Preconditioning Saddle-Point Systems arising in a Stochastic Mixed Finite Element Problem

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Joint work with:

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Related work with:

- Elisabeth Ullmann, Oliver Ernst (U. of Freiberg), David Silvester (U. of Manchester)
Outline

- Mixed SFEM on: $A(x, \omega)^{-1} u(x, \omega) - \nabla p(x, \omega) = 0$, $-\nabla \cdot u(x, \omega) = f(x)$
Outline

• Mixed SFEM on: $\mathcal{A}(x, \omega)^{-1} u(x, \omega) - \nabla p(x, \omega) = 0, \ -\nabla \cdot u(x, \omega) = f(x)$

• Solving stochastic saddle-point systems

$$\begin{pmatrix} \tilde{A} & \tilde{B}^T \\ \tilde{B} & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} g \\ f \end{pmatrix}$$

- Weak problem in $H(div; D) \otimes L^2(\Gamma)$ and $L^2(D) \otimes L^2(\Gamma)$
- Inf-sup stability
- Block-diagonal preconditioner
- Multigrid implementation
- Eigenvalue bounds
- Numerical results
Let $A(x, \omega) : D \times \Omega \rightarrow \mathbb{R}$ be a random field.

For $x \in D$, $A(\omega)$ is a random variable with finite variance; for $\omega \in \Omega$, $A(x) \in L^\infty(D)$.

We seek random fields $p(x, \omega)$, $u(x, \omega)$ such that $P$-almost everywhere $\omega \in \Omega$:

$$A(x, \omega)^{-1} u(x, \omega) - \nabla p(x, \omega) = 0,$$

$$\nabla \cdot u(x, \omega) = -f(x) \quad x \text{ in } D,$$

$$p(x, \omega) = g(x) \quad x \text{ on } \partial D_D,$$

$$u(x, \omega) \cdot n = 0 \quad x \text{ on } \partial D_N.$$
Finite Noise Assumption

We assume that the input random field can be represented by a finite number of random variables.
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Here, we consider a truncated Karhunen-Loève expansion:

\[
A^{-1}(\mathbf{x}, \omega) \approx A^{-1}_M(\mathbf{x}, \mathbf{\xi}) = \mu(\mathbf{x}) + \sum_{i=1}^{M} \sqrt{\lambda_i} c_i(\mathbf{x}) \xi_i,
\]

where \( \mathbf{\xi} = \{\xi_1(\omega), \ldots, \xi_M(\omega)\} \) are independent random variables and \( \{\lambda_i, c_i(\mathbf{x})\} \) are the eigenpairs of the correlation function \( C_{A^{-1}}(\mathbf{x}_1, \mathbf{x}_2) \).
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Note that:

\[
\int_D Var\left(A^{-1} - A_M^{-1}\right) = \int_D \sigma^2(\mathbf{x}) dD - \sum_{i=1}^{M} \lambda_i
\]
Example

Consider the covariance function

\[ C(x, z) = \sigma^2 \exp \left( -\frac{|x_1 - z_1|}{b_1} - \frac{|x_2 - z_2|}{b_2} \right), \]

\[ D = [0, 1] \times [0, 1] \] and Gaussian random variables.
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\[ D = [0, 1] \times [0, 1] \] and Gaussian random variables.

If \( b_1 = 1 = b_2 \) then 10 term KL expansion, yields relative error of 0.01

Two realisations of the resulting random field:
If \( b_1 = \frac{1}{4} = b_2 \) then a 200 term KL expansion, yields relative error of 0.08

Two realisations of the resulting random field:
Let $y_i = \xi_i(\omega) \in \Gamma_i$, and write $\Gamma = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_M$.

If the random variables are independent then the joint density function has the form:

$$
\rho(y) = \prod_i \rho_i(y_i)
$$
Let $y_i = \xi_i(\omega) \in \Gamma_i$, and write $\Gamma = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_M$.

