows: $A_n(F)$ is the free abelian group generated by pairs (L, M) of admissible n-simplices modulo the following relations:

1) (Non-degeneracy) If either L or M is degenerate then $(L, M) = 0$.
2) (Skew Symmetry) For every permutation $\sigma \in S_{n+1}$, we have $(\sigma L, M) = (-1)^{\mathrm{sgn}(\sigma)}(L, M) = (L, \sigma M)$.
3) (Left- and right-additivity) For every family of hyperplanes $(L_0, \ldots, L_{n+1})$ and n-simplex M such that $\widehat{L}^i$ is admissible for $i = 0, 1, \ldots, n+1$, we have:

$$\sum_i (-1)^i (\widehat{L}^i, M) = 0 = \sum_i (-1)^i (M, \widehat{L}^i)$$

4) (Projective Invariance) For every $g \in \mathrm{PGL}_{n+1}(F)$, we have $(gL, gM) = (L, M)$.

(And $A_0(F) = \mathbb{Z}$.)

So with this we move to just studying conigurations of hyperplanes in projective space, rather than worrying about the analytic details too.

Remark 8. When $n = 1$, we can identify $A_1(F)$ with $F^\times$ usingn the cross ratio. We're looking at ordered 4-tuples in $\mathbb{P}^1(F)$. There's a projective transformation between any such configurations if and only if they have the same cross ratio, so can transform the configuration (l_0, l_1, m_0, m_1) to $(\infty, 0, 1, \alpha)$, where $\alpha = \mathrm{r}(\infty, 0, 1, \alpha)$, and so identify this with $\alpha \in F^\times$.

Extra: It's easy to explicitly define a coproduct on the *generic* part $A^0_\bullet(F)$, it is given by the follownig simple formula:

Definition 9 (Generic Part Coproduct)**.** On the generic part $A^0_n(F)$ of $A_n(F)$ (where the hyperplanes are in general position) we can define a coproduct (in terms of its components) as follows:

$$\nu_{n-k,k} \colon A^0_n \to A^0_{n-k} \otimes A^0_k$$

is defined by;

$$(L,M) \mapsto \sum_{I,J} (-1)^{\sigma(I,J)} (L_I \mid L_{\overline{I}}, M_J) \otimes (M_J \mid L_I, M_{\overline{J}})$$

Here the sum runs over all $I = (0 < i_1 < \cdots < i_k)$ and $J = (0 < j_1 < \cdots < j_{n-k})$. $\sigma(I,J) = \operatorname{sgn}(I,\overline{I}) \operatorname{sgn}(J,\overline{J})$, where $\overline{I}$ is the complement of I, and $\operatorname{sgn}(I,\overline{I})$ is the sign of the permutation $(0,1,\ldots,n) \mapsto (I,\overline{I})$. I also write L_I to mean $(l_{i_1},\ldots,l_{i_k})$.

I should also explain the notation $(N \mid L,M)$. If $(L,M) \in A_n(F)$, then $(N \mid L,M) \in A_{n-1}(F)$. It means $(N \cap L_1, \ldots, N \cap M_n)$. If multiple hyperplanes appear before the bar, take their overall intersection.

Of more interest is how exactly the coproduct should be defined on all of $A_\bullet(F)$, to turn this into a Hopf algebra. (There is a product: products of Aomoto polylogarithms can be written as sums of other single Aomoto polylogarithms, and this carries over to $A_\bullet(F)$.) There should be a coproduct for all n, but so far we know explicitly how to define it only for $n = 2,3$, and partially for $n = 4$.

The existence of such a coproduct is important for this following conjecture relating Double Scissors congruence groups and K-theory:

Conjecture 10. *The restricted coproduct induces a complex:*

$$A_{>0} \to A_{>0} \otimes A_{>0} \to A_{>0} \otimes A_{>0} \otimes A_{>0} \to \cdots$$

whose graded n-piece

$$A_n \to \bigoplus_{k=1}^{n+1} A_k \otimes A_{n-k} \to \cdots$$

provides the isomorphism:

$$H^i_{(n)}(A_\bullet, \mathbb{Q}) \cong \operatorname{gr}^\gamma_n K_{2n-i}(F)_\mathbb{Q}$$

where γ is the γ-filtration of K-groups.

