Quantum Invariants of Knots

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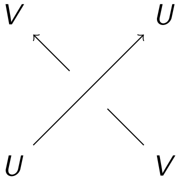
Definition

A pivot category is a category $\mathcal{C}$ with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ with

- natural isomorphisms $\alpha_{U,V,W} : U \otimes (V \otimes W) \to (U \otimes V) \otimes W$ satisfying a commutative diagram
- an object $I \in \mathcal{C}$ and natural isomorphisms $\rho_U : U \otimes I \to U$ and $\lambda_U : I \otimes U \to U$ satisfying a commutative diagram.
- an object U^* for each U and morphisms $\mathrm{ev}_U : U^* \otimes U \to I$ and $\pi_U : I \to U \otimes U^*$ and is such that the contravariant functor $U \mapsto U^*$ is an anti-equivalence of categories.

Suppose we had a family of natural isomorphisms $R_{U,V} : U \otimes V \to V \otimes U$ for each pair of objects U, V. Then we could label each strand with an object in $\mathcal{C}$ and define:



$= R_{U,V}$

$= \pi_U$

$= \mathrm{ev}_U$

Then we can define a link invariant by colouring each strand with an object in $\mathcal{C}$ and reading the map from bottom to top as a map $I \to I$. The element of $\mathrm{End}(I)$ will be a link invariant as long as $R_{U,V}$ satisfies the braid relations.

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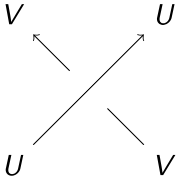

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Definition

The braid group B_n is defined as

$$\langle \sigma_1, \ldots, \sigma_n \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i-j| > 1 \rangle$$

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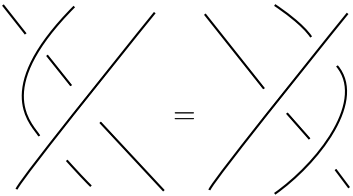

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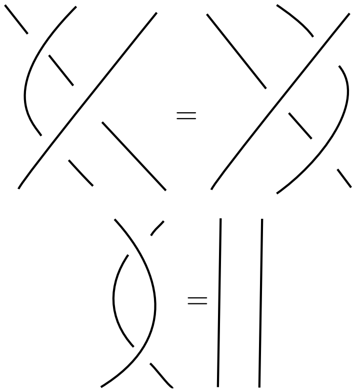

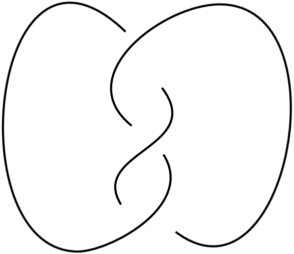

becomes

$$I \longrightarrow V^* \otimes V \otimes V \otimes V^* \xrightarrow{\mathbb{1} \otimes R^3 \otimes \mathbb{1}} V^* \otimes V \otimes V \otimes V^* \longrightarrow I$$

So how do we find pivot categories? One of the most general constructions is from the representation theory of Hopf algebras

Definition

A Hopf algebra A over a commutative ring k is a k-module equipped with k-module maps $m : A \otimes_k A \to A$, $\eta : k \to A$, $\Delta : A \to A \otimes_k A$, $\epsilon : A \to k$ and $S : A \to A$ satisfying various associativity and coassociativity axioms, and so that the algebra and coalgebra structures are compatible.

The category of A-modules form a pivot category, with $I = k$. If V, W are A-modules, then $V \otimes W$ is an A-module with $x \cdot (v \otimes w) = \Delta(x)(v \otimes w)$. Also, V^* is an A-module with $(x \cdot \phi)(v) = \phi(S(x)v)$.

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Note that the canonical k-module maps $V^* \otimes V \to k$ and $k \to V \otimes V^*$ commute with the action of A, but in general the canonical maps $V \otimes V^* \to k$ and $k \to V^* \otimes V$ do not. Also note that we could also define a dual by using S^{-1} instead of S. These two duals will be isomorphic as long as the automorphism S^2 is inner, ie. there exists invertible u such that

$$S^2(a) = uau^{-1}$$

for all $a \in A$. In this case, the map $\xi \mapsto u^{-1}\xi$ between the two kinds of dual commutes with the action of A. In this case, we also have $V^{**} \cong V$.