If the random variables are independent then the joint density function has the form:

$$\rho(y) = \prod_i \rho_i(y_i)$$

and the expectation of a random function in $y$ is defined via:

$$\langle g(y) \rangle = \int_\Gamma \rho(y)g(y) \, dy.$$
Mixed Stochastic Galerkin Formulation

Let \( y_i = \xi_i(\omega) \in \Gamma_i \), and write \( \Gamma = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_M \).

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\[
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\]

and the expectation of a random function in \( y \) is defined via:

\[
< g(y) > = \int_{\Gamma} \rho(y) g(y) \, dy.
\]

We also define the space \( L^2_\rho(\Gamma) \) of random functions which satisfy:

\[
\int_{\Gamma} \rho(y) g(y)^2 \, dy < \infty.
\]
Consider the tensor product spaces

\[ V = H_{0,N}(\text{div}; D) \otimes L^2_\rho(\Gamma) \quad \text{and} \quad W = L^2(D) \otimes L^2_\rho(\Gamma) \]

We seek \( u(x, y) \in V \) and \( p(x, y) \in W \) such that:
Mixed Stochastic Galerkin Formulation

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We seek \( u(x, y) \in V \) and \( p(x, y) \in W \) such that:

\[
\int_\Gamma \rho(y) \left( A_M^{-1} u, v \right) dy + \int_\Gamma \rho(y) (p, \nabla \cdot v) dy = \int_\Gamma \rho(y) (g, v \cdot n)_{\partial \Gamma_D} dy,
\]

\[
\int_\Gamma \rho(y) (w, \nabla \cdot u) dy = -\int_\Gamma \rho(y) (f, w) dy
\]

\( \forall \; v(x, y) \in V \) and \( w(x, y) \in W \).
Consider the tensor product spaces

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We seek \( u(x, y) \in V \) and \( p(x, y) \in W \) such that:

\[
\langle (A_M^{-1} u, v) \rangle + \langle p, \nabla \cdot v \rangle = \langle (g, v \cdot n)_{\partial \Gamma_D} \rangle,
\]

\[
\langle w, \nabla \cdot u \rangle = -\langle (f, w) \rangle
\]

\( \forall \; v(x, y) \in V \) and \( w(x, y) \in W \).
Finite-Dimensional Problem

Find \( u_{hd}(x, y) \in V_h \otimes S_d \) and \( p_{hd}(x, y) \in W_h \otimes S_d \) satisfying:

\[
\left\langle \left( A_M^{-1} u_{hd}, v \right) \right\rangle + \left\langle (p_{hd}, \nabla \cdot v) \right\rangle = \left\langle (g, v \cdot n)_{\partial \Gamma_D} \right\rangle,
\]

\[
\left\langle (w, \nabla \cdot u_{hd}) \right\rangle = -\left\langle (f, w) \right\rangle
\]

\( \forall \ v(x, y) \in V_h \otimes S_d \) and \( w(x, y) \in W_h \otimes S_d \).
Finite-Dimensional Problem

Find $u_{hd}(x, y) \in V_h \otimes S_d$ and $p_{hd}(x, y) \in W_h \otimes S_d$ satisfying:

$$\left\langle \left( A^{-1}_M u_{hd}, v \right) \right\rangle + \left\langle (p_{hd}, \nabla \cdot v) \right\rangle = \left\langle (g, v \cdot n)_{\partial \Gamma_D} \right\rangle,$$

$$\left\langle (w, \nabla \cdot u_{hd}) \right\rangle = - \left\langle (f, w) \right\rangle$$

$\forall \ v(x, y) \in V_h \otimes S_d$ and $w(x, y) \in W_h \otimes S_d$.

- $V_h \subset H(div; D), W_h \subset L^2(D)$ are a deterministic inf-sup stable pairing e.g. $RT_0(D)-P_0(D)$. 

Preconditioning Saddle-Point Systems arising in a Stochastic Mixed Finite Element Problem – p. 11/35
Finite-Dimensional Problem

Find \( u_{hd}(x, y) \in V_h \otimes S_d \) and \( p_{hd}(x, y) \in W_h \otimes S_d \) satisfying:

\[
\left\langle A^{-1}_M u_{hd}, v \right\rangle + \left\langle p_{hd}, \nabla \cdot v \right\rangle = \left\langle (g, v \cdot n)_{\partial \Gamma_D} \right\rangle,
\]

\[
\left\langle (w, \nabla \cdot u_{hd}) \right\rangle = -\left\langle (f, w) \right\rangle
\]

\( \forall v(x, y) \in V_h \otimes S_d \) and \( w(x, y) \in W_h \otimes S_d \).

