Birational geometry and arithmetic

July 2012
Basic questions

Let $F$ be a field and $X$ a smooth projective algebraic variety over $F$. 

Introduction
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- Existence
- Density
- Distribution with respect to heights
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Of particular interest are small fields:

$$F = \mathbb{F}_q, \quad \mathbb{Q}, \quad \mathbb{F}_q(t), \quad \mathbb{C}(t), ...$$
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Main idea

Arithmetic properties are governed by global geometric invariants and the properties of the ground field $F$. 

Introduction
Classification schemes

- via dimension: curves, surfaces, ...

- via degree: Fano, general type, intermediate type

- $X \subset P^n$, with $d < n+1$, $d > n+1$ or $d = n+1$ how close to $P^n$: rational, unirational, rationally connected

In small dimensions some of these notions coincide, e.g., in dimension 2 and over algebraically closed fields of characteristic zero $\text{rational} = \text{unirational} = \text{rationally connected}$

Small degree surfaces (Del Pezzo surfaces) over algebraically closed fields are rational. Cubic surfaces with a rational point are unirational.

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Over nonclosed fields $F$:

- Forms and Galois cohomology
- Brauer group $\text{Br}(X)$
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Over nonclosed fields $F$:

- **Forms and Galois cohomology**
- **Brauer group** $\text{Br}(X)$

In some cases, these are **effectively computable**.
Starting point: Curves over number fields

What do we know about curves over number fields?

- $g = 0$: one can decide when $X(F) \neq \emptyset$ (local-global principle), if $X(F) \neq \emptyset$, then $X(F)$ is infinite and one has a good understanding of how $X(F)$ is distributed.
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- $g \geq 2$: $\#X(F) < \infty$, no effective algorithm to determine $X(F)$ (effective Mordell?)
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**Theorem (2001)**

- Let $C$ be a hyperelliptic curve over $\overline{\mathbb{F}}_p$ of genus $\geq 2$ and let $C'$ be any curve. Then $C \Rightarrow C'$.
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The cover \( \tilde{C} \to C \) is explicit, so that effective Mordell for \( C_6 \) implies effective Mordell for all hyperbolic hyperelliptic curves.
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**Conjecture**

If $C, C'$ are curves of genus $\geq 2$ over $\overline{F}_p$ or $\overline{Q}$ then $C \Leftrightarrow C'$.
Prototype: hypersurfaces \( X_f \subset \mathbb{P}^n \) over \( \mathbb{Q} \)

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**Birch 1961**

If \( n \gg 2^{\deg(f)} \) and \( X_f \) is smooth then:

- if there are solutions in \( \mathbb{Q}_p \) and in \( \mathbb{R} \) then there are solutions in \( \mathbb{Q} \)
- asymptotic formulas

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- the method works over $\mathbb{F}_q[t]$ as well

Introduction: rational points on hypersurfaces
Heuristic

- Given: \( f \in \mathbb{Z} \left[ x_0, \ldots, x_n \right] \) homogeneous of degree \( \deg(f) \).
- We have \( |f(x)| = O(B^{\deg(f)}) \), for \( \|x\| := \max_j(|x_j|) \leq B \).
- May (?) assume that the probability of \( f(x) = 0 \) is \( B^{-\deg(f)} \).
- There are \( B^{n+1} \) “events” with \( \|x\| \leq B \).
- We expect \( B^{n+1-d} \) solutions with \( \|x\| \leq B \).

Hope: reasonable at least when \( n + 1 - d \geq 0 \).
Theorem

If \( \deg(f) \leq n \) then \( X_f(\mathbb{C}(t)) \neq \emptyset \).

Proof: Insert \( x_j = x_j(t) \in \mathbb{C}[t] \), of degree \( e \), into

\[
f = \sum_J f_J x^J = 0, \quad |J| = \deg(f).
\]

This gives a system of \( e \cdot \deg(f) + \text{const} \) equations in \((e + 1)(n + 1)\) variables. This system is solvable for \( e \gg 0 \), provided \( \deg(f) \leq n \).
Existence of rational points

Theorem (Esnault 2001)

Every smooth rationally connected variety over a finite field has a rational point.

Theorem (Graber-Harris-Starr 2001)

Every smooth rationally connected variety over the function field of a curve over an algebraically closed field has a rational point.

Over number fields and higher dimensional function fields, there exist local and global obstructions to the existence of rational points.
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**Hasse principle**

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**Basic examples:** Quadrics, hypersurfaces of small degree

**Counterexamples:**
- Iskovskikh 1971: The conic bundle \( X \to \mathbb{P}^1 \) given by \( x^2 + y^2 = f(t) \), \( f(t) = (t^2 - 2)(3 - t^2) \).
- Cassels, Guy 1966: The cubic surface \( 5x^3 + 9y^3 + 10z^3 + 12t^3 = 0 \).

The proofs use basic algebraic number theory: quadratic and cubic reciprocity, divisibility of class numbers.
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Existence of points
We have $X(F) \subset X(A_F) \subseteq X(A_F)^{Br}$, where $X(A_F)^{Br} := \bigcap_{A \in Br(X)} \{ (x_v) \mid \sum_v inv_v(A(x_v)) = 0 \}$.

Manin's formulation gives a more systematic approach to identifying the algebraic structure behind the obstruction.

