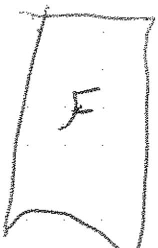
PLAN

A THETA LIFT IN $SL(2,1)$ AND LOCALLY HARMONIC MAASS FORMS

- HISTORY/MOTIVATION
- DEFINITIONS
  - LATTICE
  - GRASSMANIAN
  - SIEGEL THETA FUNCTIONS
  - HARMONIC WEAK MAASS FORMS
- MY SETTING
- THE LIFT
- LOCALLY HARMONIC MAASS FORMS
- [redacted] RELATIONSHIP WITH SHIMURA LIFT

$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$



KEY DEFINITION

First 5 minutes reminder of final slides from last talk

M_k, MODULAR FORMS $\tau = u + iv$

Key definition needed before proceeding

- $k \in \mathbb{Z}$, even, non-negative. A function $f : \mathbb{H} \to \mathbb{C}$ of weight "k"

1) f is holomorphic on $\mathbb{H}$ — i.e. "nice", "complex differentiable"
2) f is weakly modular of weight k. — $f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau)$ — $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ or subgroup
3) f is holomorphic at ∞. — No singularities

PROPERTIES

- $f(\tau+1) = f(\tau) = \boxed{\sum_{n=0}^{\infty} a_n e^{2\pi i n \tau}} = \sum_{n=0}^{\infty} a_n q^n$, $e(n\tau)$, if $a_n(0) = 0$ "cusp form" vanishes at singularities

- finite dimensional

Next I give an overview, which kind of ruins the surprise but will hopefully keep people intrigued before lots of definitions.

HISTORY / MOTIVATION — [redacted]

— Modular forms are useful — ~~as briefly~~ described a lot of interesting applications ~~last time~~, fermat, ~~do~~ physics ~~etc.~~

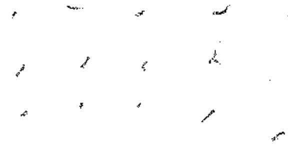
First examples of $\frac{k}{2}$ — weight M_k modular forms. (WILL DEFINE THESE SHORTLY) ~~of~~ people can't remember

— Theta functions (classical)

$$\Theta(\tau) = \sum_{x \in \mathbb{Z}} e^{i\pi \frac{(\lambda,\lambda)}{2}\tau} \quad \text{or } \lambda^2 \quad \lambda \in \mathbb{Z} \quad \in \tfrac{1}{2}$$

or x^2

or $x \in \mathbb{Z}$



L a lattice

— Kissing problem

— Representation numbers — number of ways of summing squares to get "n".

Shimura Lift

— Kickstarted theory of $\frac{k}{2}$ modular forms, 1970s

— Note can look at ~~general~~ p/q weight, but not interesting, hard to work with

— Desired a map

$$S_{k+1/2} \to M_{2k}$$

$$f(z) = \sum_{n=1}^{\infty} b(n) e^{2\pi i n z} \to \sum_{n=1}^{\infty} a(n) e^{2\pi i n z}$$

$$\sum_{n=1}^{\infty} a(n) n^{-s} := \zeta(s-k+1) \sum_{n=1}^{\infty} b(tn^2) n^{-s}$$

— Actually shown later roughly

$$f \mapsto \int_F f(z)\Theta(\tau) \qquad \text{"a theta lift"}$$

— Tunnels, congruent number problem

?

$k/2$ Forms

— Constructing Heegner points on elliptic curves, last week in the pure maths colloquium Vlad Dokchitser, BSD rank 1

— Class numbers — generating function of imaginary quadratic fields

etc.

~~Weak~~ Harmonic weak Maass forms, locally harmonic Maass forms $\boxed{H_k}$

~~$\Delta = 1$~~ no ~~cusps~~, ~~even~~
— ~~reg~~
— ~~poles~~
— ~~singularities~~
~~unimodular~~
~~even~~

— Generalises modular forms

— Ramanujans mock theta functions, is holomorphic part of one of these ($\frac{1}{2}$ weight)
1920 letter to Hardy, $1729 = 1^3 + 12^3 = 9^3 + 10^3$

— Ono, Bruinier use very similar "theta lift" to one I describe today to give finite formula of partition function

$\int_{\tau} g(\tau)\, \Theta(\tau, z)\, d\tau$

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AIM

$$\begin{array}{ccc} H_{3/2-k} & \longrightarrow & LH_{2-2k} \\ \text{surjective} \downarrow & & \downarrow \\ S_{k+1/2} & \xrightarrow{\text{Shimura}} & M_{2k} \end{array}$$

— Surprise

REM $k = 1, 2, 3 \ldots$

$Q(rx) = r^2 Q(x)$

$(x, y) = Q(x+y) - Q(x) - Q(y)$

DIFFERENT "k"

k here $= 1, 2, 3, 4, 5 \ldots$

DEFINITIONS LATTICE — need this to define thetas

We define a lattice:

