Reductive algebraic groups & parabolic induction. / 6-Feb-2014.

Part - I | Basics of algebraic groups.

Throughout this talk:

i: $\mathbb{F}_q$ — a finite field with q-elements, q a power of some prime p, ℓ a prime $\neq p$

ii: Every variety is over $\overline{\mathbb{F}}_q$, when we say it is over $\mathbb{F}_q$, it means $X = X^0 \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q$ for some var. $/\mathbb{F}_q$.
This is the language of Weil, not Grothendieck.
Sometimes we use both languages, when there won't cause confusions.

<u>Definition 1</u>: An alg. gr. G over $\overline{\mathbb{F}}_q$ is by def. an alg. var. $/\overline{\mathbb{F}}_q$ with a gr. structure s.t. the multiplicate and inverse are var. mor.s.

<u>Example 2</u>: (i): $GL_n(\overline{\mathbb{F}}_q) = \mathrm{Spec}\left(\overline{\mathbb{F}}_q[x_{ij}, y]_{i,j} \Big/ \det((x_{ij}))\cdot y - 1 \right)$

is an alg. gr..

(ii): $SL_n(\overline{\mathbb{F}}_q) = \mathrm{Spec}\left(\overline{\mathbb{F}}_q[x_{ij}]_{i,j} \Big/ \det((x_{ij})) - 1 \right)$

is an alg. gr.

(iii): An exercise showing $\mathbb{Z}[x_{ij}] \big/ \det(x_{ij}) - 1 \not\cong \mathbb{Z}[y_1, \cdots, y_{n^2-1}]$

$|SL_2(q)| = q(q-1)(q+1)$

<u>Theorem 3</u>: Any affine algebraic group is isomorphic to a closed subgroup of GL_n, so they are also called linear algebraic group.

(List) Definit'n[4]: (Kind of informal but more intuitive def'n's)

(i): Torus: alg. gr.s iso. to $\mathbb{G}_m \times \mathbb{G}_m \times \ldots \times \mathbb{G}_m$.

(ii): $R_u(G)$ (unipotent radical): The unique max. clo. conn. nor. uni. subgr. of G. (Here unipotent means the eigenvalues are $=1$)

(iii): If $R_u(G)=1$ then we call G a reductive algebraic group. ~~Most~~ Many interesting algebraic groups are reductive: GL_n, SL_n, PSL_n, U_n, ...

(iv): Borel subgroups are of form:

$$\begin{bmatrix} \times & \times & \times & \cdots \\ & \times & \times & \times \\ & & \ddots & \vdots \\ & & & \times \end{bmatrix}$$ (conjugates).

Its generalizat'ns (parabolic = parabolic)

$$\begin{bmatrix} \boxed{x} & \times & \times \\ & \boxed{x} & \ast \\ & & \boxed{\ddots} \end{bmatrix}$$

Max. torus in a Borel: ~~d~~ conjs of diag.s.

~~Max~~ Its generalizat'n = Levi: $\begin{bmatrix} \boxed{x} & & \\ & \boxed{x} & \\ & & \boxed{x} \end{bmatrix}$

(v): Levi decomposit'n: $P = LU = \left\{\begin{bmatrix} \boxed{x} & & \\ & \boxed{x} & \\ & & \boxed{x} \end{bmatrix}\right\} \times \left\{\begin{bmatrix} I_1 & \times & \times \\ & I_2 & \vdots \\ & & \ddots \end{bmatrix}\right\}$

(vi): Weyl gr. (unique up to iso.): $W(T) := N(T)/T$ (in a conn. red. alg. gr.) is a Coxeter gr.

~~I~~ Idea on showing GL_n is reductive[5]: Lie-Kolchin implies $R_u(GL_n)$ is a ~~e~~ clo. subgr. of upper tri'gles, also unipotent gr.s are solv. ......

Theorem[6]: Bruhat decomposit'n: (For conn. red. alg. gr. G)

$$G = \coprod_{w \in W} BwB$$

Part II | Harish-Chandra Induct'n.

The philosophy of parabolic induct'n is that, the representat'ns of a finite red. conn. alg. gr. G^F should be parametrized ~~by inducing induce~~ via induct'n from rep.s of smaller gr.s of "same type".

<u>Example 7</u>: Consider $GL_n(\overline{\mathbb{F}}_q)$, for some very big n. So, if we can get a good understanding of $GL_m(\overline{\mathbb{F}}_q)$ for some small m, then we ~~can get a~~ know the rep.s of $\begin{bmatrix} GL_{m_1} & & \\ & \ddots & \\ & & GL_{m_k} \end{bmatrix}$, which is a Levi. subgr. of GL_n for $n = \sum m_i$ and, by trivially lifting those rep.s to

$$\mathrm{Rep}(GL_n) \ni \mathrm{Ind}\,\bar{\theta} \leftsquigarrow \bar{\theta} \in \mathrm{Rep}\left(\begin{bmatrix} GL_{m_1} & * & * \\ & \ddots & * \\ & & GL_{m_k} \end{bmatrix}\right) \leftsquigarrow \theta \in \mathrm{Rep}\left(\begin{bmatrix} GL_{m_1} & & \\ & \ddots & \\ & & GL_{m_k} \end{bmatrix}\right)$$

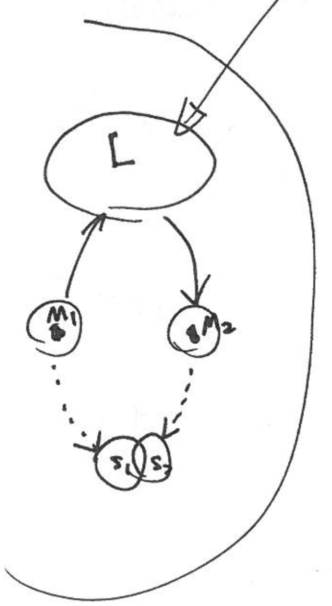
a parabolic of GL_n, and ~~by inducing~~ then by inducing the rep.s to GL_n, ~~we~~ the parabolic induct'n philosophy says this procedure provide a good knowledge of the rep.s of GL_n.

