1-form Laplacian on a Graph-like Manifold

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Metric Graph

Metric Graph

Given a discrete graph $G = (V, E, \partial)$ where $\partial : E \to V \times V$ such that $e \mapsto (\partial_- e, \partial_+ e)$ and a length function $\ell : E \to (0, +\infty)$, $e \mapsto \ell_e > 0$, we identify $e \sim I_e := [0, \ell_e]$. A *metric graph* X_0 is

$$X_0 := \dot{\bigcup_{e \in E}} I_e / \sim$$

where $\sim$ identifies end points of I_e.

Metric Laplacian

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- *L^2-space* $L^2(\Lambda^1(X_0))$:

$$L^2(\Lambda^1(X_0)) = \bigoplus_{e \in E} L^2(\Lambda^1(I_e))$$

$$\|\alpha\|^2_{L^2(\Lambda^1(X_0))} := \sum_{e \in E} \|\alpha_e\|^2_{L^2(\Lambda^1(I_e))}$$

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- *exterior derivative:*

$$d : \operatorname{dom} d \longrightarrow L^2(\Lambda^1(X_0)), \quad (f_e)_{e\in E} = (f_e'\, ds)_{e\in E}$$

$$\operatorname{dom} d = H^1(X_0) \cap C(X_0)$$

- *adjoint operator:*

$$d^*\alpha = -(f_e)'_{e\in E} \quad \operatorname{dom} d^* = \{\alpha \in H^1(\Lambda^1(X_0)) \,\Big|\, \sum_{e\in E} \vec{\alpha}_e(v) = 0\}$$

$$\vec{\alpha}_e(v) = \begin{cases} -f_e(0)\; ds & v = \partial_- e \\ f_e(l_e)\; ds & v = \partial_+ e \end{cases}$$

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- *Laplacian on 1-forms on* X_0:

$$\Delta^1_{X_0}\alpha = dd^*\alpha = -\alpha''$$

$$\operatorname{dom} \Delta_{X_0} = \{\alpha \in \operatorname{dom} d^* \mid d^*\alpha \in \operatorname{dom} d\}$$

Graph-like Manifold

Graph-like Manifold

Given a metric graph X_0, we associate a family of compact and connected manifolds $\mathrm{X}_\varepsilon = (X, g_\varepsilon)$, $\varepsilon > 0$, of dimension $n \geq 2$ built as

$$\mathrm{X}_\varepsilon = \dot{\bigcup_{e \in E}} X_{\varepsilon,e} \cup \dot{\bigcup_{v \in V}} X_{\varepsilon,v}$$

$$X_{\varepsilon,v} \cap X_{\varepsilon,e} = \begin{cases} \emptyset & e \notin \{e \in E \,|\, v = \partial_\pm e\} \\ Y_{\varepsilon,e} & e \in \{e \in E \,|\, v = \partial_\pm e\} \end{cases} \tag{1}$$

Here: $X_{\varepsilon,v} = (X_v, \varepsilon^2 g_v)$, $Y_{\varepsilon,e} = (Y_e, \varepsilon^2 h_e)$ and $X_{\varepsilon,e} = (X_e, g_{\varepsilon,e}) = (I_e \times Y_e, ds^2 + \varepsilon^2 h_e)$ with X_v, X_e compact Riemannian manifolds with boundary of dimension n and Y_e a closed manifold of dimension $n-1$.

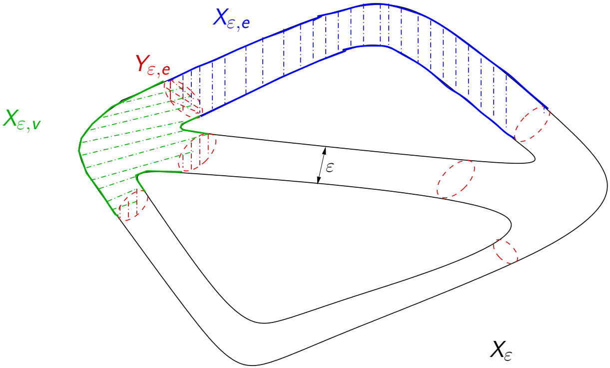

Figure : An example of a 2-dimensional graph-like manifold

Hodge Laplacian

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- *L^2-space:*

$$L^2(\Lambda^1(X_\varepsilon)) = \left\{ w \in \Lambda^1(X_\varepsilon) \mid \|w\|^2_{L^2(\Lambda^1(X_\varepsilon))} := \int_{X_\varepsilon} w \wedge *w < \infty \right\}$$

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- *Hodge Laplacian on 1-forms on X_ε:*

$$\Delta^1_{X_\varepsilon} = dd^* + d^*d$$

with d and d^* are the classical exterior derivative and co-derivative on a manifold.

