Modular Forms and The Kissing Number Problem

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October 31, 2013

The Game Plan

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- The Problem

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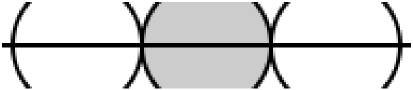
Question: How about the case $n = 1$?
Answer: $\tau_1 = 2$



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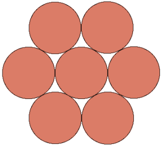

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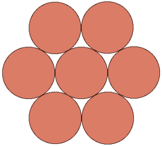

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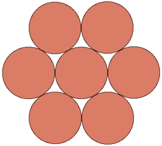

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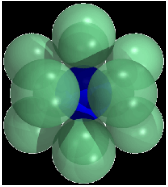

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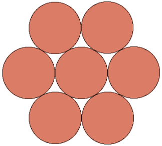

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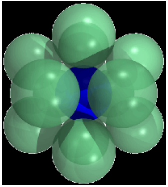
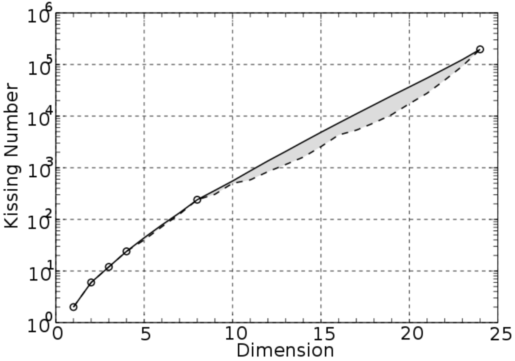


The case $n = 3$ is actually already quite hard!

Even Higher Cases



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1	2	
2	6	
3	12	
4	24[4]	
5	40	44
6	72	78
7	126	134
8	240	
9	306	364
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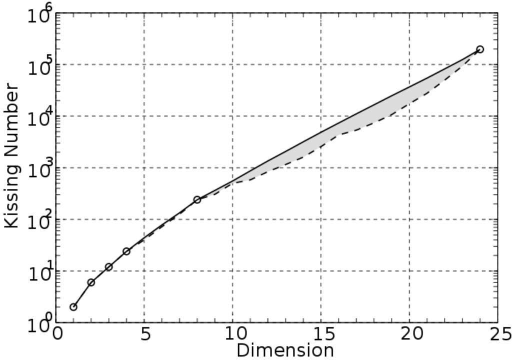

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Today we will look at the special case $n = 24$

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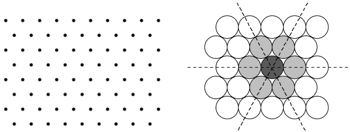

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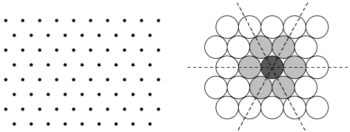

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Lattices (in our case) are subsets of $\mathbb{R}^n$ and equipped with the standard inner product (dot product) denoted (x, y) for $x, y \in \mathbb{R}^n$ with norm $(x, x) = ||x||^2$. (any inner product on a real vector space is a positive-definite symmetric bilinear form).

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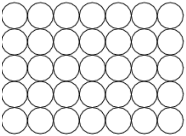
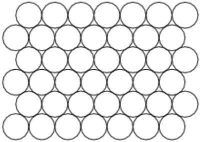

square packing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- L_{24} is even
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What to know how many length 4 vectors are there in L_{24}?

Modular Forms for Dummies

Definition: Modular Form

Let k be an integer. A function $f : \mathbb{H} \mapsto \mathbb{C}$ is a modular form of weight k if
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Barry Mazur: "Modular forms are functions on the complex plane that are inordinately symmetric. They satisfy so many internal symmetries that their mere existence seem like accidents. But they do exist."

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"Unreasonable effectiveness of modular forms"

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- Eisenstein Series of weight 12

$$E_{12}(\tau) = 1 + \frac{65530}{691} \sum_{n=1}^{\infty} \sigma_{11}(n) q^n$$

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where $\tau(n)$ can be written as

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- As dimension of $\dim(M_{12}) = 2 \quad M_{12} = \mathbb{C}E_{12} \oplus \mathbb{C}\Delta$

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Given a lattice L (in our positive definite case) we can define it's theta series as

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- In the case $n = 2$ and for the square lattice $\mathbb{Z}(1,0) \oplus \mathbb{Z}(0,1)$

$$\Theta_{\mathbb{Z}(1,0)\oplus\mathbb{Z}(0,1)} = \sum_{m=0}^{\infty} r_2(n) q^{m/2}$$

where $r_2(n)$ is the number of ways of writing a number as a sum of squares

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The kissing number problem in 24 dimensions has a solution of $\tau_{24} = 196560$

The Proof

- $\Theta_{L_{24}} \in M_{12}$ by theorem on previous slide as the Leech lattice is even and unimodular

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- So for $m = 2$ there are $N_4 = \frac{65530}{691}(\sigma_{11}(2) - \tau(2)) = 196560$ vectors of norm 4 in the Leech lattice.
- But there already exists an upper bound of 196560 so this is indeed the kissing number in 24 dimensions.

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- Other things:
  Generate class numbers of imaginary quadratic fields, main conjecture of Iwasawa theory, Langlands program, identities between divisor functions, physics (Calabi-Yau varieties, Kac Moody algebras, moonshine, number of BPS states in $N = 4$ string theories), polymer chemistry allegedly.

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S_k, cusp forms
H_k, harmonic weak Maass forms
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Will have too wait for another time!