Part-1 Zeta Functions on Finite type schemes.

X — a finite-type scheme $/\mathbb{Z}$.

$|X|$ — closed points of X.

Definition: (Arithmetic Zeta)

~~The arithme~~ A formal product $\zeta_X(s) = \prod_{x \in |X|} \left(1 - (\# k(x))^{-s}\right)^{-1}$

($k(x)$ is the residue fi. at x).

To see it is well-defined, ~~du~~ we need to show $\# k(x)$ is finite. Actually $k(x)$ can only be a num. fi. or a finite field by basic alg. geom. setting. So only need to show ~~#~~ $k(x)$ is not a num. fi.

To show $k(x)$ is not a num. fi., basic alg. geom. argument shows we only need to show

K is not finite-type over $\mathcal{O}_K$.
~~$K / \mathcal{O}_K$~~

I.e. $K \not\cong \mathcal{O}_K[t_1, \cdots, t_n]/I$.

Two methods to see this:

alg. geom method — ① "Flat finite-type mor.s between noe. sch.s are open."
So the im. of $\operatorname{Spec} K$ in $\operatorname{Spec} \mathcal{O}_K$ is open, impossible. [EGA IV. 2.4.6].

num. theoretic method — ② "Unique prime id. factorizat. of frac. id.s".
A little more tedious but not difficult indeed.

Example

When $X = \operatorname{Spec} \mathcal{O}_K$, $\zeta_X(s)$ is the Dedekind ~~fu~~ zeta funct.
Particularly, when $\mathcal{O}_K = \mathbb{Z}$, we recover the Riemann zeta.

$$\zeta_X(s) = \prod_{\mathfrak{p}} \left(1 - (\# \mathcal{O}_K/\mathfrak{p})^{-s}\right)^{-1}$$

$$\zeta_{\mathbb{Z}}(s) = \prod_{p} \left(1 - (\# \mathbb{Z}/p)^{-s}\right)^{-1}$$

Example

When $X/\mathbb{F}_q$ is a var. over a finite fi. $\mathbb{F}_q$.

$$\zeta_X(s) = \prod_{x \in |X|} \left(1 - \left(q^{\deg(x)}\right)^{-s}\right)^{-1}$$

$$= \prod_{x \in |X|} \left(1 - t^{\deg(x)}\right)^{-1} = Z(X, t)$$

$$(t = q^{-s}).$$

Theorem

$$Z(X,t) = \exp\left(\sum_{r=1}^{\infty} N_r \frac{t^r}{r}\right), \quad N_r = \# X(\mathbb{F}_{q^r})$$

(as a formal power series $/\mathbb{Q}$)

Coro :

$$N_n = \left(\left(\log Z(X,t) \right)^{(n)} \Big|_{t=0} \right) \Big/ (n-1)!$$

So, if we have a well-under study of $Z(X,t)$.
We got a well method for approaching $\{N_r\}_r$, the distribut of the rational points of X.

Ideas on proof of the theorem:

① Take "log" on both side, ② do Taylor expans
③ do differentc, ④ do ~~combi~~ combinat trick.

See Beilinson's lecture notes (by Boyarchenko) on a descrypt of Frobenius - orbit & characterize of ζ.

Part-2 | Weil Conjectures (1948) ~~[Solved]~~

① Rationality:

$$Z(X,t) = \frac{P_1(t)\cdot P_3(t)\cdot \ldots \cdot P_{2d-1}(t)}{P_0(t)\cdot P_2(t)\cdot \ldots \cdot P_{2d}(t)}$$

is a rat. func. $/\mathbb{Q}$.
$d = \dim X$, X a smooth proj. geom. con. var.

Here P_i is of certain form expected.
And the degree of P_i is denoted β_i - Betti num.

② Functional Equat:

$$Z\left(X, \frac{1}{q^d t}\right) = \left(-q^{\frac{d}{2}} t\right)^{\chi} \cdot Z(X, t).$$

$$\chi = \sum_{i=0}^{2d} (-1)^i \beta_i$$ — Euler-Poincaré characteristic.

③ Betti num.:

If X lifts to a smooth proj. var. Y/num. fi. then we can arrange the β_i to be the ith Betti num. of $Y_{\mathbb{C}}$ as a complex manifold.

④ Riemann hypothesis analogue:

One can arrange the factors so that

$$P_i(t) \in \mathbb{Z}[t] \quad \& \quad |\alpha_{ij}| = q^{\frac{i}{2}}$$

for $\forall\ 0 \leq i \leq 2d$. And this arrangement is compatible with the Betti number.

$$\left(\alpha_{ij} \text{ appeared in } P_i(t) = \prod_j (1 - \alpha_{ij} t) \right)$$

Note that in case X is a curve, by ~~var-chag~~ variable-change $t = q^{-s}$, the above actually says all zeros ~~& poles~~ of $Z(X, q^{-s})$ lie on the vertical line $x = \frac{1}{2}$. So we call it a Riemann hypothesis analogue.

Applicat:

If $Z(X, t)$ is rat., then by the linear recurs relat of ~~[illegible]~~ rat. func., if finitly of $\{N_r\}_r$ are known then the whole $\{N_r\}_r$ will be known.

Part-3: Rationality & Weil cohomology

~~We~~
Weil observed a "nice" cohomology theory should be helpful.