- \( V_h \subset H(\text{div}; D) \), \( W_h \subset L^2(D) \) are a deterministic inf-sup stable pairing e.g. \( RT_0(D)-P_0(D) \).

- \( S_d \subset L^2(\Gamma) \) is set of multivariate polynomials in \( M \) random variables. Choose from:
  1. total degree \( d \) (generalised polynomial chaos) of dimension \( N_\xi = \frac{(M+d)!}{M!d!} \)
  2. degree \( d \) in each random variable of dimension \( N_\xi = (d+1)^M \)
Abstract Saddle-Point Problem

We seek $u_{hd}(x, y) \in V_h \otimes S_d$, and $p_{hd}(x, y) \in W_h \otimes S_d$ s.t.:

$$a(u_{hd}, v) + b(p_{hd}, v) = \left\langle (g, v \cdot n)_{\partial \Gamma_D} \right\rangle,$$

$$b(w, u_{hd}) = -\left\langle (f, w) \right\rangle$$

∀ $v(x, y) \in V_h \otimes S_d$ and $w(x, y) \in W_h \otimes S_d$
Abstract Saddle-Point Problem

We seek $u_{hd}(x, y) \in V_h \otimes S_d$, and $p_{hd}(x, y) \in W_h \otimes S_d$ s.t.: 

\[
\begin{align*}
    a(u_{hd}, v) + b(p_{hd}, v) &= \langle (g, v \cdot n)|_{\partial \Gamma_D} \rangle, \\
    b(w, u_{hd}) &= -\langle (f, w) \rangle 
\end{align*}
\]

\[
\forall \; v(x, y) \in V_h \otimes S_d \text{ and } w(x, y) \in W_h \otimes S_d
\]

which leads to a symmetric indefinite system of the form:

\[
\begin{pmatrix}
    \tilde{A} & \tilde{B}^T \\
    \tilde{B} & 0
\end{pmatrix}
\begin{pmatrix}
    u \\
    p
\end{pmatrix}
= 
\begin{pmatrix}
    g \\
    f
\end{pmatrix}
\]

of dimension $N_x \times N_\xi$ where $N_x = N_u + N_p$. 

Preconditioning Saddle-Point Systems arising in a Stochastic Mixed Finite Element Problem – p. 12/3
Matrix structure

\[ V_h = \text{span} \{ \varphi_i(x) \}_{i=1}^{N_u}, \quad W_h = \text{span} \{ \phi_j(x) \}_{j=1}^{N_p}, \quad S_d = \text{span} \{ \psi_k(y) \}_{k=1}^{N_\xi} \]

with \( \{ \psi_k(y) \} \) orthonormal w.r.t \( \langle \cdot, \cdot \rangle \), the saddle-point matrix has the structure:

\[
\begin{pmatrix}
I \otimes A_0 + \sum_{k=1}^{M} G_k \otimes A_k & I \otimes B^T \\
I \otimes B & 0
\end{pmatrix}
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0 & I \otimes B
\end{pmatrix}
\]

where

\[ [A_0]_{ij} = \int_D \mu(x) \varphi_i(x) \cdot \varphi_j(x) \quad [A_k]_{ij} = \sqrt{\lambda_k} \int_D c_k(x) \varphi_i(x) \cdot \varphi_j(x) \]

and

\[ [B]_{ij} = \int_D \nabla \cdot \varphi_i(x) \phi_j(x) \quad [G_k]_{rs} = \langle y_k \psi_r(y), \psi_s(y) \rangle \]
Examples

\[ M = 2, d = 2 \text{ (left) and } M = 4, d = 2 \text{ (right)} \]
Well-posedness

The well-posedness of the stochastic saddle-point problem can be analysed using the standard Brezzi-Babuska stability criteria.

