Regularity of Solutions to Obstacle Problems

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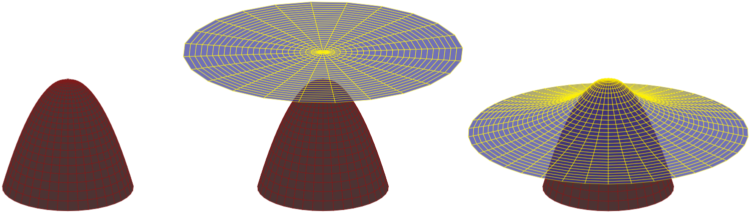
Obstacle Problems



?

Ingredients

An obstacle problem is the problem of modelling a membrane suspended over an obstacle and fastened to a wire on the boundary of a domain. We need

1. a domain $\Omega \subset \mathbb{R}^n$,
2. an obstacle $\psi : \overline{\Omega} \to \mathbb{R}$,
3. the wire $\phi : \partial\Omega \to \mathbb{R}$ and
4. an energy functional as an infinitesimal indicator for the behaviour of the membrane.

But: The Energy functional must be global!

Naive Mathematical Formulation

Assume that $\Omega \subset \mathbb{R}^n$ is an open domain with Lipschitz boundary $\partial\Omega \in C^{0,1}$, $\phi, \psi \in C^{0,1}(\overline{\Omega})$ and assume that

$$\mathscr{F}(u) = \int_\Omega f(x, u(x), Du(x))\, dx$$

is an energy functional on $C^1(\Omega)$ for some $f \in C^0(\Omega \times \mathbb{R} \times \mathbb{R}^n)$. Find a function $u \in C^1(\Omega) \cap C^0(\overline{\Omega})$ that minimizes $\mathscr{F}$ in the class

$$K := \{v \in C^1(\Omega) \cap C^0(\overline{\Omega}) \mid v \geq \psi, v|_{\partial\Omega} = \phi\}\, .$$

Example

Let $\Omega = B_3(0) \subset \mathbb{R}^n$, $\psi(x) = 1 - |x|^2$. Find $u : \overline{\Omega} \to \mathbb{R}$ such that

1. $u = 0$ on $\partial B_3(0)$,
2. $u \geq \psi$ everywhere in $B_3(0)$ and
3. $\mathscr{F}(u) = \inf\{\mathscr{F}(v) \mid v \in K\}$ where

$$K := \{v \in C^1(\Omega) \cap C^0(\overline{\Omega}) \mid v|_{\partial\Omega} = 0, v \geq \psi\}$$

and

$$\mathscr{F}(v) = \int_\Omega |Dv|^2 \, dx \, .$$

Example (cont.)

We would like to apply the direct method of the calculus of variations to solve this prolem:

1. Take a sequence (v_n) such that $v_n \in K$ and $\lim_{n\to\infty} \mathscr{F}(v_n) = \inf_{v\in K} \mathscr{F}(v)$,
2. extract a convergent subsequence such that $v_{n_k} \to v$,
3. show that $\mathscr{F}(v) \leq \liminf_{k\to\infty} \mathscr{F}(v_{n_k})$.

PROBLEMS:

- Compactness.
- Lower Semi-continuity.

Intermezzo: Sobolev Spaces 101

The Sobolev Space $W^{k,p}(\Omega)$ is the completion of

$$\{u \in C^k(\Omega) \mid D^\alpha u \in L^p(\Omega) \ \ \forall \ |\alpha| \leq k\}$$

with respect to the norm

$$||u||_{k,p} := \left(\sum_{|\alpha| \leq k} ||D^\alpha u||_{p,\Omega}^p \right)^{\frac{1}{p}} .$$

Alternatively, $u \in W^{1,p}(\Omega)$ are the functions $u \in L^p(\Omega)$ such that there are $u_1, ..., u_n \in L^p(\Omega)$ such that

$$\int_\Omega u \, D_i \eta \, dx = - \int u_i \eta \, dx \qquad \forall \ \eta \in C_c^\infty(\Omega)$$

and $u \in W^{k,p}(\Omega)$ iff $u, u_1, ..., u_n \in W^{k-1,p}(\Omega)$ for higher k.

Intermezzo: Sobolev Spaces 101 (cont.)