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The universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is a quotient of

$$T(\mathfrak{g}) = \oplus_n \mathfrak{g}^{\otimes n}$$

by the two-sided ideal generated by $x \otimes y - y \otimes x - [x, y]$.

This can be made into a Hopf algebra with

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \quad S(x) = -x, \quad \epsilon(x) = 0$$

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Unfortunately, if V is a finite-dimensional representation of a simple Lie algebra $\mathfrak{g}$, the only morphisms $V \otimes V \to V \otimes V$ are the identity and the map that interchanges the two tensor factors.

This means that the square of $R_{V,V}$ is the identity, so the same map is associated to a positive crossing or a negative crossing. So every knot is assigned the same invariant.

The problem is the cocommutativity of Δ: if we didn't have $\Delta^{op} = \Delta$, then the flip map wouldn't be a morphism of $U(\mathfrak{g})$-modules, so there might be a non-trivial map instead.

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The problem of turning commutative things into non-commutative things (or cocommutative things into non-cocommutative things) has been studied by algebraic geometers, quantum physicists and others.

Definition

A deformation of a Hopf algebra $(A, \eta, \mu, \epsilon, \Delta, S)$ over a field k is a topological Hopf algebra $(A_h, \eta_h, \mu_h, \epsilon_h, \Delta_h, S_h)$ over the ring $k[[h]]$ such that

- A_h is isomorphic to $A[[h]]$ as a $k[[h]]$-module
- $\mu_h \equiv \mu \mod h$, $\Delta_h \equiv \Delta \mod h$.

Theorem

If $\mathfrak{g}$ is semi-simple, every deformation of $U(\mathfrak{g})$ is isomorphic to $U(\mathfrak{g})[[h]]$ as an algebra. Moreover, if $\mathfrak{g}$ is simple, every cocommutative deformation of $U(\mathfrak{g})$ is trivial.

So we may restrict our attention to the 'usual' algebra structure, but the coalgebra structure must be changed.

The standard deformation $U_h(\mathfrak{sl}(2))$ is generated by $\{H, X^+, X^-\}$ with relations

$$[H, X^+] = 2X^+, \quad [H, X^-] = -2X^-, \quad [X^+, X^-] = \frac{e^{hH} - e^{hH}}{e^h - e^{-h}}.$$

The Hopf algebra structure is given by:

$$\Delta_h(H) = H \otimes 1 + 1 \otimes H, \quad \Delta_h(X^+) = X^+ \otimes e^{hH} + 1 \otimes X^+$$

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$$S_h(H) = -H, \quad S_h(X^+) = -X^+ e^{-hH}, \quad S_h(X^-) = -e^{hH} X^-$$

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Since $U_h(\mathfrak{sl}(2))$ is not cocommutative, the flip map is no longer a morphism of representations. However, there is still a relationship between Δ_h and Δ_h^{op}.

Theorem

The element $R_h \in U_h(\mathfrak{sl}(2)) \hat{\otimes} U_h(\mathfrak{sl}(2))$ defined by

$$R_h = \sum_{n=0}^{\infty} A_n(h) e^{\frac{1}{2}h(H \otimes H)} (X^+)^n \otimes (X^-)^n$$

is invertible and satisfies

$$\Delta_h^{op}(a) = R_h \Delta_h(a) R_h^{-1}$$

for all $a \in U_h(\mathfrak{sl}(2))$.

We call R_h the universal R-matrix.

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If $u_h = \mu(S \otimes \mathbb{1})(\sigma R_h)$, *then*

$$S^2(a) = u_h a u_h^{-1}$$

for all $a \in U_h(\mathfrak{sl}(2))$. *Also, if we let* $v_h = e^{-hH} u_h$, *then* v_h *is central and*

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Definition

For any Lie algebra $\mathfrak{g}$ associated to a symmetrisable Cartan matrix $(a_{ij})_{i,j=1,\ldots,n}$, $U_h(\mathfrak{g})$ is defined as the algebra over $\mathbb{C}[[h]]$ topologically generated by H_i, X_i^+, X_i^- for $i = 1, \ldots, n$ subject to

$$[H_i, H_j] = 0, \quad [H_i, X_j^{\pm}] = \pm a_{ij} X_j^{\pm}$$

$$X_i^+ X_j^- - X_j^- X_i^+ = \delta_{i,j} \frac{e^{d_i h H_i} - e^{-d_i h H_i}}{e^{d_i h} - e^{-d_i h}}$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \left[\begin{array}{c} 1 - a_{ij} \\ k \end{array} \right]_{e^{d_i h}} (X_i^{\pm})^k X_j^{\pm} (X_i^{\pm})^{1-a_{ij}-k} = 0.$$