Extra: So let's look at what happens when $n = 2$. To work out the coproduct we only need the ν_{11} component, so we compute:

$$\sum_{I,J} (-1)^{\sigma(I,J)} (L_I \mid L_{\overline{I}}, M_J) \otimes (M_J \mid L_I, M_{\overline{J}})$$

where $I = \{1\},\{2\}$ and $J = \{1\},\{2\}$. Giving

$$\begin{aligned}&(L_1 \mid L_0L_2, M_0M_1) \otimes (M_1 \mid L_0L_1, M_0M_2) + (-1) \times (L_1 \mid L_0L_2, M_0M_2)\\ &\quad \otimes (M_2 \mid L_0L_1, M_0M_1) + (-1) \times (L_2 \mid L_0L_1, M_0M_1)\\ &\quad \otimes (M_1 \mid L_0L_2, M_0M_2) + (L_2 \mid L_0L_1, M_0M_2) \otimes (M_2 \mid L_0L_2, M_0M_1)\end{aligned}$$

Any generic configuration in $A_2(F)$ can be put in the form:

$$(ZYX, Y = X + Z, Y = a_1X + b_1Z, Y = a_2X + b_2Z)$$

(where $a_i, b_i \neq 0, 1$, all pairwise distinct, and $a_1b2 - a2b1 \neq 0$.)

Plugging this into the above coproduct formula we get:

$$\frac{b_1}{a_1} \otimes \frac{(a_1-1)(a_1b_2-b_1a_2)}{(a_1-a_2)(a_1-b_1)} + (-1) \times \frac{b_2}{a_2} \otimes \frac{(a_2-1)(a_1b_2-b_1a_2)}{(a_1-a_2)(a_2-b_2)}$$
$$+ (-1) \times b_1 \otimes \frac{(a_1-1)(b_1-b_2)}{(a_1-a_2)(b_1-1)} + b_2 \otimes \frac{(a_2-1)(b_1-b_2)}{(a_1-a_2)(b_2-1)}$$

Extra: So let's go through an example of this for $n = 2$. Recall the dilogarithm configuration we had previously. It's given by $L = (Z, Y, X)$ and $M = (X + Y - Z, Y - tZ, Z - X)$. Which we can put into the form:

$$(ZYX, Y = X + Z, Y = 1/tZ, Y = X)\,.$$

Unfortunately this is not a generic configuration, for example there are three lines intersecting in a point at $[1, 1, 0]$, so the above coproduct formula doesn't apply. If we try to use it, then we get lots of division by zero, and infinity popping up.

However, proceeding in a formal sense, first looking at the configuration $(ZYX, Y = X+Z, Y = \epsilon_1 X + 1/tZ, Y = X + \epsilon_2 Z)$ which is generic, one can apply the coproduct formula and then let $\epsilon_1, \epsilon_2 \to 0$, and this goes give the right answer $t \otimes (1 - t)$ in this case. And so this bears more investigation.

I'll briefly outline how much trouble one has to go to, to define the coproduct properly in the non-generic case for $n = 2$.

Firstly define chain complex $C^\bullet(L, M) = C^\bullet(L, M)_0 \oplus C^\bullet(L, M)_2 \oplus C^\bullet(L, M)_4$, where

$$\begin{aligned} C^\bullet(L, M)_0 &= 0 \to C_0^{-2} \to C_0^{-1} \to C_0^0 \to 0 \\ C^\bullet(L, M)_2 &= 0 \to C_2^{-1} \to C_2^0 \to C_2^{-1} \to 0 \\ C^\bullet(L, M)_4 &= 0 \to C_4^0 \to C_4^1 \to C_4^2 \to 0 \end{aligned}$$