L of signature (b^+, b^-) — V a rational vector space, quadratic

i.e. "a free $\mathbb{Z}$ module $L \subset V$" $V = L \otimes \mathbb{Q}$ $V(\mathbb{R}) = V \otimes \mathbb{R}$

$\boxed{L \cong \mathbb{Z}^m}$ usually

with an attached quadratic form $Q(x) = \frac{1}{2}(x, x)$ bilinear form

$\boxed{\text{isometric}}$ to $\mathbb{R}^{b^+, b^-} = \mathbb{R}^{b^+ + b^-}$ $Q(x) = x_1^2 + \cdots x_{b^+}^2 - x_{b^++1}^2 \cdots -$

ISOMETRIC
~~isomorphism~~
injective linear map
$\sigma : V \to V'$
$Q(\sigma(x)) = Q(x)$

- Assume L even — $Q(x) \in \mathbb{Z}$
- Assume L unimodular — $\det(T) = 1, -1$ $T = \begin{pmatrix} (b_1, b_1) & (b_1, b_2) & \cdots \end{pmatrix}$

big cheat here, simplifies exposition for today. i.e. scalar valued forms

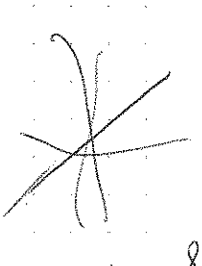
GRASSMANIAN

$$Gr(L) = \{ Z \subset V(\mathbb{R}) \mid \dim Z = b^- \text{ and } Q|_Z < 0 \}$$

"set of negative definite b^--dimensional subspaces"

can think of this as a bijection to $\mathbb{H}$ actually in my case

in case $b^- = 1$



$Q_z(\lambda) = Q(\lambda_{z^\perp}) - Q(\lambda_z)$ for a fixed $z = x + iy$

$\lambda = \lambda_z + \lambda_{z^\perp}$

always positive

take line of length 1, 2 shell hyperboloid

like $\mathbb{H}$

SIEGEL THETA FUNKTION

the quadratic form is

"Generalises earlier version, which diverged if negative definite as $v \to \infty$"

$$\Theta_L(\tau, z, p) = v^{\frac{b^-}{2}+m^-} \sum_{\lambda \in L} p(\lambda)\, e\left(Q(\lambda)u + Q_z(\lambda) i v\right)$$

$\lambda = \lambda_1 b_1 + \lambda_2 b_2 \ldots$ basis of L

"polynomial" degree m^+, m^- in $\lambda_1, \lambda_2 \lambda_3 \ldots$

- weight $\frac{b^+ - b^-}{2} + m^+ - m^-$ modular form in τ.
-

Invariant under $O(L)$ (preserves quadratic), polynomial later will create a modular form weight in $z = x+iy$

- Surjective homo, $\mathrm{Spin}(V) \to SO^+(V)$, acting via conjugation

group of linear maps, isometries preserving Q

NOTE ON 1/2 WEIGHT FORMS

- $g(\gamma\tau) = (c\tau+d)^{1/2} g(\tau)$ leads to contradictions choice of square root
- $g(\gamma\tau) = \underbrace{\left(\frac{c}{d}\right)^{2k+1} \epsilon_d^{-2k-1}}_{\text{junk}} (c\tau+d)^{k+1/2} g(\tau)$ ← can define half weight forms like this

$= SL_2(\mathbb{Q})$ in our case

subgroup of this

- actually fractional weight forms should be under "central extensions of $SL_2(\mathbb{Z})$, ie. braid group B_3"
- $Mp_2(\mathbb{Z})$ unique connected double cover of $SL_2(\mathbb{Z})$

$$Mp_2(\mathbb{R}) = (\gamma, \phi_\gamma) \quad \gamma \in SL_2(\mathbb{R}) \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
$$\phi_\gamma(\tau)^2 = (c\tau + d)$$

$$(\gamma, \phi_\gamma(\tau))(\gamma', \phi_{\gamma'}(\tau)) = (\gamma\gamma', \phi_\gamma(\gamma'\tau)\, \phi_{\gamma'}(\tau))$$

- so junk is projected from reps of $Mp_2(\mathbb{Z})$
- I work in case non-unimodular, so have "vector valued forms"
- unique rep, "weil representation"

L'/L has cosets e_h

$$\rho_L(T)(e_h) = e(Q(h))\, e_h$$
$$\rho_L(S)(e_h) = \frac{e((b^- - b^+)/8)}{\sqrt{|L'/L|}} \sum_{\lambda \in L'/L} e(-(h,\lambda))\, e_\lambda$$

NATURAL. from Heisenberg group

automorphisms of Heisenberg fix centre, are Symplectic groups, gives action, using stone von neumann ~~theorem~~

$f(\tau)$ = HARMONIC WEAK MAASS FORMS - Generalise modular forms.

