Knot Concordance

Jonathan Grant

Definition

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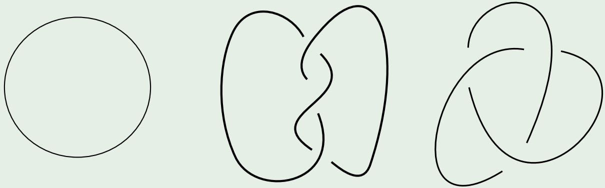

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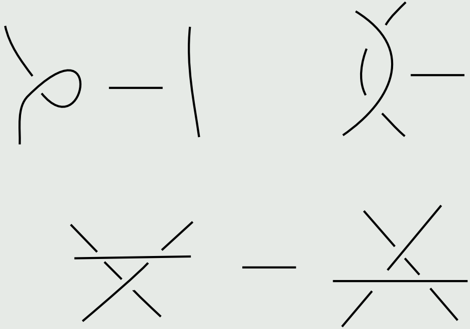
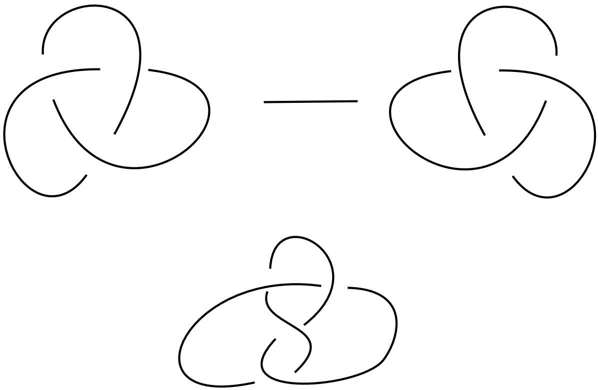
Theorem (Reidemeister)

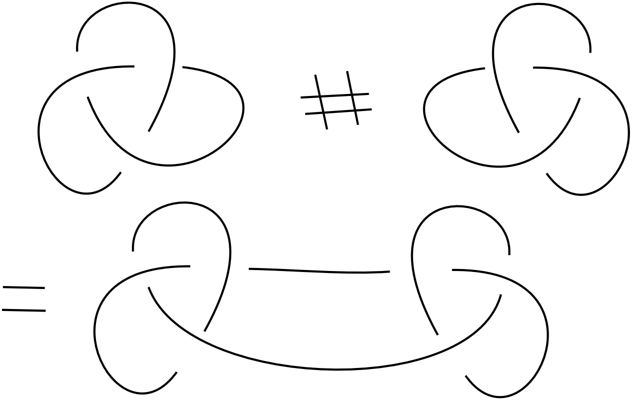
Any two diagrams of the same knot are related via the following moves:



1. Given a knot K, we can form its mirror image $\bar{K}$:



Given two knots K_1 and K_2 we can also form their connected sum $K_1 \# K_2 = K_2 \# K_1$:



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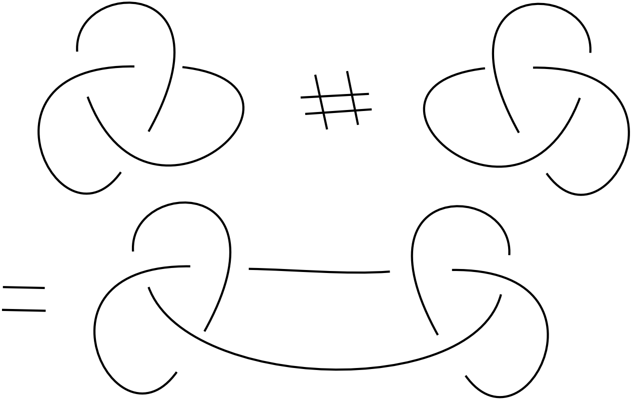

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Theorem

For any knots A and B, if $A\#B$ is equal to the unknot, then both A and B are also equal to the unknot.

Proof.

Suppose $A\#B = B\#A = U$. Then

$$A = A\#(B\#A)\#(B\#A)\#\cdots = (A\#B)\#(A\#B)\#\cdots = U$$

so $A = U$. Similarly, $B = U$. □

This trick is called the Mazur swindle.

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2. Knots give a concrete way of describing lots of 3- and 4-dimensional spaces
3. Every (closed) 3-dimensional manifold can be described by cutting out a tube around a knot, and then gluing it back in 'with a twist'
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Definition

Two knots K_1 and K_2 are said to be concordant (and we write $K_1 \sim K_2$ if there is an embedding $f : S^1 \times [0,1] \to S^3 \times [0,1]$ such that $f(S^1 \times 0) = K_1$ and $f(S^1 \times 1) = K_2$.

Clearly concordance is an equivalence relation, and the equivalence classes are called concordance classes.

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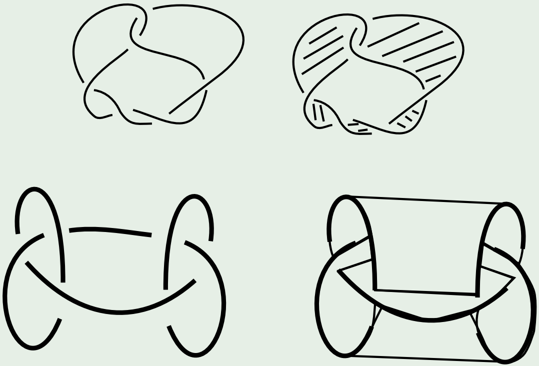
Theorem

A knot K is slice if and only if there is a an embedding of the disc into B^4 with boundary equal to K.

