Geometry of Periodic Monopoles

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28th November 2013

GandAlF

[based on arXiv: 1212.4481, 1309.7013 (with Richard Ward) and 1311.6354]

Outline

- introduction
  - boundary conditions & topology
  - parameters & moduli
  - Nahm transform & spectral curve
- moduli space
  - identifying the Nahm/Hitchin moduli
  - calculating the metric
  - dynamics on the moduli space
  - the effect of the size parameter
- summary, references

Setup

- Periodic monopoles satisfy Bogomolny equations on $\mathbb{R}^2 \times S^1$ with $\zeta = \rho\, e^{i\theta} = x + iy$ and $z \sim z + 2\pi$

$$F = *D\Phi.$$

- Φ and A are valued in $\mathfrak{su}(2)$, with boundary conditions for $\rho \to \infty$ [Cherkis & Kapustin '01]

$$\Phi \to i\pi \log(\rho/C)\sigma_3 \qquad A_{x,y} \to 0 \qquad A_z \to i\chi/\pi\sigma_3.$$

- Topological (magnetic) charge k is the number of zeroes of Φ, also

$$k = \int_{T^2_\infty} \frac{\mathrm{tr}(F\Phi)}{4\pi|\Phi|}.$$

Parameters and moduli

- There are four parameters we are allowed to vary $\to$ moduli:
  - relative xy positions (determined by $K \in \mathbb{C}$),
  - z offset,
  - relative phase.
- Define the z-holonomy through

$$\partial_z V(\zeta) = (A_z + \mathrm{i}\Phi)V(\zeta) \quad \text{with} \quad V(0) = \mathbf{1}_2.$$

  This quantity is only sensitive to the K modulus.
- Other parameters have an infinite L^2 norm:
  - monopole size $|C|$,
  - orientation $\arg(C)$.
- In contrast to monopoles in $\mathbb{R}^3$, variations of the centre of mass have infinite L^2 norm, so can only consider the *relative* moduli space, $\mathcal{M}$. [Cherkis & Kapustin '02]

Nahm transform

- A generalisation of the ADHM construction of instantons.
- Relates dimensional reductions of the self-dual Yang-Mills equations on reciprocal 4-tori. [Braam & van Baal '89]
- Here, monopole space $\mathbb{R}^2 \times S^1$ suggests Hitchin equations on $\mathbb{R} \times S^1$ with $r \in \mathbb{R}$, $t \sim t+1$ and $s = r + \mathrm{i}t$,

$$2f_{rt} = \mathrm{i}[\phi, \phi^\dagger] \qquad\qquad D_{\bar{s}}\phi = 0.$$

- Spectral data defined from either side of the transform: [Cherkis & Kapustin '01]

$$\det(\mathrm{e}^{\beta s} - V(\zeta)) = 0 \qquad \equiv \qquad \det(\zeta - \phi(s)) = 0.$$

- This fixes gauge invariants of ϕ,

$$\mathrm{tr}(\phi) = 0 \qquad\qquad \det(\phi) = -C^2(2\cosh(2\pi s) - K).$$

Solving the Nahm/Hitchin equations

Charge 2 solutions are gauge-equivalent to [Harland & Ward '09]

$$\phi = \begin{pmatrix} 0 & \mu_+ \mathrm{e}^{\psi/2} \\ \mu_- \mathrm{e}^{-\psi/2} & 0 \end{pmatrix} \qquad a_{\bar{s}} = a\sigma_3 + \alpha\phi$$

with $\mu_+\mu_- = C^2(2\cosh(2\pi s) - K)$, and the Hitchin equations become

$$\Delta \mathrm{Re}(\psi) = 2\left(|\mu_+|^2 \mathrm{e}^{\mathrm{Re}\psi} - |\mu_-|^2 \mathrm{e}^{-\mathrm{Re}\psi}\right) \qquad a = -\tfrac{1}{4}\partial_{\bar{s}}\psi$$

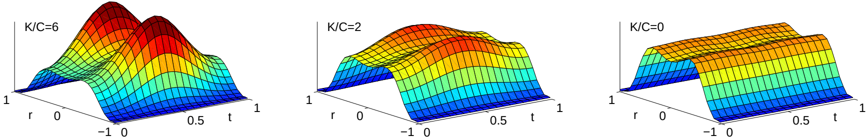
where α has been set to zero by symmetry ($\alpha \neq 0$ encodes z offset and phase). $\mathrm{Im}(\psi)$ chosen to ensure ϕ is periodic.