If $\partial\Omega \in C^{0,1}$, then

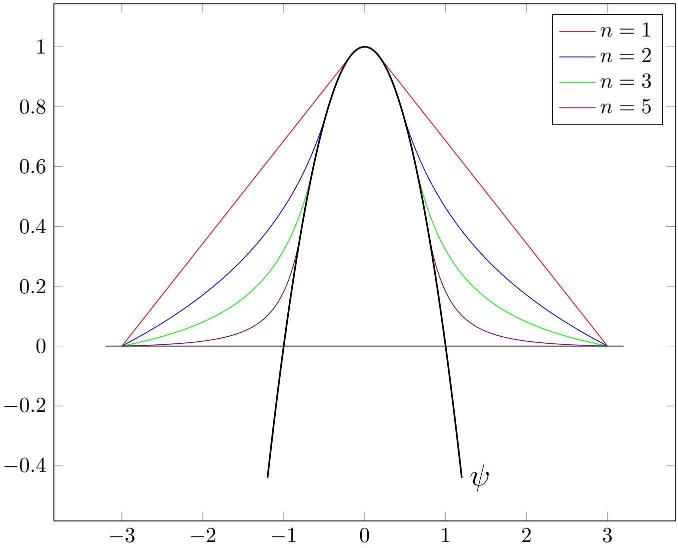
- for $p < \infty$ bounded subsets of $W^{1,p}(\Omega)$ are weakly compact,
- if $p < n$ then $W^{1,p}(\Omega) \to L^q(\Omega)$ for all $q \leq p^* = \frac{np}{n-p} > p$ and the embedding is compact for $q < p^*$,
- if $p > n$ then $W^{1,p}(\Omega) \to C^{0,\alpha^*}(\overline{\Omega})$ for all $\alpha^* = 1 - \frac{n}{p}$.

By induction we proceed to higher order spaces. **Reminder:**

$$|u|_{0,\alpha} = |u|_0 + \sup_{x \neq y \in \overline{\Omega}} \frac{|u(x) - u(y)|}{|x-y|^\alpha}$$

and if $\partial\Omega \in C^{0,1}$ then $C^{k,\alpha}(\overline{\Omega}) \to C^{k,\beta}(\overline{\Omega})$ compactly and with closed image for $\alpha > \beta$.
Reminder 2: $f_n \rightharpoonup f$ converges weakly iff $\langle \lambda, f_n \rangle \to \langle \lambda, f \rangle$ for all continuous linear functionals λ.

Regularity for Obstacle Problems?



Variational Inequalities

$$\begin{aligned} 0 &\leq \frac{d}{dt}\bigg|_{t=0} \mathscr{F}(tu + (1-t)v) \\ &= \int_\Omega \frac{d}{dt}\bigg|_{t=0} f(x, u + t(v-u), Du + t\, D(v-u))\, dx \\ &= \int_\Omega (\partial_z f) \cdot (v-u) + (\partial_{\xi_i} f) \cdot D_i(v-u)\, dx \\ &= \langle Au + Hu, v-u \rangle \end{aligned}$$

with

$$Au = -D_i(\partial_{\xi_i} f(x, u, Du)), \qquad Hu = \partial_z f(x, u, Du)$$

as formal expressions in $(W_0^{1,p}(\Omega))' = W^{-1,p'}(\Omega)$.

Elliptic Differential Operators

A quasi-linear differential operator in divergence form is an operator $Q = A + H : W^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ with

$$Au = -\mathrm{div}(a(x, u, Du)), \quad Hu = h(x, u, Du)$$

acting by multiplication and integration (by parts) for a Lipschitz vector field a and function h. Q is called elliptic if the matrix

$$a^{ij}(x, z, \xi) = \frac{\partial a^i}{\partial \xi_j}(x, z, \xi)$$

is positive definite. For a functional $\mathscr{F}$ this corresponds to convexity in the gradient since $a^{ij} = \partial_{\xi_i} \partial_{\xi_j} f$.

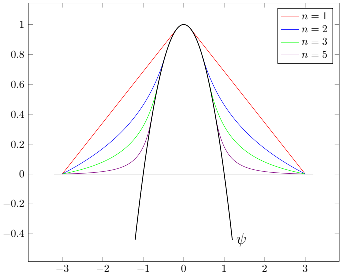
Regularity for obstacle problems? (cont.)