Theorem

$U_h(\mathfrak{g})$ becomes a topological Hopf algebra with

$$\Delta_h(H_i) = H_i \otimes 1 + 1 \otimes H_i, \quad \Delta_h(X_i^+) = X_i^+ \otimes e^{d_i h H_i} + 1 \otimes X_i^+$$

$$\Delta_h(X_i^-) = X_i^- \otimes 1 + e^{-d_i h H_i} \otimes X_i^-$$

and

$$S_h(H_i) = -H_i, \quad S_h(X_i^+) = -X_i^+ e^{-d_i h H_i}, \quad S_h(X_i^-) = -e^{d_i h H_i} X_i^-$$

$$\epsilon_h(H_i) = \epsilon_h(X_i^\pm) = 0$$

There are also analogous elements R_h, u_h and v_h in this case.

The classification of finite-dimensional representations of $U_h(\mathfrak{g})$ is no harder than that of $U(\mathfrak{g})$.

Lemma

If V_h is a representation of $U_h(\mathfrak{g})$, then V_h/hV_h is a representation of $U(\mathfrak{g})$. If V is a representation of $U(\mathfrak{g})$, then $V[[h]]$ is a representation of $U_h(\mathfrak{g})$. If the representations are finite-dimensional, these operations are mutually inverse, and send indecomposable representations to indecomposable representations.

In other words, the category of $U(\mathfrak{g})$ representations is equivalent to the category of $U_h(\mathfrak{g})$ representations (as long as we resetrict to finite-dimensional and free representations).

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Let $\rho : U_h(\mathfrak{g}) \to \mathrm{End}(V)$ be a representation. As we remarked before, in general the map $V \otimes V^* \to k$ is not a morphism of $U_h(\mathfrak{g})$-modules. However, since we have an element $v_h^{-1} u_h$ with $S^2(a) = v_h^{-1} u_h a u_h^{-1} v_h$ we get $V \cong V^{**}$, so we could consider the map

$$V \otimes V^* \to V^{**} \otimes V^* \to k.$$

An element $f \in \mathrm{End}(V) \cong V \otimes V^*$ is mapped to $\mathrm{trace}(\rho(v_h^{-1} u_h) f)$.

Definition

The quantum trace tr_q is defined as

$$\mathrm{tr}_q(f) = \mathrm{trace}(\rho(v_h^{-1} u_h) f).$$

The quantum dimension $\dim_q(V)$ of V is defined as

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Quantised universal enveloping algebras usually appear in the literature in a simplified form:

Definition

The associative algebra $U_q(\mathfrak{sl}(2))$ over $\mathbb{C}(q)$ is generated by E, F, K, K^{-1} with relations

$$KK^{-1} = K^{-1}K = 1, \quad KE = q^2 EK, \quad KF = q^{-2} FK$$

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

This is related to $U_h(\mathfrak{sl}(2))$ by $E = X^+$, $F = X^-$, $q = e^h$, and $K = e^{hH}$. Another technical advantage to this is we can specialise q to any complex number, which we could not do with U_h.

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The Hopf algebra structure becomes

$$\Delta(E) = E \otimes K + 1 \otimes E, \quad \Delta(F) = F \otimes 1 + K^{-1} \otimes F$$

$$\Delta(K) = K \otimes K$$

$$S(E) = -EK^{-1}, \quad S(F) = -KF, \quad S(K) = K^{-1}$$

$$\epsilon(E) = \epsilon(F) = 0, \epsilon(K) = 1.$$

The simplified algebra no longer has all the structure of $U_h(\mathfrak{sl}(2))$, but we can still formally write

$$R = q^{H\otimes H/2}\sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}(q-q^{-1})^n}{[n]!}E^n\otimes F^n$$

$$u = q^{-H^2/2}\sum_{n=0}^{\infty}q^{3n(n-1)/2}\frac{(q-q^{-1})^n}{[n]!}F^nK^{-n}E^n$$

$$v = q^{-H^2/2}\sum_{n=0}^{\infty}q^{3n(n-1)/2}\frac{(q-q^{-1})^n}{[n]!}F^nK^{-n-1}E^n$$