- $\det(\Phi)$ has two zeroes. Two distinct solutions: we can place both zeroes in μ_+ or one in each of $\mu_\pm$.
- For $|C| \gg 1$ and/or $|K| \gg 2$, solution is $\mathrm{Re}(\psi) = \log\left(|\mu_-|/|\mu_+|\right)$.

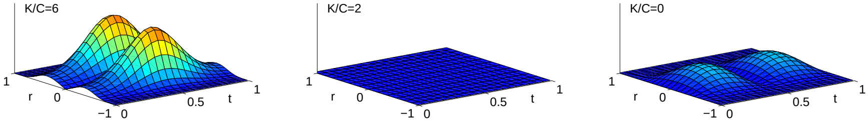
Lumps on Cylinder

For $K \in \mathbb{R}$, zeroes of $\det(\phi)$ correspond roughly to peaks in $|f_{rt}|$,

- 'zeroes together'



- 'zeroes apart'



Here $C = 1$. The size/period ratio now determined by $1/C$.

Moduli on the Cylinder

- For large $|K|$, we have 'lumps' at $2\pi s_{\pm} = \pm \cosh^{-1}(K/2)$.
- Treat these as delta-functions.
- Approximate fields away from $s_{\pm}$,

$$\phi = C\sqrt{2\cosh(2\pi s) - K}\sigma_3, \qquad a_r(r,t) = 0, \qquad a_t(0,t) = \mathrm{i}\theta\sigma_3.$$

- Peaks have a phase angle in the σ_1/σ_2 plane. Relative phase ω.
- The constant θ gives the t-holonomy at $r = 0$,

$$U = \mathcal{P}\exp\left(\int_0^1 a_t(0,t)dt\right) \qquad 2\cos(\theta) = \mathrm{tr}(U).$$

- The moduli ω and θ are defined up to a choice of sign.
- Take $\mathrm{Re}(K) > 0$ for simplicity.

Moduli Space Approximation

- The low energy dynamics of monopoles can be understood as geodesic motion on the moduli space, $\mathcal{M}$.
- A tangent vector in $\mathcal{M}$ to the solution $(\phi, a_{\bar{s}})$ is $V_1 = (\delta_1\phi, \delta_1 a_{\bar{s}})$, which must be orthogonal to the gauge orbits and satisfy Hitchin equations

$$4D_{\bar{s}}\,(\delta_1 a_s) \;=\; [\phi, \delta_1\phi^\dagger] \qquad\qquad D_{\bar{s}}\,(\delta_1\phi) \;=\; [\phi, \delta_1 a_{\bar{s}}].$$

- Three other solutions $V_i = (\delta_i\phi, \delta_i a_{\bar{s}})$,

$$V_2 \;=\; \mathrm{i}V_1, \qquad V_3 \;=\; (2\delta_1 a_s, \tfrac{1}{2}\delta_1\phi^\dagger), \qquad V_4 \;=\; \mathrm{i}V_3$$

satisfying

$$\langle V_i, V_j\rangle \;=\; \frac{1}{2}\,\mathrm{Re}\int \mathrm{tr}\left((\delta_i\phi)(\delta_j\phi)^\dagger + 4(\delta_i a_{\bar{s}})(\delta_j a_{\bar{s}})^\dagger\right) dr\,dt \;=\; p^2\delta_{ij}$$

Metric on the Moduli Space

A perturbation satisfying all the above is $V_1 = \epsilon\,(\frac{1}{2}h\sigma_3, 0)$ with

$$h \;=\; C(-\det(\phi))^{-1/2} \;=\; (2\cosh(2\pi s) - K)^{-1/2}.$$

Change coordinates such that $V_i \equiv (\delta_i \mathrm{Re}(K), \delta_i \mathrm{Im}(K), \delta_i\theta, \delta_i\omega)$, then

$$Q \;=\; \frac{1}{p}\begin{pmatrix} \delta_1 K_r & \delta_2 K_r & \delta_3 K_r & \delta_4 K_r \\ \delta_1 K_i & \delta_2 K_i & \delta_3 K_i & \delta_4 K_i \\ \delta_1\theta & \delta_2\theta & \delta_3\theta & \delta_4\theta \\ \delta_1\omega & \delta_2\omega & \delta_3\omega & \delta_4\omega \end{pmatrix}$$

and

$$g \;=\; (QQ^{\intercal})^{-1}\,.$$

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$$Q \;=\; \frac{1}{p}\begin{pmatrix} \delta_1 K_r & & & \\ & \delta_2 K_i & & \\ & & \delta_3\theta & \delta_4\theta \\ & & \delta_3\omega & \delta_4\omega \end{pmatrix}$$

and

$$g \;=\; (QQ^{\mathsf{T}})^{-1}\,.$$

Metric on the Moduli Space

- Recall the solution away from $s_\pm$ had $a_t(0,t) = \mathrm{i}\theta\sigma_3$

  $\Rightarrow$ determine $\delta\theta$ directly from $\delta a_{\bar{s}}$.