If we introduce the set of coincidence

$$I := \{x \in \overline{\Omega} \mid u(x) = \psi(x)\}$$

we can easily see that

1. $Au + Hu = 0$ in $\Omega \setminus I$,
2. Obviously $Au + Hu = A\psi + H\psi$ almost everywhere in I and
3. $Au + Hu \geq 0$.



Main Theorem

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega \in C^{2,\alpha}$,
$a^i, h \in C^{0,1}(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$,
$A + H$ an elliptic operator and $\phi, \psi \in C^{1,1}(\overline{\Omega})$. Set

$$K := \{v \in W^{1,\infty}(\Omega) \mid v \geq \psi, v|_{\partial\Omega} = \phi\}$$

and assume $u \in K$ solves

$$\langle Au + Hu, v - u\rangle \geq 0 \qquad \forall\ v \in K\ .$$

Then $u \in W^{2,p}(\Omega)$ for all $p < \infty$ and hence $u \in C^{1,\alpha}(\overline{\Omega})$ for all $\alpha < 1$.

Generalizations

1. The theorem remains true for domains with sharp edges away from the non-smooth parts. The solution u is always $C^{1,\alpha}$ in the interior and close to smooth parts of the boundary. For a proof we have to deal with a relatively complicated localization process.
2. The norm of u can be controlled in known local quantities.
3. The theorem remains true for $\alpha = 1$ if $\partial\Omega \in C^{3,\alpha}$, $a \in C^{2,1}$ and $\phi \in C^{2,1}(\overline{\Omega})$ (or $\phi > \psi$ on $\partial\Omega$). This is much harder to show than the case $\alpha < 1$ and the proof is highly technical. The function

$$f(x) := \sum_{n=0}^{\infty} 2^{-n} s(2^n x), \quad s(x) = \min_{m \in \mathbb{Z}} |x - m|$$

lies in $C^{0,\alpha}[0,1]$ for all $\alpha < 1$, but not in $BV(0,1) \subset C^{0,1}[0,1]$.

Proof. Step I: Simplification

We can assume that h is bounded and a^{ij} is uniformly positive definite by using cut-off functions. Choose M such that $|u|_\infty + |Du|_\infty^2 + 1 \leq M$ and functions

$$\theta(t) = \begin{cases} t & |t| \leq M \\ M+1 & |t| \geq M+1 \end{cases}, \qquad |\theta| \leq M+1$$

$$w(t) = \begin{cases} 1 & 0 \leq t \leq 2M \\ 0 & t \geq 3M \end{cases}, \qquad 0 \leq w, w' \leq 1$$

$$g(t) = \begin{cases} 0 & 0 \leq t \leq M \\ 1 & t \geq 2M \end{cases}, \qquad 0 \leq g, g' \leq 1$$

Proof. Step I: Simplification (cont.)

Now set

$$\begin{aligned} \tilde{a}^i(x,z,\xi) &= a^i(x,\theta(z),\xi)\cdot w\left(|\xi|^2\right) + k\cdot g\left(|\xi|^2\right)\cdot \xi^i \\ \tilde{h}(x,z,\xi) &= h(x,\theta(z),\xi)\cdot w\left(|\xi|^2\right) \end{aligned}$$

for some $k > 0$. The operators are nicer now without having changed in a neighbourhood of the solution. Most importantly, for large enough γ the operator $A + H + \gamma$ is coercive, i.e.

$$\langle (A+H+\gamma)u - (A+H+\gamma)v, u-v\rangle > 0 \qquad \forall\; u \neq v \in W^{1,2}(\Omega)\,,$$

so for fixed $u_\infty \in K$ the variational inequality

$$\langle Au + Hu + \gamma(u-u_\infty), v-u\rangle \geq 0 \qquad \forall\; v \in K$$

has only one solution.