Let $V = \langle v_-, v_+ \rangle$ be the 2-dimensional representation of $U_q(\mathfrak{sl}(2))$. This is defined by

$$E(v_-) = v_+, F(v_-) = 0, K(v_-) = q^{-1}v_-$$

$$E(v_+) = 0, F(v_+) = v_-, K(v_+) = qv_+.$$

In fact, the element $v^{-1}u$ acts the same as K here, so

$$\dim_q(V) = q + q^{-1}$$

Hence the invariant associated to the unknot is $q + q^{-1}$ in this case.

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Hence the invariant associated to the unknot is $q + q^{-1}$ in this case.

The R-matrix in this case acts on the basis $\langle v_+ \otimes v_+, v_+ \otimes v_-, v_- \otimes v_+, v_- \otimes v_- \rangle$ of $V \otimes V$ as

$$q^{\frac{3}{2}} \begin{pmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & 0 & q^{-2} & 0 \\ 0 & q^{-2} & q^{-1} - q^{-3} & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix}$$

and the element v acts as multiplication by $q^{-3/2}$. We therefore see that

$$q^{\frac{1}{2}} R - q^{-\frac{1}{2}} R^{-1} = (q - q^{-1}) \mathbb{1}_{V \otimes V}$$

and that a twist of a strand is multiplication by $q^{-3/2}$.

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We can turn this from a framed invariant to a genuine knot invariant by multiplying R by $q^{-3/2}$. Then a twist of a strand is just the identity, and

$$q^2 R - q^{-2} R^{-2} = (q - q^{-1}) \mathbb{1}_{V \otimes V}.$$

This is exactly the skein relation for the Jones polynomial.

A special feature of V is that $V \cong V^*$ via the map

$$f : V \to V^* : v_+ \mapsto v_-^*, \quad v_- \mapsto -q^{-1} v_+^*.$$

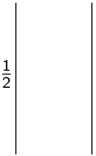
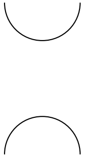
This means that upward-oriented strands are assigned the same colouring as downward-oriented strands (ie. the Jones polynomial is an invariant of unoriented knots). We can interpret cups and caps as

$$= (\mathbb{1} \otimes f^{-1}) \circ \pi_V : \mathbb{C}(q) \to V \otimes V$$

$$= \mathrm{ev} \circ (f \otimes \mathbb{1}) = \mathrm{tr}_q \circ (\mathbb{1} \otimes f) : V \otimes V \to \mathbb{C}(q)$$

Then the map associated to an unoriented circle is multiplication by $-q - q^{-1}$.

By explicit computation, we see that the R-matrix acts in the same way as

$q^{\frac{1}{2}}$  $+ q^{-\frac{1}{2}}$ 

which is exactly what the Kauffman bracket associates to a crossing.

Why we are interested in quantum invariants of knots:

- Very little is known about them: even in the case of the Jones polynomial it is unknown if the invariant detects the unknot.
- They may contain some topological information about 3- or 4-manifolds
- Quantum invariants generally seem to be fairly strong invariants
- Every quantum invariant can be categorified.

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Khovanov homology associates to every knot K a bigraded abelian group $Kh^{i,j}(K)$ so that

$$J(K) = \sum_{i,j} (-1)^i q^j \operatorname{rank} Kh^{i,j}(K).$$

The advantage of this is that $Kh(K)$ is a stronger invariant of knots, gives us access to methods in homological algebra, and also that $Kh(K)$ is functorial: for any cobordism $\Sigma : K \to K'$, there exists a map $Kh(K) \to Kh(K')$. Using this, it is possible to extract a lower-bound for the slice-genus of a knot using the Rasmussen s-invariant.

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Theorem (Webster, 2010)

For any simple Lie algebra $\mathfrak{g}$ and representation V, there is a homology theory $H_{\mathfrak{g},V}$ of bigraded vector spaces so that

$$\sum_{i,j} (-1)^i q^j \dim H^{i,j}_{\mathfrak{g},V}$$

is the quantum polynomial invariant of K associated to $(\mathfrak{g}, V)$.