Define the following norms on the tensor product spaces:

\[
\|q_{hd}\|_{div \otimes L^2}^2 = \left\langle \|q_{hd}\|_{div(D)}^2 \right\rangle, \quad q_{hd} \in V_h \otimes S_d
\]

\[
\|w_{hd}\|_{L^2 \otimes L^2}^2 = \left\langle \|w_{hd}\|_{L^2(D)}^2 \right\rangle, \quad w_{hd} \in W_h \otimes S_d
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\[
\| w_{hd} \|_{L^2 \otimes L^2}^2 = \left\langle \| w_{hd} \|_{L^2(D)}^2 \right\rangle, \quad w_{hd} \in W_h \otimes S_d
\]

If we choose:

- \( V_h := RT_0(D) \) (lowest-order Raviart-Thomas elements)
- \( W_h := P_0(D) \) (piecewise constants)

and assume that:

\[
0 < a_{min} \leq A^{-1}_M(x, y) \leq a_{max} < +\infty, \quad \text{a.e. in } D \times \Gamma
\]

then, the following results can be proved independently of the choice of \( S_d \subset L^2_\rho(\Gamma) \).
Well-posedness

• \( a(\cdot, \cdot) \) and \( b(\cdot, \cdot) \) are continuous bilinear forms

• Ellipticity

\[
a(\mathbf{v}_{hd}, \mathbf{v}_{hd}) \geq a_{min} \| \mathbf{v}_{hd} \|_{\text{div} \otimes L^2}^2 \quad \forall \mathbf{v}_{hd} \in Z_{hd}
\]

where

\[
Z_{hd} = \{ \mathbf{v}_h \in V_h \otimes S_d \text{ s.t. } b(\mathbf{v}_{hd}, w_{hd}) = 0, \forall w_{hd} \in W_h \otimes S_d \}
\]

• Theorem (inf-sup stability)

There exists a constant \( \tilde{\beta} > 0 \) depending only on the domain \( D \) and the Raviart-Thomas interpolation operator (and therefore independent of \( h, M \) and \( d \)) such that:

\[
\sup_{\mathbf{v}_{hd} \in V_h \otimes S_d \setminus \{0\}} \frac{b(\mathbf{v}_{hd}, w_{hd})}{\| \mathbf{v}_{hd} \|_{\text{div} \otimes L^2} \| w_{hd} \|_{L^2 \otimes L^2}} \geq \tilde{\beta} \| w_{hd} \|_{L^2 \otimes L^2} \quad \forall w_{hd} \in W_h \otimes S_d.
\]
Define deterministic matrices $D \in \mathbb{R}^{N_u \times N_u}$, and $M \in \mathbb{R}^{N_p \times N_p}$ via:

$$[D]_{ij} = \int_D \nabla \cdot \varphi_i \nabla \cdot \varphi_j, \quad [M]_{rs} = \int_D \phi_r \phi_s.$$ 

We then have matrix representations of the following stochastic norms:

$$\| v_{hd} \|_{\text{div}, A^{-1} \otimes L^2}^2 = v^T (\tilde{A} + \tilde{D}) v \quad \text{where } \tilde{D} = I \otimes D$$

$$\| w_h \|_{L^2 \otimes L^2}^2 = w^T \tilde{M} w \quad \text{where } \tilde{M} = I \otimes M.$$
Define deterministic matrices $D \in \mathbb{R}^{N_u \times N_u}$, and $M \in \mathbb{R}^{N_p \times N_p}$ via:

$$[D]_{ij} = \int_D \nabla \cdot \varphi_i \nabla \cdot \varphi_j, \quad [M]_{rs} = \int_D \phi_r \phi_s.$$ 

We then have matrix representations of the following stochastic norms:

$$\| \mathbf{v}_{hd} \|_{d_{\text{div}}, A^{-1} \otimes L^2}^2 = \mathbf{v}^T \left( \tilde{A} + \tilde{D} \right) \mathbf{v} \quad \text{where} \quad \tilde{D} = I \otimes D$$

$$\| \mathbf{w}_h \|_{L^2 \otimes L^2}^2 = \mathbf{w}^T \tilde{M} \mathbf{w}, \quad \text{where} \quad \tilde{M} = I \otimes M.$$ 