and

$$\begin{aligned} C_0^{-2} &= \mathbb{Z}\langle(\mathbb{P}^2)\rangle \\ C_0^{-1} &= \mathbb{Z}\langle(M_i)\rangle \\ C_2^{-1} &= \mathbb{Z}\langle(L_i)\rangle \\ C_2^0 &= \mathbb{Z}\langle(\mathbb{P}^2), (V \text{ - } LM\text{-vertices})\rangle \\ C_2^1 &= \mathbb{Z}\langle(M_i)\rangle \\ C_4^1 &= \mathbb{Z}\langle(L_i)\rangle \\ C_4^2 &= \mathbb{Z}\langle(\mathbb{P}^2)\rangle \end{aligned}$$

and

$$\begin{aligned} C_0^4 &= \ker((l, L_i) \mapsto (l) \colon L \text{ flags} \to L \text{ vertices}) \\ C_0^4 &= \operatorname{coker}((m) \mapsto \sum_{M_i \ni m} (m, M_i) \colon M \text{ vertices} \to M \text{ flags}) \end{aligned}$$

With appropriate differentials... (See section 2.2, page 397 onwards of Govindachar for details.)

Then look at $G_i = H^0 C_i$. The elements of G_2 can be given in terms of generators and relations. It is generated by triangles $(L_{ijk}) = (L_{ij}) + (L_{jk}) + (L_{ki})$ where $L_{ij} = (L_i \cup L_j, l_i) - (L_i \cup L_j, L_j) \in C_4^0(L, M)$.

An element $z \in G_1$ is a sum $\sum a_V(V) + n\mathbb{P}^2$ (with some conditions). We can look at $z_i = \sum_{V \in M_i} a_V(V)$ to get a divisor on the line M_i.

From a triangle Δ and such z we can define

$$\phi(\Delta, z) = (-1)^{\deg(z)} \frac{f_j(z_i) f_k(z_j) f_i(z_k)}{f_k(z_i) f_i(z_j) f_j(z_k)} \in F^\times$$

where f_i are the equations determining Δ. This extends to the map

$$\phi_{01} \colon \widehat{G_0(L, M)} \otimes G_1(L, M) \to F^\times$$

The finally the coproduct of (L, M) is defined by

$$\nu_{11}(L, M) = \sum_{j=1}^{k} \phi_{01}^{LM}(e \otimes f_j) \otimes \phi_{01}^{ML}(g \otimes f^j)$$

where e is the vector in $\widehat{G_0(L, M)}$ determined by M, $g \in G_2(L, M) \cong \widehat{G_0(M, L)}$ and f_j is a basis for $G_1(L, M)$ with f^j the dual basis of $\widehat{G_1(L, M)}$.

With this coproduct, and a product, the $A_n(F)$ form a Hopf algebra. The product I haven't really talked about directly, although you can imagine taking the product of two Aomoto polylogarithms, and multiplying the integrals to get an expression for the product.

One question that we can ask is whether the map $(L, M) \to (M, L)$ is actually the antipode of the Hopf algebra. As yet we don't know.

What does this have to do with Zagier's polylogarithm conjecture?

Goncharov's method was to use this geometrical point of view to give an explicit description for the regulator $r_n \colon K_{2n-1}(\mathbb{C}) \to \mathbb{R}$ for $n = 2, 3$, in terms of dilogs and trilogs respectively. Relating the double scissor congruence groups / Aomoto polylogarithms to other geometric configurations and to the Bloch / Goncharov polylogarithm complex.

Given such a description in terms of polylogs, Borel's theorem would give the result about $\zeta_F(2)$ or $\zeta_F(3)$:

Theorem 11 (Borel)**.** *For a number field F:*

$$K_{2n-1}(F) \text{ has rank } r(n+1) = \begin{cases} r_1 + r_2 & \text{if } n \text{ odd} \\ r_2 & \text{if } n \text{ even} \end{cases}.$$

And the image of $K_{2n-1}(F)$ in $\mathbb{R}^{r(n)}$ under the regulator map is a lattice with co-volume $\zeta_F(m)/\pi^{nr(n)}\sqrt{|\Delta_F|}$.