Nice: ① Poincaré duality: $H^i(X) \times H^{2d-i}(X) \longrightarrow H^{2d}(X) \cong K$.
② Künneth formula: $H^*(X) \otimes_K H^*(Y) \cong H^*(X \times Y)$
③ Cycle gr. maps: $cl_X^r : C^r(X) \longrightarrow H^{2r}(X)$.

i: Some additional axioms: e.g. vanish for $i > 2d$.

ii: $H^*(-)$: a contravariant functor from cat. of k-vars to some subcat. of K-vec. sp-s.

Example:

De Rham cohomology, singular cohom
(these are Weil cohom. / $\mathbb{C}$, or field of char $= 0$)

The comology theory enters into our situation (over a finite field!) is called the l-adic cohomology.

Everything begins with the observation that a ~~Left~~ Lefschetz-type fixed pt. thm should be helpful. ~~Thanks to the axioms we expected for the Weil axioms cohomology~~. We have such a theorem is a formal consequence from the Weil axioms (nontrivially!)

Theorem (Lefschetz fixed pt. for Frobenius)

$$N_r = \sum_{i=0}^{2d} (-1)^i \operatorname{Tr}\left(Fr_q^r \mid H^i(X) \right)$$

‖ why it is called a fixed pt. thm.?
‖ Note that $N_r = \# X(\mathbb{F}_{q^r}) = \# X(\overline{\mathbb{F}}_q)^{Fr_q^r}$.
‖ (Frob.: $Fr_q^r : a \mapsto a^{q^r}$ / $\overline{\mathbb{F}}_q$)

Let's prove the rationality!

~~Let~~ $\alpha_{i,j} \in \overline{K}$ $(j = 1, \dots, \beta_i)$. (β_i can be zero!)

the eigenvalues of $Fr_q \mid H^i(X)$

Then by linearity:

$$\operatorname{Tr}\left(Fr_q^r \mid H^i(X) \right) = \sum_{j=1}^{\beta_i} \alpha_{i,j}^r$$

Proof (of rat.):

$$Z(X,t) = \exp\left(\sum_{r=1}^{\infty} N_r \frac{t^r}{r} \right) = \exp\left(\sum_{r=1}^{\infty} \sum_{i=0}^{2d} (-1)^i \operatorname{Tr}\left(Fr_q^r \mid H^i \right) \frac{t^r}{r} \right)$$

$$= \exp\left(\sum_{i=0}^{2d} (-1)^i \left(\sum_{j=1}^{\beta_i} \sum_{r=1}^{\infty} \alpha_{ij}^r \frac{t^r}{r} \right) \right)$$

(Taylor expan.!) $$= \exp\left(\sum_{i=0}^{2d} (-1)^i \left(\log \prod_{j=1}^{\beta_i} (1 - \alpha_{ij} t)^{-1} \right) \right)$$

$$= \frac{P_1(t) \cdot P_3(t) \cdots P_{2d-1}(t)}{P_0(t) \cdot P_2(t) \cdots P_{2d}(t)}$$

(Have we proved them?)

(continued):

A supplement lemma

If $f(t) \in K(t) \cap \mathbb{Q}[[t]]$, then $f(t) \in \mathbb{Q}(t)$.

Proof: See [Bourbaki, Algebra] ~~Chap. 4~~ □

Part-4 ~~Eta~~ | Étale topology & étale cohomology.

~~# Reason for Zariski top. not general good enough: ellipse & hyperbola (How about étale?)~~

I ~~don't want to~~ won't give the precise def.s, which may be too technical

① étale morphisms ———— morphisms that induces iso.s at tangent spaces.

② étale top. (Grothendieck top.) —— A ~~sys~~ system of families of étale morphism with certain properties.
(on some cat. C) (mimic the open covering)

③ étale site ———— "C" + "the étale topology".
(on some cat. C)

④ Étale sheaves ———— contravariant functors from the site to $\mathcal{A}b$ with unique ~~glu~~ glueing property.

⑤ ~~If~~ When X is scheme. Let C be cat. of étale mor.s of ~~Let C be~~ X-schemes. Denote by $Sh(X_{\acute{e}t})$ the cat. of étale sheaves. It is a typical topos.

__Definition__ (étale cohomology).

The derived functors of taking global sections

$$\Gamma(X,-) : Sh(X_{ét}) \to Ab$$

Denote them by $H^r(X_{ét}, -)$.

__Definition__ (ℓ-adic cohomology) ($\ell \neq 0$ in $\mathbb{F}_q$)

$$H^r(X_{ét}, \mathbb{Q}_\ell) := \left(\varprojlim_n H^r(X_{ét}, \mathbb{Z}/\ell^n) \right) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell .$$

|| This is the expected Weil cohomology that worked in our scheme. ~~(that's also)~~

Turn to relation between Galois cohomology

① Finite Galois extensions. $k \hookrightarrow k'$ are étale.

② $\mathcal{F} \mapsto \varinjlim_{k'} \Gamma(\mathrm{Spec}(k'), \mathcal{F}) =: M_{\mathcal{F}}$ is an equivalent functor between $Sh(X_{ét})$ and $Mod(G)$ (disc G-mod) where $X = \mathrm{Spec}(k)$, $G = \mathrm{Gal}(\bar{k}^{sep}/k)$

__Theorem__

$$H^r(X_{ét}, \mathcal{F}) \cong H^r(G, M_{\mathcal{F}}).$$

|| So étale cohomology generalize the Galois cohom, which is the case of a point!

|| étale is defined for any loc. noe. sch. not only varieties.
~~and~~
|| See Milne for some form of generalizations of arithmetic duality,
|| Galois has the advantage being more elementary, étale cohom is more heuristic and more machinery, geom.