Note that the discrete inf-sup condition tells us that:

$$\tilde{\beta}^2 \min \left( 1, \frac{1}{a_{\text{max}}} \right) \leq \frac{\mathbf{w}^T \tilde{B} \left( \tilde{A} + \tilde{D} \right)^{-1} \tilde{B}^T \mathbf{w}}{\mathbf{w}^T \tilde{M} \mathbf{w}} \quad \forall \mathbf{w} \in \mathbb{R}^{N_p N_\xi} \setminus \{0\}.$$
Consider the ‘ideal’ preconditioner

\[
P = \begin{pmatrix}
\tilde{A} + \tilde{D} & 0 \\
0 & \tilde{M}
\end{pmatrix} = \begin{pmatrix}
\tilde{A} + \tilde{B}^T \tilde{M}^{-1} \tilde{B} & 0 \\
0 & \tilde{M}
\end{pmatrix}
\]
Eigenvalue bounds

Consider the ‘ideal’ preconditioner

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P = \begin{pmatrix}
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\end{pmatrix}
= \begin{pmatrix}
\tilde{A} + \tilde{B}^T \tilde{M}^{-1} \tilde{B} & 0 \\
0 & \tilde{M}
\end{pmatrix}
\]

Theorem

The eigenvalues of

\[
\begin{pmatrix}
\tilde{A} & \tilde{B}^T \\
\tilde{B} & 0
\end{pmatrix}
\begin{pmatrix}
u \\
p
\end{pmatrix}
= \lambda
\begin{pmatrix}
\tilde{A} + \tilde{D} & 0 \\
0 & \tilde{M}
\end{pmatrix}
\begin{pmatrix}
u \\
p
\end{pmatrix}
\]

are bounded and lie in the union of the intervals,

\[
\left[ -1, -\frac{\beta^2}{a_{max}} \right] \cup \{1\}
\]
Example

Let $D = [0, 1] \times [0, 1]$, with mixed bcs. We choose an exponential covariance function for the random input with $\mu(x) = 1$ and $\sigma(x) = 0.2$. 
Mean of numerical solution

Pressure (left), Flux (right)
Variance of numerical solution

Pressure (left), y component (middle) and x component (right) of the Flux
In this example $a_{max} = O(1)$

<table>
<thead>
<tr>
<th></th>
<th>$h = \frac{1}{16}$</th>
<th></th>
<th>$h = \frac{1}{32}$</th>
<th></th>
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</thead>
<tbody>
<tr>
<td></td>
<td>d</td>
<td>$N_\xi$</td>
<td>Iter</td>
<td>dimension</td>
</tr>
<tr>
<td>M=4</td>
<td>1  5</td>
<td></td>
<td>6</td>
<td>6,560</td>
</tr>
<tr>
<td></td>
<td>2  15</td>
<td></td>
<td>6</td>
<td>19,650</td>
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<td></td>
<td>3  35</td>
<td></td>
<td>6</td>
<td>45,920</td>
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<td></td>
<td>4  70</td>
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<td>6</td>
<td>91,840</td>
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<tr>
<td>M=5</td>
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<td></td>
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</tr>
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</tr>
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</tr>
<tr>
<td>M=6</td>
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<td></td>
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<td>2  28</td>
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<td>36,736</td>
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<tr>
<td></td>
<td>3  84</td>
<td></td>
<td>6</td>
<td>110,208</td>
</tr>
<tr>
<td></td>
<td>4  210</td>
<td></td>
<td>6</td>
<td>275,520</td>
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</tbody>
</table>
(Exact) Preconditioned Minres iterations

With $h = \frac{1}{32}, M = 4$ and $d = 2$ fixed and varying ratio $\frac{\sigma}{\mu}$

<table>
<thead>
<tr>
<th>$\frac{\sigma}{\mu}$</th>
<th>0.1</th>
<th>0.2</th>
<th>0.4</th>
<th>0.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Iter</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
</tr>
</tbody>
</table>

