An Introduction to Khovanov Homology

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GAndAlF / RADAGAST

20th March 2014

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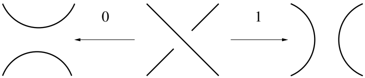

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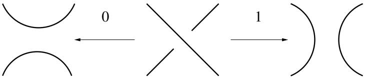

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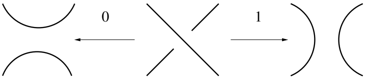

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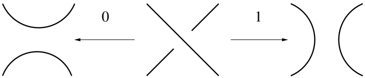

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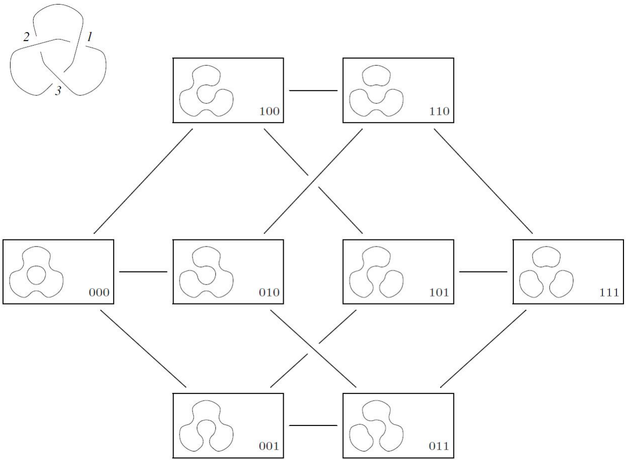
2. Apply a $(1+1)$-dimensional TQFT to this cube. This associates a vector space to every circle, and map between vector spaces to a cobordism between circles.

Example: The trefoil



Cobordisms on edges

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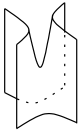

So every cobordism will contain this saddle-cobordism.

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$$m : V \otimes V \to V$$

$$v_+ \otimes v_+ \mapsto v_+$$

$$v_+ \otimes v_- \mapsto v_-$$

$$v_- \otimes v_+ \mapsto v_-$$

$$v_- \otimes v_- \mapsto 0$$

$$\Delta : V \to V \otimes V$$

$$v_+ \mapsto v_+ \otimes v_- + v_- \otimes v_+$$

$$v_- \mapsto v_- \otimes v_-$$

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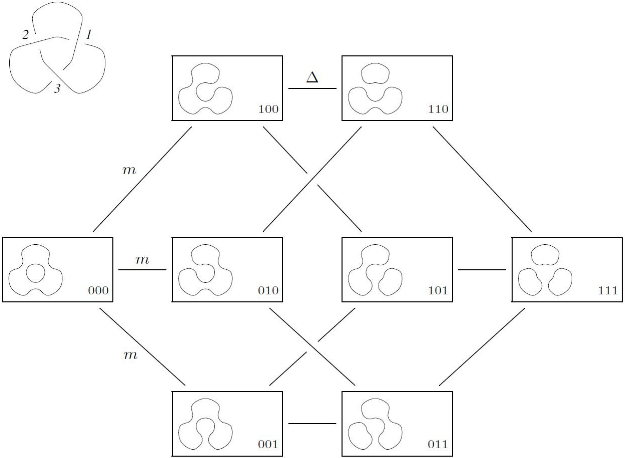

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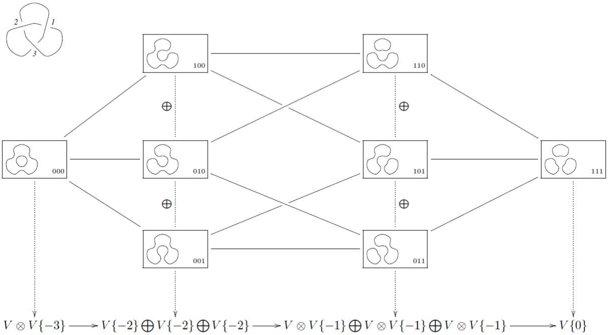

Since the quantum grading does not change passing over edges, this complex splits as a direct sum of complexes; one for each quantum grading. This gives a bi-graded chain complex.

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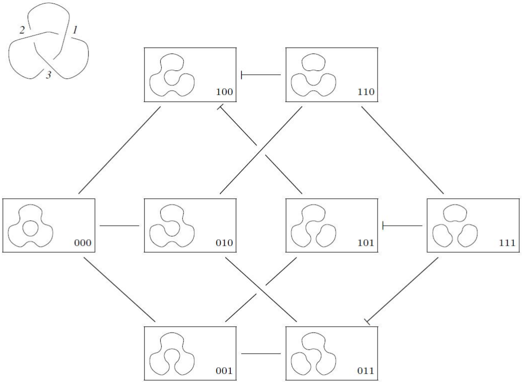

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$j \setminus i$	-3	-2	-1	0
-1				$\mathbb{Z}$
-3				$\mathbb{Z}$
-5		$\mathbb{Z}$		
-7		$\mathbb{Z}/2\mathbb{Z}$		
-9	$\mathbb{Z}$			

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To prove invariance of Khovanov homology, you take two diagrams D_1 and D_2 of the same knot (so differ by a sequence of Reidemeister moves), and produce a quasi-isomorphism on $C^{*,*}(D_1) \to C^{*,*}(D_2)$.

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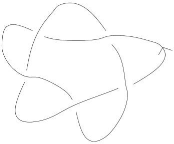
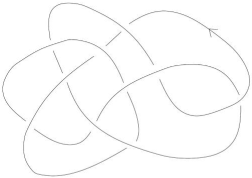
Significance of $Kh^{*,*}$

Khovanov homology is a stronger invariant than the Jones polynomial. To see this, consider the knots 5_1 and 10_{132}:





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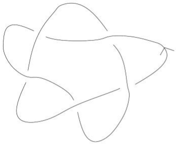

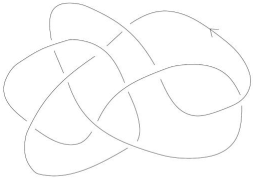

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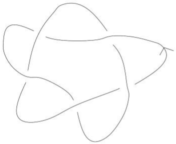

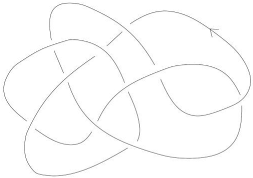

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j \ i	-5	-4	-3	-2	-1	0
-3						$\mathbb{Q}$
-5						$\mathbb{Q}$
-7				$\mathbb{Q}$		
-9						
-11		$\mathbb{Q}$	$\mathbb{Q}$			
-13						
-15	$\mathbb{Q}$					

j \ i	-7	-6	-5	-4	-3	-2	-1	0
-1							$\mathbb{Q}$	$\mathbb{Q}$
-3								$\mathbb{Q}$
-5					$\mathbb{Q}$	$\mathbb{Q} \oplus \mathbb{Q}$		
-7				$\mathbb{Q}$				
-9				$\mathbb{Q}$	$\mathbb{Q}$			
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