- A change $\delta a_{\bar{s}}$ affects the progpagation of f_- to s_+ (but leaves $f_\pm$ unchanged), with γ a path between $s_\pm$ and $\omega = \mathrm{tr}(f_+\hat{f}_-)$.

  $$\partial_\gamma f_- + [a_\gamma + \delta a_\gamma, f_-] \;=\; 0 \qquad \Rightarrow \qquad \delta\omega \;=\; 4\int_\gamma \delta\mathbf{a}\cdot d\boldsymbol{\ell}$$

  This quantity is path-dependent due to twisting of θ.

- The constant p is given by

  $$p^2 \;=\; \frac{\epsilon^2}{4}\int |h|\, dr\, dt \;=\; \frac{\epsilon^2}{4}\int \frac{dr\, dt}{|2\cosh(2\pi s) - K|} \;\approx\; \epsilon^2\frac{\log(4|K|)}{4\pi|K|}.$$

Metric on the Moduli Space

Performing integrals for $K = |K|e^{2\pi i\eta}$ and $|K| \gg 2$, get metric

$$ds^2 = \frac{\log(4|K|)}{4\pi|K|}\left(C^2|dK|^2 + 4|K|d\theta^2\right) + \frac{\pi}{\log(4|K|)}\left(d\omega - 4\eta d\theta\right)^2 .$$

- Agreement with the metric obtained from the monopole side of the Nahm transform using physical arguments. [Cherkis & Kapustin '03]
- Identify
  - $\theta =$ twice the z-offset,
  - $\omega =$ twice the relative phase between monopoles.

Surfaces of constant θ, ω

Recall the two different solutions to the Nahm/Hitchin equations:

- 'zeroes together' has $\omega = 0$,
- 'zeroes apart' has $\omega = \pi$.

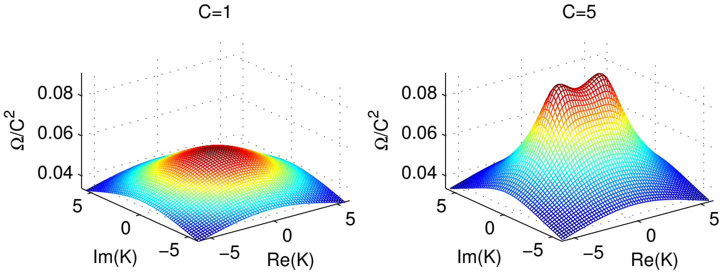
In both cases we can have $\theta = 0$ or $\theta = \pi$. These surfaces are isometric.

- For 'zeroes together' simply corresponds to choice of $\text{Im}(\psi) = \pm 2\pi t$.
  Analogue to Atiyah-Hitchin cone (next slide).

- For 'zeroes apart' the surfaces are connected on $K \in [-2, 2]$.
  "Double trumpet", 3 dimensional scattering.
  Double scattering: outgoing chains shifted along z by π and rotated.

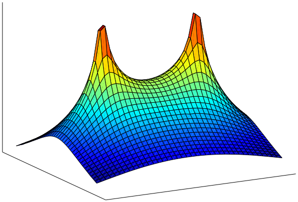
Surfaces of constant θ, ω

The metric on surfaces with $\omega = 0$ can be computed numerically for all K,



For $|C| \to \infty$, the conformal factor is

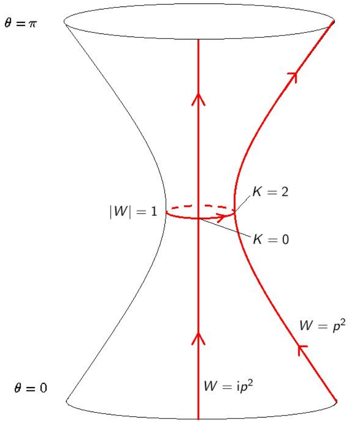
$$\Omega = \frac{1}{4} \int \frac{dr\, dt}{|2\cosh(2\pi s) - K|}.$$



Agrees with 'spectral approximation'.
Scattering in xy plane.