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(2) $\Delta_k(f) = 0$

where $\Delta_k = -v^2\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right)+ikv\left(\frac{\partial}{\partial u}+i\frac{\partial}{\partial v}\right)$

- not finite dimensional
- $f = f^+ + f^-$, $f^+ = \sum_{n \geq -\infty} c^+(n)e(n\tau)$, $f^- = \sum_{n<0} c^-(n)\Gamma(1-k, |4\pi n v|)e(n\tau)$

holomorphic part | non-holomorphic

- for $\boxed{k \geq 2}$ f^- vanishes

$f \in H_k$ OPERATORS

- we have lowering, raising operators

$R_k := 2i\frac{\partial}{\partial \tau} + kv^{-1}$ $\quad L_k := -2iv^2\frac{\partial}{\partial \bar{\tau}}$

$R_k(f)$ weight $k+2$ $\quad L_k(f)$ weight $k-2$

$\xi_k(f) = v^{k-2}\overline{L_k f(\tau)} = R_{-k}v^k\overline{f(\tau)}$ basically the same

- Theorem

$\xi_k : H_k \to S_{2-k}$ surjectively Jens + Bruinier

- Background done

- My work based on german thesis, case $k=1$. he did this in MY SETTING

- V sig $(2,1)$
- $V = \mathrm{Mat}_2(\mathbb{Q})$, traceless
- $Q(X) = -\det(X)$
- $L = \begin{pmatrix} b & -a \\ c & -b \end{pmatrix}$ $a, b, c \in \mathbb{Z}$
- $Gr(L) \cong \longleftrightarrow z \longleftrightarrow$ Hyperbolic space $\quad z = x + iy$
- $\forall$ $\lambda \in V \Rightarrow \lambda = \lambda_1(z)b_1(z) + \lambda_2(z)b_2(z) + \lambda_3(z)b_3(z)$ orthonormal basis

$$\lambda_3(z) = \frac{-1}{\sqrt{2}y}(c|z|^2 - bx + a)$$

MY THETA

$$\Theta_k(\tau,z) = v^{3/2}\sum_{\lambda\in L} \lambda_3(z)\overbrace{\underbrace{(\lambda_1(z)+i\lambda_2(z))^{k-1}}_{\text{Shimura theta } \boxed{\lambda_1(z)-i\lambda_2(z)}}}^{\#_z(\lambda_1)} e(Q(\lambda)u + Q_z(\lambda)iv)$$

$$Q_z(\lambda) = \lambda_1(z)^2 + \lambda_2^2(z) - \lambda_3^2(z)$$

- ask the audience
- weight $\frac{3}{2} - 1 + k - 1 - 1 = \underline{k - \frac{3}{2}}$ in τ, $Mp_2(\mathbb{R})$
- weight $2-2k$ in z $\left(\#_z(\gamma.\lambda) = \#_z(\lambda)(cz+d)^{-2} \right)$
- in $\Gamma_0(N) \subset Spin(V)$ $= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ $c \equiv 0 \bmod (N)$

THETA LIFT

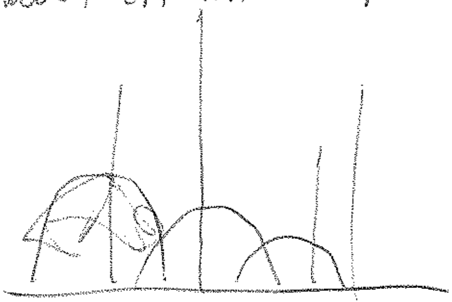
$$\Phi_k(z,f) = \int^{reg}_{\tau\in F} f(\tau)\, \Theta(\tau,z)\, \underbrace{\frac{du\,dv}{v^2}}_{\boxed{\text{actually peterson scalar product}}}$$

- ~~In our case ($f \in H_k$)~~
- $f \in H_k$, then Θ in $\tau \in M_k$ to make sense
- Also as ($f \in H_k$) can grow exponential fast $v \to \infty$
- So take $\lim_{t\to\infty} \int_{F_t}$
- Can prove (some work) this converges
- In our case $\underline{f \in H_{3/2-k}}$

- $\Phi_1(z,f)$ clearly of weight $\underline{2-2k}$ $\underline{\text{(but what is it?)}}$

EXCITING SINGULARITIES

- Smooth on $\mathbb{H}$ except



- Step / Jump singularities

polynomial jump

i.e. jump of "z" or $\lambda_1(z)+i\lambda_2(z)^{k-1} = (cz^2-bz+a)^{k-1}$

- Why? -
- When showing convergence, turns out only finite number of vectors λ contribute to potential singularities and $\lambda \perp z$