Example

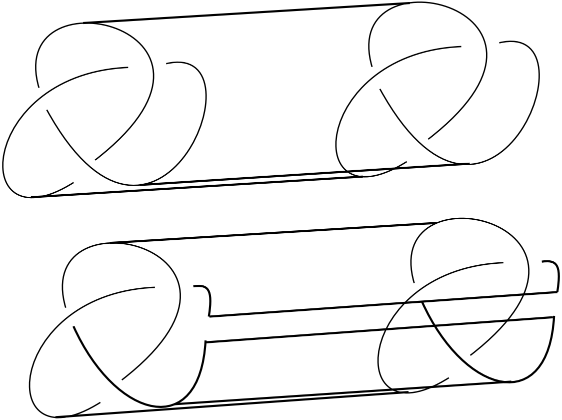


Theorem

The set of concordance classes of knots form an abelian group under the operation of connected sum, with the inverse of a knot given by its mirror image.

Proof.

If K_1 is concordant to K_1' and K_2 is concordant to K_2', then by cutting a vertical strip out of each of the concordance cylinders, we can glue them together to see that $K_1 \# K_2$ is concordant to $K_1' \# K_2'$ so the operation is well-defined.
To show that K and $\bar{K}$ are inverses, we must show that $K \# \bar{K}$ is slice. Take the embedding of the cylinder such that both boundaries are K. Then cutting a strip out of the cylinder produces a disc with boundary $K \# \bar{K}$. □



- Clearly the concordance class of the figure 8 knot has order 2 in the concordance group, since it is equal to its own mirror image (ie. it is amphichiral).
- Very little is known about the concordance group: it is known it contains $\mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty$.
- It is an open question whether the concordance group contains any $\mathbb{Z}_n$ summands for $n > 2$ (it is conjectured that it does not).

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A knot may not bound a disc in B^4, but it will always bound some embedded (orientable) surface in B^4.

Definition

We define the slice genus $g_*(K)$ of a knot to be the minimal genus of a surface embedded in B^4 with boundary equal to the knot.

To be more specific:

- Smooth slice genus: require the surface to be smoothly embedded
- Topological slice genus: require the surface to be 'locally flatly' embedded

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Theorem

Every knot bounds a disc topologically embedded in B^4.

Proof.

The 'cone' of S^1 is defined to be $S^1 \times [0,1]/(S^1 \times 1)$, which is homeomorphic to a disc. Similarly, the cone of S^3 is homeomorphic to B^4. Hence given our knot, given by an embedding $f : S^1 \to S^3$, we can define an embedding $Cf : D^2 \to B^4$ that bounds the knot. □

The slice genus is often hard to compute in general, but it is possible to extract information about it from algebraic invariants.

Definition

The Alexander polynomial is defined by $\Delta(U) = 1$ and

$$\Delta(\nwarrow\!\!\!\!\nearrow) - \Delta(\nearrow\!\!\!\!\nwarrow) = (t - t^{-1})\Delta(\uparrow\uparrow)$$

Example

For example, the Alexander polynomial of the trefoil is

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Theorem (Freedman)

If $\Delta(K) = 1$, K is topologically slice.

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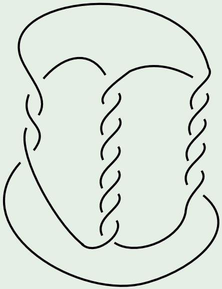

Khovanov homology associates an abelian group $Kh(K)$ to each knot K. From $Kh(K)$ it is possible to extract a number s, called the Rasmussen s invariant. This s has the property that

$$|s(K)| \leq 2g_*(K)$$

for any knot K, where $g_*(K)$ is the smooth slice genus. Equality holds for some classes of knots.

Example

The s invariant of the Pretzel knot K in the previous example is $s = 2$. Therefore the smooth slice genus of K is ≥ 1. So K is topologically slice, but not smoothly slice.

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We can use this to construct an exotic $\mathbb{R}^4$: this is a space that is homeomorphic to $\mathbb{R}^4$, but not diffeomorphic (ie. a non-standard smooth structure on $\mathbb{R}^4$).

Let K be a knot. Construct a space X_K by gluing $D^2 \times D^2$ along the knot K.

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X_K has a smooth embedding into $\mathbb{R}^4$ if and only if K is smoothly slice, and X_K has a topological embedding into $\mathbb{R}^4$ if and only if K is topologically slice.

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Let K be a knot that is topologically slice but not smoothly slice (such as the Pretzel knot before). Let $\rho : X_K \to \mathbb{R}^4$ be a topological embedding.

There is a smooth structure on $\mathbb{R}^4 \setminus \rho(\text{int}(X_K))$. The smooth structures on $\partial(\mathbb{R}^4 \setminus \rho(\text{int}(X_K)))$ and ∂X_K are diffeomorphic because there is a unique smooth structure on any closed 3-manifold.

Hence we can glue X_K into $\mathbb{R}^4 \setminus \rho(\text{int}(X_K))$ by identifying their boundaries to form a new space R, which is homeomorphic to $\mathbb{R}^4$ and comes equipped with a smooth structure.

If there was a diffeomorphism $\phi : R \to \mathbb{R}^4$, it would restrict to an embedding $\phi_{|X_K} : X_K \to \mathbb{R}^4$, which is impossible as K is not smoothly slice. Hence R is an exotic $\mathbb{R}^4$. □

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