Aim and Strategy

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Then, we have

$$\Delta^1_{X_\varepsilon}\alpha = dd^*(df) + d^*d(d^*\beta) = \Delta^{1,\mathrm{ex}}_{X_\varepsilon}(df) + \Delta^{1,\mathrm{co\text{-}ex}}_{X_\varepsilon}(d^*\beta)$$

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Hence, we study the two new Laplacian separately.

Results

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Theorem 1

Let X_ε be a graph-like manifold with underlined graph X_0. Let $\lambda_j^{ex,1}(\mathsf{X}_\varepsilon)$, $\lambda_j^1(X_0)$ be the j-th eigenvalue on exact 1-forms on X_ε and on 1-forms on X_0, respectively. Then,

$$\lambda_j^{ex,1}(\mathsf{X}_\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} \lambda_j^1(X_0) \qquad \text{for all } j > 0$$

Results

Theorem 2

Let X_ε be a graph-like manifold of dimension $n \geq 3$ shrinking to a metric graph X_0 as $\varepsilon \to 0$. Then, the first eigenvalue of the Laplacian defined on co-exact 1-forms on X_ε satisfies

$$\lambda_1^{co-ex,1}(X_\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} \infty$$

Therefore, in the limit, the spectrum of the Laplacian on co-exact 1-forms contains just 0 and all the other eigenvalues tend to ∞ as $\varepsilon \to 0$

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We have:

$$\begin{array}{ccccc}
\lambda_j^{ex,1}(\mathrm{X}_\varepsilon) & \overset{(a)}{=} & \lambda_j^0(\mathrm{X}_\varepsilon) & \Longrightarrow & \lambda_j^{ex,1}(\mathrm{X}_\varepsilon) \xrightarrow[\varepsilon \to 0]{} \lambda_j^1(X_0) \\
 & & (c) \Big\downarrow \varepsilon \to 0 & & \\
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(a) is given applying d^* to the eigen-exact 1-forms on X_ε and d to the eigenfunctions on X_ε.

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(c) is due to convergence results by Exner and Post in [2].

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- Let $\lambda_j^{co-ex,1}(\mathsf{X}_\varepsilon), \lambda_j^{ex,2}(\mathsf{X}_\varepsilon)$ be the j-th eigenvalues, $j > 0$, on co-exact 1-forms and exact 2-forms on X_ε.

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  Hence, we choose to work with 2-exact forms.

Theorem 3

Let M be a n-dimensional compact Riemannian manifold without boundary and let $\{U_i\}_{i=1}^m$ be an open cover of M. Let $U_{ij} = U_i \cap U_j$ and let $\mu(U_i)$, resp. $\mu(U_{ij})$, be the smallest positive eigenvalue of the Laplacian acting on exact 2-forms on U_i, resp. on 1-forms on U_{ij}, satisfying absolute boundary conditions. Further, assume $H^1(U_{ij}) = 0$ for all i, j. Then, the first eigenvalue of the Laplacian on exact 2-forms on M satisfies

$$\lambda_1^{ex,2}(M) \geq \frac{2^{-3}}{\sum_{i=1}^{m} \left(\frac{1}{\mu(U_i)} + \sum_{j=1}^{m_i} \left(\frac{w_{n,2} c_\rho}{\mu(U_{ij})} + 1 \right) \left(\frac{1}{\mu(U_i)} + \frac{1}{\mu(U_j)} \right) \right)}$$

where m_i is the number of j, $j \neq i$ for which $U_i \cap U_j \neq \emptyset$ and $c_\rho, w_{n,2}$ are constants.

See [3, Lemma 2.3]

Assume $n \geq 3$ and choose the open cover $\{U_{\varepsilon,i}\}_{i \in E,V}$ of X_ε as the disjoint union of edge and vertex neighbourhoods in (1) in such a way that they overlap a bit.