With $h = \frac{1}{32}, M = 4$ and $d = 2$ and $\frac{\sigma}{\mu} = 0.1$ so that only $a_{max}$ is varying

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$10^{-2}$</th>
<th>$10^{-1}$</th>
<th>$10^{0}$</th>
<th>$10^{1}$</th>
<th>$10^{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Iter</td>
<td>4</td>
<td>4</td>
<td>6</td>
<td>9</td>
<td>22</td>
</tr>
</tbody>
</table>
Practical Implementation

We need a fast solver for systems with the coefficient matrix:

$$\tilde{A} + \tilde{D} = I \otimes \left( A_0 + B^T M^{-1} B \right) + \left( \sum_{k=1}^{M} G_k \otimes A_k \right)$$

which represents a weighted stochastic $H(div; D) \otimes L^2_\rho(\Gamma)$ operator:

$$\tilde{\mathcal{H}}_A : RT_0(D) \otimes S_d(\Gamma) \rightarrow RT_0(D) \otimes S_d(\Gamma)$$

defined via:

$$\left( \tilde{\mathcal{H}}_A \mathbf{v}_{hd}, \mathbf{v}_{hd} \right) = \int_{\Gamma} \rho(\mathbf{y}) \left( \int_D A_M^{-1} \mathbf{v}_{hd} \cdot \mathbf{v}_{hd} + \nabla \cdot \mathbf{v}_{hd} \nabla \cdot \mathbf{v}_{hd} \, dD \right) \, d\mathbf{y}.$$ 

Note that this is not an elliptic operator.
**Eigenvalue bounds**

**Theorem:** Suppose there exists a matrix $\tilde{V}$ satisfying

$$\theta \leq \frac{v^T (\tilde{A} + \tilde{D}) v}{v^T \tilde{V} v} \leq \Theta \leq 1$$

with positive constants $\theta$ and $\Theta$. The eigenvalues of:

$$\begin{pmatrix}
\tilde{A} & \tilde{B}^T \\
\tilde{B} & 0
\end{pmatrix} \begin{pmatrix}
u \\
p
\end{pmatrix} = \lambda \begin{pmatrix} \tilde{V} & 0 \\
0 & \tilde{M}
\end{pmatrix} \begin{pmatrix} u \\
p
\end{pmatrix}$$

lie in the union of the intervals,

$$\left[-1, -\frac{1}{2} \left(\theta (1 - \alpha) - \sqrt{\theta^2 (\alpha - 1)^2 + 4\alpha \theta}\right)\right] \cup [\theta, 1]$$

where $\alpha = \frac{\tilde{\beta}^2}{a_{max}}$ is the corresponding bound for the ideal preconditioner.
Geometric $H(\text{div})$ Multigrid

We approximate the action of $\left(\tilde{A} + \tilde{D}\right)^{-1}$ via a specialised multigrid V-cycle.
Geometric H(div) Multigrid

We approximate the action of \((\bar{A} + \bar{D})^{-1}\) via a specialised multigrid V-cycle.

Geometric $H(\text{div})$ Multigrid

We approximate the action of $(\tilde{A} + \tilde{D})^{-1}$ via a specialised multigrid V-cycle.


The main idea is to only vary the spatial discretisation from grid to grid whilst keeping the stochastic discretisation fixed.
Geometric H(div) Multigrid

We approximate the action of \((\tilde{A} + \tilde{D})^{-1}\) via a specialised multigrid V-cycle.


The main idea is to only vary the spatial discretisation from grid to grid whilst keeping the stochastic discretisation fixed.