Proof. Step II: Penalization (first idea)

$$Au + Hu = \begin{cases} 0 & u > \psi \\ A\psi + H\psi & u = \psi \end{cases} = \Theta(u - \psi) \cdot (A\psi + H\psi) \geq 0$$

with $\Theta(z) = \begin{cases} 1 & z > 0 \\ 0 & z = 0 \end{cases}$. Problem: This equation is highly irregular.

$$\Theta_\epsilon(z) := \begin{cases} 1 & z \geq \epsilon \\ \frac{z}{\epsilon} & 0 < z < \epsilon \\ 0 & z \leq 0 \end{cases}$$

and

$$Au_\epsilon + Hu_\epsilon = \Theta_\epsilon(u_\epsilon - \psi) \cdot \max\{A\psi + H\psi, 0\}, \quad u_\epsilon|_{\partial\Omega} = \phi$$

with the maximum to get $u_\epsilon \geq u$ for free.

Proof. Step II: Penalization (first idea, cont.)

Then we need existence of solutions $u_\epsilon \in H^{2,p}(\Omega)$ (for large p), $||u_\epsilon||_{2,p} \leq C$ for $C \in \mathbb{R}$ independently of ϵ to deduce that up to a subsequence

$$u_\epsilon \rightharpoonup v \in H^{2,p}(\Omega)$$

by weak compactness, thus $u_\epsilon \to v$ strongly in $H^{1,2}(\Omega)$ by compact embeddings. For our operators then $Au_\epsilon + Hu_\epsilon \rightharpoonup Av + Hv$ weakly in $H^{-1,2}(\Omega)$, hence

$$\langle Au_\epsilon + Hu_\epsilon, w - u_\epsilon \rangle \to \langle Av + Hv, w - v \rangle \, .$$

Here we get stuck in the general case. For $A + H = -\Delta$ and $\psi < \phi$ on $\partial\Omega$ one can show that the left term is asymptotically non-negative, thus - as the problem has only one solution - $u = v \in H^{2,p}(\Omega) \to C^{1,\alpha}(\overline{\Omega})$, but in our case...

Proof. Step II: Penalization

... we need a different penalization. Let $\beta(z) = \begin{cases} -z^2 & z \leq 0 \\ 0 & z \geq 0 \end{cases}$ and $\mu > 1$. Assume that u_∞ solves the obstacle problem and set

$$\begin{cases} Au_\mu + Hu_\mu + \gamma(u_\mu - u_\infty) + \mu\beta(u_\mu - \psi) & = 0 \text{ in } \Omega \\ u_\mu & = \phi \text{ on } \partial\Omega \end{cases} .$$

Then by design for $v \in K$ we have

$$\begin{aligned} \langle Au_\mu \quad + \quad & Hu_\mu + \gamma(u_\mu - u_\infty), v - u_\mu\rangle = -\mu\langle\beta(u_\mu - \psi), v - u_\mu\rangle \\ = \quad & \mu \int_{u_\mu < \psi} (u_\mu - \psi)^2 \cdot (v - u_\mu)\, dx \\ \geq \quad & 0\,. \end{aligned}$$

Proof. Step II: Penalization (cont.)

Proof that $\lim_{\mu\to\infty} u_\mu \in K$. Set

$$Lv := -a^{ij}(x,u,Du)D_iD_jv - (\partial_z a^i)(x,u,Du)D_iv + \gamma v$$

Then

$$Lu = (\partial_i a^i)(x,u,Du) - h(x,u,Du) + \gamma u_\infty - \mu\beta(u-\psi)$$

and by the maximum principle, if $u < \psi$ then

$$\begin{aligned} 0 &\leq L(\psi - u) \\ &= C(a,h,\psi) - \mu(u-\psi)^2 \,. \end{aligned}$$

The remaining estimates can be derived similarly. Existence follows via Schauder's fixed point theorem and suitable a priori estimates in $W^{2,p}$.

The case $\alpha = 1$

The case $\alpha = 1$ is more difficult for several reasons:

- From $||Au||_\infty$ we can only deduce bounds on $||u||_{2,p}$, not $||u||_{2,\infty}$, so we have to work them out directly,
- $H^{2,\infty}$ is neither reflexive nor separable, nor do smooth functions lie dense, so we always have to work with $H^{2,p}$ and consider show independence while $p \to \infty$,
- While for $p < \infty$ we only get $H^{1,p} \to C^{0,1-\frac{n}{p}}$ in the case $p = \infty$ we have the converse direction by Rademacher's theorem:

$$H^{2,\infty} = C^{1,1}\,.$$

This space is much more powerful than other Hölder spaces, so additional complications are to be expected.

Thank You!