$\operatorname{tr}\begin{pmatrix} b/2 & -a \\ c & -b/2 \end{pmatrix}\begin{pmatrix} -x & x^2+y^2 \\ -1 & x \end{pmatrix}$

$c|z|^2-bz+a=0$

and $\int_{v=1}^{\infty} e(Q(\lambda_z)iv)\, v^{-1/2}dv$ ~~has singularity~~

$= \Gamma(\tfrac{1}{2}, -2\pi Q(\lambda_z))$ has singularity at $0 = Q(\lambda_z)$

$\lambda \perp z$, λ fixed then z defines a $\boxed{\text{geodesic}}$

and $C^+(n,h)$

$-\infty < n < 0$

so finitely many geodesics

LOCALLY HARMONIC MAASS FORMS

- Defined [redacted] recently
- Same as H_k but have singularities LH_k [redacted]
- $\Phi_k(z,f) \in LH_{2-2k}$
- Proof (Very Sketchy)

$$4\Delta_{k-3/2}\Theta_k(\tau,z) = \Delta_{2-2k}\Theta_k(\tau,z) + (\ldots-4k)\Theta_z(\tau,z)$$

in τ in z

- [redacted] proof using Casimir element, Lie Algebra, chain rule
- and $\int_{F_t} f\, \overline{\Delta_k g}\, v^{k-2}du\,dv - \int_{F_t} \Delta_k f\, \overline{g}\, v^{k-2}du\,dv = \int_{\partial F_t} \ldots f g$

STOKES   stuff

- So $\Delta_{2-2k}\Phi_k(\tau,z) = \int f.\Delta_{2-2k}\Theta = \int \Delta_{3/2-k}(f)\,\Theta = 0$

EXPANSION

- Can check

$$\begin{array}{ccc} H_{3/2-k} & \longrightarrow & LH_{2-2k} \\ \downarrow & & \downarrow \\ S_{k+1/2} & \longrightarrow & M_{2k} \end{array}$$

[redacted] about with ξ and Cimbarg Shimura $\Theta \Longleftrightarrow$ my Θ

~~Shimura $\Phi \Longleftrightarrow$ my Φ~~

so $\Phi_{Shim}(\xi_{3/2-k}(g)) = \xi_{2-2k}(\Phi_k(g))$

- Also would like explicit expansion
  - lots lots of work, first write as poincare series

~~- have to carefully re~~ $\Theta(\tau) = \sum_{(c,d)=1} (c\tau+d)^k g\left(\frac{a\tau+b}{c\tau+d}\right)$

then $\int_{F_T}$ so $\int_0^\infty \int_{-1/2}^{1/2} g(\tau) \overline{g(\tau)} \frac{du\,dv}{v}$

Shimura

$\Phi(\xi_{3/2-k}(g))$

$= \int \xi(g)\Theta' = \int g\, \xi\Theta'$

$= \xi\left(\int g\, \Theta\right)$

$=$ my way

$$- \sum_{m\geq 1} \sum_{0\leq h\leq k} \binom{k-1}{h} \frac{1}{(h+1)} (my)^{k-1-h} B_{1+h}(-mx)\, c^+\left(-\frac{m^2}{4}, \frac{rm}{2}\right)$$

$$+ \sum_{m\geq 1} \sum_{n\geq 1} \sum_{0\leq i\leq 2k-2} (my)^k (\pi m n y)^{k-2-i} (\text{stuff})\, e(-nm\bar{z})\, c^-\left(-\frac{m^2}{4}, \frac{rm}{2}\right)$$

$$+ \text{constant } (L\text{-function at } k)\, \Gamma(\ )$$

$$+ \frac{1}{y} \int g\, \Theta = \left(\sum_{m\geq 1} m^{1+k} c^+\left(-\frac{m^2}{4}, \frac{rm}{2}\right)\right)$$

Θ - 1-dim

- Full Expansion of lift. (Time permitting)

Shimura $g \in M_{k+1/2}$ $\quad g = \sum c(n,h) e(h\tau)$

$$= \text{constant} + \sum_{n\geq 1} \sum_{\substack{d\geq 1 \\ d\mid n}} d^{k-1} c\left(\frac{n^2}{d^2 4}, \frac{hr}{2d}\right) e(nz)$$

Expansion of shimura lift

after ξ

$$\Phi(\xi_{3/2-k}(g)) = A_{q,r}(\xi_{2-2k}(g)) - (\pi)^{k-1/2} \sum_{n\geq 1} \sum_{\substack{d\geq 1 \\ d\mid n}} \frac{n^{2k-1}}{d^k} \overline{c^-\left(\frac{n^2}{d^2 4}, \frac{hr}{2d}\right)}\, e(nz)$$