The Linked Twist Map Approach to Fluid Mixing

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Joint work with Steve Wiggins and Julio Ottino
### Dynamical systems and fluids

#### Fluids
- incompressible fluid
- Poincaré section
- region of unmixed (stationary) fluid
- islands forming barriers to mixing
- “chaotic”

#### Dynamical systems
- invertible, area-preserving dynamical system
- Discrete time map, $f : M \rightarrow M$
- invariant (periodic) set $f(A) = A$
- KAM theory
- existence of a horseshoe
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Horseshoes in fluids

from [Chien, Rising, Ottino, JFM 170 355-77 (1986)]
Dynamical systems, ergodic theory and fluids

**Topological**
- topological space
- behaviour of individual trajectories
- dense orbit

**Measure-theoretic**
- measure space
- need an invariant measure — Lebesgue measure $\mu$
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Ergodicity

Definition

\[ f \text{ is ergodic if } \mu(A) = 0 \text{ or } 1 \text{ whenever } f(A) = A. \]

Birkhoff ergodic thm \implies \text{“time averages = spatial averages”}

Central notion is \textit{indecomposability}

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Mixing

Definition

\( f \) is **mixing** if

\[
\lim_{n \to \infty} \frac{\mu(f^n(A) \cap B)}{\mu(B)} = \mu(A)
\]

Intuitive definition is that upon iteration, sets become asymptotically independent of each other.
The Bernoulli property

Bernoulli means “statistically indistinguishable from coin tosses”

The Ergodic Hierarchy
Bernoulli $\implies$ Mixing $\implies$ Ergodicity

... plus lots more!
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Bernoulli $\Rightarrow$ Mixing $\Rightarrow$ Ergodicity

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Linked Twist Maps on the torus

Define annuli $P$ and $Q$ on the torus $\mathbb{T}^2$ which intersect in region $S$. 
Linked Twist Maps on the Torus

The horizontal annulus $P$ has a horizontal twist map....
Linked Twist Maps on the torus

\[ F(x, y) = (x + f(y), y) \]

for points in \( P \)

\( f(y) \) could be linear...
Linked Twist Maps on the Torus

...or not, but must be monotonic
After $F$, apply a vertical twist

$$G(x, y) = (x, y + g(x))$$

to points in $Q$. Again $g$ must be monotonic.

The combined map $H(x, y) = G \circ F$ is the linked twist map.
Mixing properties of LTM on the torus

Domain is two intersecting annuli with two distinct regions of intersection.
Linked Twist Maps on the plane

The action of a twist map is to take a line...
Linked Twist Maps on the plane

... and twist it around the annulus.
An egg-beater can be viewed as either linked twist map on the plane, or on the torus. from [Ottino, J, Sci. Am., 260, 56–67 (1989)]
The Blinking Vortex

Streamlines in the first half of the advection cycle

Streamlines in the second half of the advection cycle
The Blinking Vortex

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Microfluidics — patterned walls

The Partitioned Pipe Mixer

The Rotated Arc Mixer

Microfluidics — electroosmotic flow

DNA Hybridization

Fill hybridization chamber with "target" solution of mRNA
DNA Hybridization

Attach "probes" (DNA strands) to silicon surface in hybridization chamber
DNA Hybridization

Introduce syringes to form a source-sink pair
DNA Hybridization

Streamlines from source to sink
DNA Hybridization

Second set of streamlines
DNA Hybridization

-linked twist maps

Future Directions

Examples of Mixers that fit the framework

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LTM Approach to Fluid Mixing
DNA Hybridization

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DNA Hybridization

\[ f(y) = ry(1 - y) \]

\[ g(x) = rx(1 - x) \]
DNA Hybridization

Short pumping time $\rightarrow$ Long pumping time

from [J.M. Hertzsch, R. Sturman & S. Wiggins, 2006]
DNA Hybridization
DNA Hybridization

Off-centre sources and sinks
DNA Hybridization

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Future Directions

- Monotonicity
- Transversality
- Speed of mixing
- Diffusion
Duct flows

- Schematic view of a duct flow with concatenated mixing elements
- Red and blue blobs of fluid mix well under a small number of applications
- Changing only the position of the centres of rotation can have a marked effect on the quality of mixing
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