<u>Definition</u>[8]: Let G be a rat. red. conn. gr. $/\mathbb{F}_q$. L a rat. levi. of a rat. para. P, and θ a rep. of L^F, then the inducing rep. $R_L^G\theta$ of G^F given by the bi-module

$\mathbb{C}[G^F/U^F]$ (G^F-mod.-L^F) is called the Harish-Chandra induced rep. of θ.

$\theta \mapsto R_L^G\theta$ is a functor, called Harish-Chandra induction functor.

<u>Theorem</u>[8]: (i): We have a Mackey formula for R_L^G.
(ii): R_L^G is independent of choice of P.
(iii): One can give the ~~rat. levis of G (including G itself) a partial order~~ pairs (L,δ), $\delta \in \mathrm{Irr}(L^F)$ a partial order naturally, for all rat. L contained in a rat. para.. An mini. pair is called a cuspidal representation. Any $\chi \in \mathrm{Irr}(G^F)$ appeared in $R_L^G\theta$ for some cuspidal rep. $\theta \in \mathrm{Irr}\, L^F$.



With the notion of cuspidal representation, one then obtain, due to Harish Chandra, a first approach towards the classification of irreducible representations, of G^F.

Inducing rep.s given by bimodule: (M a $\mathbb{C}[G]$-mod - $\mathbb{C}[H]$)

$$E \mapsto M \otimes_{\mathbb{C}[H]} E = M \otimes_{\mathbb{C}} E \Big/ \langle mh\otimes e - m\otimes he \rangle$$

Part III | Rationality and Deligne-Lusztig thy.

The idea of parabolic induction on classifying $\mathrm{Irr}(G^F)$ is awesome. However, there do exist some $G/\mathbb{F}_q$, s.t. the rat. Levis does not contained in any rat. parabolic, ~~and in th~~ so, there should be a modification of parabolic induction, and this ~~brough~~ brought the genius construction of Deligne-Lusztig.

~~Example 10~~

~~Again~~

Example 10: Consider $GL_n(\overline{\mathbb{F}}_q)$, the most natural rational structure on it is given by $GL_n(\overline{\mathbb{F}}_q) = GL_n(\mathbb{F}_q) \underset{\mathbb{F}_q}{\otimes} \overline{\mathbb{F}}_q$. However, there "do" exist another rational structure:

$$GL_n(\overline{\mathbb{F}}_q) = U_n \underset{\mathbb{F}_{q^2}}{\otimes} \overline{\mathbb{F}}_q, \text{ where}$$

$$U_n = \left\{ A \in GL_n(\mathbb{F}_{q^2}) \;\middle|\; A \cdot {}^tA^{[q]} = I \right\},$$

where $A^{[q]}$ is the ~~$A \mapsto$~~ $(a_{ij}) \xmapsto{F} (a_{ij}^q)$. where F is the usual Frob., ~~then~~ Thus:

$$U_n = GL_n(\overline{\mathbb{F}}_q)^{F'},$$

where $F' : A \longmapsto ({}^tA^{[q]})^{-1}$. One can show $\begin{bmatrix} GL_{m_1} & & \\ & \ddots & \\ & & GL_{m_k} \end{bmatrix}$ are not in any rat. para. under this F'-rational structure.

Note that $\begin{bmatrix} GL_{m_1} & & \\ & \ddots & \\ & & GL_{m_k} \end{bmatrix}$ is still a rat. Levi. of

$\begin{bmatrix} GL_{m_1} & * & * \\ & \ddots & * \\ & & GL_{m_k} \end{bmatrix} = P$, but P is not rat. ~~w~~ wrt.

to F', since F' maps upper triangle to lower triangle.

Definition.[11]: Given L a rat. Levi. of a para. P, $P = L \cdot U$ the Levi. decomp., then we can associate to this Levi. decomp. a variety $\tilde{X}_L$, called the Delyne-Lusztig variety ~~of~~. Its l-adic cohom gr.s form a virtual bimodule

$H_c^*(\tilde{X}_L, \overline{\mathbb{Q}}_l)$ (G^F-mod-L^F). The induced ~~mod~~ rep.s of G^F from L^F via this bimodule is called the ~~Har~~ Delyne-Lustig construct, denoted ~~by~~ $\theta \mapsto R^G_{L \subseteq P}\theta$. ~~&~~ call $R^G_{L \subseteq P}$ the Lusztig functor. In most case we only need to consider $L = T$ a max. torus.

? Almost same notat with Harish chandra functor

~~Prop.~~ Theorem[12]: When $L = T$ is a max. torus, $R^G_{L \subseteq P}$ is independent of P, and it has a Macky formula...

Theorem![13]: Any $\chi \in Irr(G^F)$ appeared in some $R^G_{T \subseteq P}\theta$ for $\theta \in Irr\, T^F$.

Theorem![14]: When L is in some rat. para. The Delyne-Lusztig ~~functor~~ ~~cost~~ ind. coincide with the Harish-Chandra ~~functor~~ ind.