Key ingredients:

- Prolongation: \(\tilde{P} = I \otimes P^h_H\) where \(P^h_H\) is a standard spatial prolongation operator
- Restriction operator \(\tilde{R} = \tilde{P}^T = I \otimes R^H_h\)
- Smoother: additive Schwarz method (block Jacobi)
Let \( \tilde{H}_h = \tilde{A} + \tilde{D} \) be the stochastic \( H(div) \) matrix associated with a fixed spatial mesh \( T_h \), decomposed into vertex-based patches:

The smoothing operator (in matrix form) is defined via:

\[
\tilde{S}_h = \eta \sum_k \tilde{P}_h^k \tilde{H}_h^{-1}
\]
Additive Schwarz Smoothing

where

$$\tilde{P}_h^k = (I \otimes R_k^T) \tilde{H}_{h,k}^{-1} (I \otimes R_k) \tilde{H}_h.$$  

Then, for $v \in \mathbb{R}^{N_u N_\xi}$ we have:

$$\tilde{S}_h v = \eta \sum_k (I \otimes R_k^T) \tilde{H}_{h,k}^{-1} (I \otimes R_k) v$$

where $\tilde{H}_{h,k}$ represents a local ‘patch-version’ of the matrix $\tilde{H}_h$. 
Additive Schwarz Smoothing

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where $$\tilde{H}_{h,k}$$ represents a local ‘patch-version’ of the matrix $$\tilde{H}_h$$.

Smoothing requires multiple decoupled solves with $$\tilde{H}_{h,k}$$. In the stochastic problem:

$$\tilde{H}_{h,k} = I \otimes (A_{0,k} + D_{0,k}) + \sum_{i=1}^{M} G_i \otimes A_{i,k}$$

and so the dimension of each local matrix is $$N_\xi N_k$$.
Additive Schwarz Smoothing

where

\[ \tilde{P}_h^k = (I \otimes R_k^T) \tilde{H}_{h,k}^{-1} (I \otimes R_k) \tilde{H}_h. \]

Then, for \( \nu \in \mathbb{R}^{N_u N_\xi} \) we have:

\[ \tilde{S}_h \nu = \eta \sum_k (I \otimes R_k^T) \tilde{H}_{h,k}^{-1} (I \otimes R_k) \nu \]

where \( \tilde{H}_{h,k} \) represents a local ‘patch-version’ of the matrix \( \tilde{H}_h \).

Smoothing requires multiple decoupled solves with \( \tilde{H}_{h,k} \). In the stochastic problem:

\[ \tilde{H}_{h,k} = I \otimes (A_{0,k} + D_{0,k}) + \sum_{i=1}^M G_i \otimes A_{i,k} \]

and so the dimension of each local matrix is \( N_\xi N_k \).

This is tractable for a few thousand stochastic degrees of freedom.
Multigrid Convergence

Theorem

Let $\tilde{V}$ denote the matrix corresponding to the inverse of the multigrid V-cycle operator described above. Then,

$$\theta \leq \frac{v^T (\tilde{A} + \tilde{D}) v}{v^T \tilde{V} v} \leq 1$$

where

$$\theta = 1 - \frac{C}{C + 2\nu}$$

depends only on the number of smoothing steps $\nu$ and $a_{min}$ and $a_{max}$. 
Multigrid Convergence

Theorem

Let \( \tilde{V} \) denote the matrix corresponding to the inverse of the multigrid V-cycle operator described above. Then,

\[
\theta \leq \frac{\nu^T (\tilde{A} + \tilde{D}) \nu}{\nu^T \tilde{V} \nu} \leq 1
\]

where

\[
\theta = 1 - \frac{C}{C + 2\nu}
\]

depends only on the number of smoothing steps \( \nu \) and \( a_{min} \) and \( a_{max} \).

Combining this result with eigenvalue bound for preconditioned saddle-point system, we have a solver that is optimal w.r.t all discretisation parameters.
Example 1

\[ P = \begin{pmatrix} \tilde{V} & 0 \\ 0 & \tilde{M} \end{pmatrix} \]

- 1 multigrid V-cycle per minres iteration; 1 pre and 1 post smoothing step;
- Uniform random variables; \( \mu(x) = 1, \sigma = 0.1 \) (\( \Rightarrow a_{max} = O(1) \))

<table>
<thead>
<tr>
<th>d</th>
<th>M = 1</th>
<th>M = 2</th>
<th>M = 3</th>
<th>M = 4</th>
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<tr>
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<td>2</td>
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<tr>
<td>( h = \frac{1}{64} )</td>
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<td>17</td>
<td>17</td>
<td>17</td>
</tr>
<tr>
<td>-</td>
<td>2</td>
<td>17</td>
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<td>-</td>
<td>4</td>
<td>17</td>
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</tbody>
</table>
Example 2

- Fixed discretisation parameters: $h = \frac{1}{16}$, $M = 4$, $p = 2$.