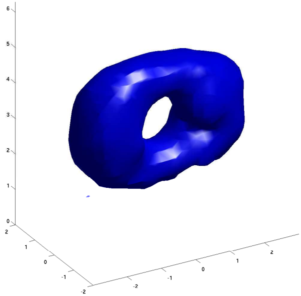
Surfaces of constant θ, ω

For $\omega = \pi$, symmetries fix geodesic submanifolds with $K \in \mathbb{R}$ and $K \in \mathrm{i}\mathbb{R}$. In terms of the coordinate $K = W + W^{-1}$ there are three distinct geodesics (here $p > 0$),



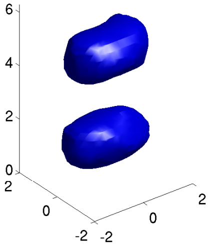
Geodesics on the Double Trumpet - radial

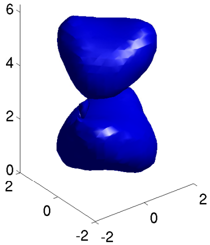
- The trajectory with $W \in \mathbb{R}$ and $W > 0$ describes monopoles incoming along the x-axis at $z = 0$ and scattering along the z axis. They then scatter with those in adjacent periods and depart parallel to the x-axis and shifted by π along z.
- For $W \in i\mathbb{R}$ monopoles are incoming along $x = y$. They scatter first along z, and again along $x = -y$. We thus have a shift and a rotaion by $\pi/2$.



Geodesics on the Double Trumpet - $|W| = 1$

- The $|W| = 1$ geodesic describes a chain of equally separated monopoles (at $z = \pm\pi/2$) whose shape oscillates in size. For $C = 1$ and $W = 1$, i:



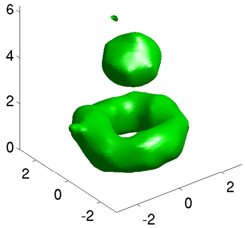


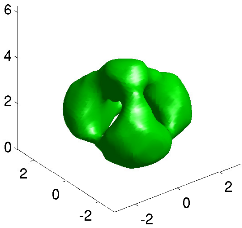
- $W = \pm 1$ encode a charge one chain taken over two periods.

[Harland & Ward '09]

Higher Charge Chains

The above can all be generalised to higher charges, e.g. for charge 3,





Summary & Outlook

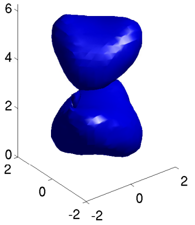
- Reproduce the metric describing the dynamics of charge 2 periodic monopoles from Nahm data, identifying asymptotic moduli.
- Investigate novel examples of monopole scattering.
- Small size C: describe as small well separated monopoles, large C: 'spectral approximation'. The intermediate region is also interesting.
- Other geodesic submanifolds? e.g. expect a symmetry fixing $\theta = \pi/2$.
- Can we do the same for monopole walls? (work in progress)

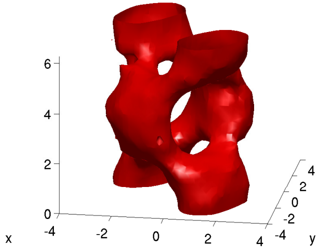
END

- Braam & van Baal, *Nahm's Transformation for Instantons*, Commun. Math. Phys. **122** (1989) 267
- Cherkis & Kapustin, *Nahm transform for periodic monopoles and N=2 super Yang-Mills theory*, hep-th/0006050
- ——— , *Hyper-Kähler metrics from periodic monopoles*, hep-th/0109141
- ——— , *Periodic monopoles with singularities and N=2 super QCD*, hep-th/0011081
- Harland & Ward, *Dynamics of Periodic Monopoles*, 0901.4428
- Maldonado, *Periodic monopoles from spectral curves*, 1212.4481
- ——— , *Higher charge periodic monopoles*, 1311.6354
- Maldonado & Ward, *Geometry of Periodic Monopoles*, 1309.7013
- Ward, *Periodic monopoles*, hep-th/0505254

Geodesics on the Double Trumpet - $|W| = 1$

For larger $|C|$ the picture is more complicated. Here $C = 1, 2$ with $W = \mathrm{i}$.





Geodesics on the Double Trumpet - $|W| = 1$

As $|C|$ is increased the energy density distribution approaches that of the 'spectral approximation' (i.e. when approximate monopole fields are read off from the spectral curve). Here, for $W = \mathrm{i}$ ($K = 0$) and $C = 4, 6$.