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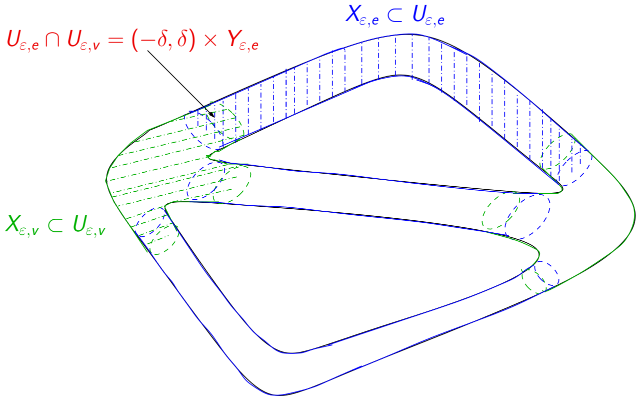

Figure : The open cover of X_ε

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Then,

$$\lambda_1^{ex,2}(X_\varepsilon) \geq \frac{2^{-3}}{\sum_{v \in V} \left(\frac{1}{\mu_{1,2}(X_{\varepsilon,v})} + \sum_{e \in E_v} \left(\frac{w_{n,2} c_\rho}{\mu_{1,1}(X_{\varepsilon,e})} + 1 \right) \left(\frac{1}{\mu_{1,2}(X_{\varepsilon,v})} + \frac{1}{\mu_{1,2}(X_{\varepsilon,e})} \right) \right)}$$

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where $\mu_{k,p}(\cdot)$ denotes the k-eigenvalue on p-forms satisfying absolute boundary conditions and $w_{n,2}$ and c_ρ are constants.

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$$\mu_{1,2}(X_{\varepsilon,v}) = \frac{\mu_{1,2}(X_v)}{\varepsilon^2} \quad \mu_{1,i}(X_{\varepsilon,e}) = \frac{c_i(\varepsilon)}{\varepsilon^2}, \quad c_i(\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} \lambda_{1,i}(Y_e) \quad i = 1, 2$$

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Therefore,

$$\lambda_1^{co-ex,1}(\mathrm{X}_\varepsilon) = \lambda_1^{ex,2}(\mathrm{X}_\varepsilon) \geq \frac{const}{\varepsilon^2} \underset{\varepsilon\to 0}{\longrightarrow} \infty$$

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Hence, the whole spectrum escapes to infinity.

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- If $H^1(Y_{\varepsilon,e}) \neq 0$ for some $e \in E$, we obtain the same result proving $\lambda_j^{ex,2}(\mathrm{X}_\varepsilon) \to \infty$ as $\varepsilon \to 0$ for $j \geq N > 1$ and $\lambda_j^{ex,2}(\mathrm{X}_\varepsilon) \to 0$ for smaller j's (again [3, Lemma 2.3]).

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- The bound for $\lambda_1^{ex,2}(\mathsf{X}_\varepsilon)$ extends to any exact p-forms assuming that $H^{p-1}(Y_{\varepsilon,e}) = 0$.

  Therefore, we have a complete description of the spectrum of the Laplacian on any degree forms.

Spectral gap

Spectral gap

Given a metric graph X_0 with a spectral gap (a, b) in its 1-form Laplacian, i.e., (a, b) does not belong to the spectrum, then the associated graph-like manifold X_ε has a spectral gap close to (a, b) in its 1-form Laplacian for ε small enough.

Examples

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Attach the graph $\widetilde{X_0}$ to each vertex of X_0.

This process opens up gaps in the spectrum of its 1-form Laplacian, and so in the spectrum of the 1-form Laplacian of its associated graph-like manifold.

Example 2: Ramanujan graphs, part 1

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Let G be a Ramanujan graph, i.e., G is a k-regular graph with n vertices such that

$$\mu_1(G) = \max_{|\mu_i(G)| \leq k} \mu_i(G) \leq \frac{2\sqrt{k-1}}{k}$$

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The discrete (normalized) Laplacian Δ_G on functions has a spectral gap

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Due to the relation $\mu = 1 - \cos(\sqrt{\lambda}) \in \Delta_G \Leftrightarrow \lambda \in \Delta_{X_0}$ between discrete and metric graph (see [1]), it follows that $\sigma(\Delta_{X_0}) = \sigma(\Delta^1(X_0)$ and therefore $\sigma(\Delta^1_{X_\varepsilon})$ has a spectral gap given by $I = (0, b_\varepsilon)$ with b_ε close to $\arccos^2\left(1 - \frac{2\sqrt{k-1}}{k}\right)$.

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Let the length of the edge neighborhoods shrink according to $n^{-\frac{1}{2\alpha}}$, $\alpha \geq 1$.
Set $\varepsilon = n^{-\frac{1}{\beta d}}$, $\beta \geq 1$

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Let the length of the edge neighborhoods shrink according to $n^{-\frac{1}{2\alpha}}$, $\alpha \geq 1$.

Set $\varepsilon = n^{-\frac{1}{\beta d}}$, $\beta \geq 1$

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- There is an increasing spectral gap that grows according to a power of n.

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THANK YOU FOR YOUR ATTENTION!