- Varying $\frac{\sigma}{\mu}$

Preconditioned minres iterations:

<table>
<thead>
<tr>
<th>$\frac{\sigma}{\mu}$</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
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<tr>
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<td>6</td>
<td>6</td>
<td>6</td>
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<tr>
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<td>17</td>
<td>17</td>
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Multigrid constants

<table>
<thead>
<tr>
<th>$\frac{\sigma}{\mu}$</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
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<tbody>
<tr>
<td>$\theta$</td>
<td>0.4576</td>
<td>0.4564</td>
<td>0.4548</td>
<td>0.4527</td>
<td>0.4495</td>
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<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
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</tr>
</tbody>
</table>
Example 3

- Fixed discretisation parameters: $h = \frac{1}{16}$, $M = 4$, $p = 2$.
- Vary $a_{\text{max}}$ by varying $\mu$ and setting $\sigma = \frac{\mu}{10}$.

Preconditioned minres iterations:

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$10^{-3}$</th>
<th>$10^{-2}$</th>
<th>$10^{-1}$</th>
<th>$10^0$</th>
<th>$10^1$</th>
<th>$10^2$</th>
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<tbody>
<tr>
<td>Ideal</td>
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<td>3</td>
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<td>5</td>
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Multigrid constants

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$10^{-3}$</th>
<th>$10^{-2}$</th>
<th>$10^{-1}$</th>
<th>$10^0$</th>
<th>$10^1$</th>
<th>$10^2$</th>
<th>$10^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta$</td>
<td>0.4552</td>
<td>0.4550</td>
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<td>1.0000</td>
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</tbody>
</table>
Summary

- Solving well-posed stochastic saddle-point problem

- Stochastic inf-sup stability result leads to nice eigenvalue bounds for $H(\text{div})$ preconditioners

- Practical implementation based on deterministic Arnold-Falk-Winther multigrid

- Analysis of extended multigrid method available

- Preconditioner for saddle-point system is optimal w.r.t spatial and stochastic discretisation parameters

- Overall performance does depend on $a_{min}$ and $a_{max}$

- Experiments with modified (cheaper) smoothers look promising
Alternative Preconditioning Scheme

We know that optimal preconditioners mimic the mapping properties of the underlying saddle-point operator. Here, there are two options.
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An alternative ideal preconditioner is given by:

\[ P = \begin{pmatrix} \tilde{A}_{diag} & 0 \\ 0 & \tilde{B} \tilde{A}^{-1}_{diag} \tilde{B}^T \end{pmatrix} \]
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\end{pmatrix}
\]

The blocks of this matrix represent norms in which an alternative inf-sup condition can be established. In particular,

\[w^T \tilde{B} \tilde{A}^{-1}_{\text{diag}} \tilde{B}^T w = \langle \| w_{h,d} \|_{1,h,A}^2 \rangle\]

represents the expectation of a (weighted) mesh-dependent \(H_1(D)\) norm.
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\[
w^T \tilde{B} \tilde{A}^{-1}_{\text{diag}} \tilde{B}^T w = \left\langle \| w_{h,d} \|_{2_{1,h,A}}^2 \right\rangle
\]

represents the expectation of a (weighted) mesh-dependent \( H_1(D) \) norm.

Pro: Standard multigrid methods can be used. Con: Obtaining \( \tilde{A}_{\text{diag}} \) that yields robustness w.r.t PDE coefficients is difficult.
References

\[ P = \begin{pmatrix} \tilde{A} + \tilde{D} & 0 \\ 0 & \tilde{M} \end{pmatrix} \]


\[ P = \begin{pmatrix} \tilde{A}_{diag} & 0 \\ 0 & \tilde{B} \tilde{A}^{-1}_{diag} \tilde{B}^T \end{pmatrix} \]