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Brauer-Manin obstruction

\[
\begin{align*}
\text{Br}(X_F) & \longrightarrow \bigoplus_v \text{Br}(X_{F_v}) \\
0 & \longrightarrow \text{Br}(F) \longrightarrow \bigoplus_v \text{Br}(F_v) \stackrel{\sum_v \text{inv}_v}{\longrightarrow} \mathbb{Q}/\mathbb{Z} \longrightarrow 0,
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Effectivity of Brauer-Manin obstructions

The obstruction group for geometrically rational surfaces

\[ \frac{\text{Br}(X_{\bar{F}})}{\text{Br}(F)} = H^1(\text{Gal}(\bar{F}/F), \text{Pic}(\bar{X})). \]
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Kresch-T. 2006

Let \( X \subset \mathbb{P}^n \) be a geometrically rational surface over a number field \( F \). Then there is an effective algorithm to compute \( X(\mathbb{A}_F)^{\text{Br}} \).
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**Kresch-T. 2010**

Let \( X \subset \mathbb{P}^n \) be a surface over a number field \( F \). Assume that

- the geometric Picard group of \( X \) is torsion free and is generated by finitely many divisors, each with a given set of defining equations
- \( \text{Br}(X) \) can be bounded effectively.

Then there is an effective algorithm to compute \( X(\mathbb{A}_F)^{\text{Br}} \).
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- $\text{Pic}(\overline{X})$, together with the Galois action;
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- Finiteness of $\text{Br}(X)/\text{Br}(F)$ for all K3 surfaces over number fields (Skorobogatov-Zarhin 2007)

Existence of points
Effectivity of Brauer-Manin obstructions

Main ingredients:
- effective Kuga-Satake correspondence, relies on an effective construction of the Bailey-Borel compactification of the moduli space of polarized K3 surfaces;
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- the work of Masser-Wüstholz on the effective Tate conjecture for abelian varieties;
- effective GIT, Matsusaka, Hilbert Nullstellensatz, etc.
Let $X$ be a smooth intersection of two quadrics in $\mathbb{P}^4$ over $\mathbb{Q}$ (a Del Pezzo surface of degree 4).
Computing the obstruction group

Let $X$ be a smooth intersection of two quadrics in $\mathbb{P}^4$ over $\mathbb{Q}$ (a Del Pezzo surface of degree 4). The Galois action on the 16 lines factors through the Weyl group $W(D_5)$ (a group of order 1920).

**Bright, Bruin, Flynn, Logan 2007**

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- Implement an algorithm to compute the BM obstruction and provide more examples of Iskovskikh type.
Conjecture (Colliot-Thélène–Sansuc 1980)

Let $X$ be a smooth projective rationally-connected surface over a number field $F$, e.g., an intersection of two quadrics in $\mathbb{P}^4$ or a cubic in $\mathbb{P}^3$. Then

$$X(F) = \overline{X(\mathbb{A}_F)^{\text{Br}}}.$$
Uniqueness of the Brauer-Manin obstruction

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In particular, existence of rational points on rationally-connected surfaces would be decidable.
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Degree 4 Del Pezzo surfaces admitting a conic bundle $X \to \mathbb{P}^1$. 

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Colliot-Thélène–Sansuc–Swinnerton-Dyer 1987:
Degree 4 Del Pezzo surfaces admitting a conic bundle $X \to \mathbb{P}^1$. The conjecture is open for general degree 4 Del Pezzo surfaces.
Do we believe this conjecture?

Recall that a general Del Pezzo surface $X$ has points locally, and that

$$\text{Br}(X)/\text{Br}(F) = 1.$$
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The Galois group action on the exceptional curves has to be small to allow an obstruction; this is counterintuitive.
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**Elsenhans–Jahnel 2007:** Thousands of examples of cubic surfaces over $\mathbb{Q}$ with different Galois actions, the conjecture holds in all cases.
Del Pezzo surfaces over $\mathbb{F}_q(t)$

**Theorem (Hassett-T. 2011)**

Let $k$ be a finite field with at least $2^2 \cdot 17^4$ elements and $X$ a general Del Pezzo surface of degree 4 over $F = k(t)$ such that its integral model

$$\mathcal{X} \to \mathbb{P}^1$$

is a complete intersection in $\mathbb{P}^1 \times \mathbb{P}^4$ of two general forms of bi-degree $(1, 2)$. Then $X(F) \neq \emptyset$. 

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The idea of proof will follow....
Potential density: Zariski density after a finite extension of the field.

Let $X \subset \mathbb{P}^1 \times \mathbb{P}^3$ be a general hypersurface of bidegree $(1, 4)$. Then the K3 surface fibration $X \to \mathbb{P}^1$ has a Zariski dense set of sections, i.e., such K3 surfaces over $\mathbb{F} = \mathbb{C}(t)$ have Zariski dense rational points; more generally, this holds for general pencils of K3 surfaces of degree $\leq 18$ (Hassett-T. 2008).

Same holds, if $X \subset \mathbb{P}^1 \times \mathbb{P}^3$ is given by a general form of bidegree $(2, 4)$ (Zhiyuan Li 2011).
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Higher dimensions

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Counting rational points

Counting problems depend on:

- a projective embedding $X \hookrightarrow \mathbb{P}^n$;
- a choice of $X^\circ \subset X$;
- a choice of a height function $H : \mathbb{P}^n(F) \to \mathbb{R}_{>0}$.
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**Main problem**

\[
N(X^\circ(F), B) = \# \{ x \in X^\circ(F) \mid H(x) \leq B \} \sim c \cdot B^a \log(B)^{b-1}
\]
The geometric framework

**Conjecture (Manin 1989)**

Let $X \subset \mathbb{P}^n$ be a smooth projective Fano variety over a number field $F$, in its anticanonical embedding.
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where $b = \text{rk} \text{Pic}(X)$.

We do not know, in general, whether or not $X(F)$ is Zariski dense, even after a finite extension of $F$. Potential density of rational points has been proved for some families of Fano varieties, but is still open, e.g., for hypersurfaces $X_d \subset \mathbb{P}^d$, with $d \geq 5$. 

Counting points
Let $F = \mathbb{F}_q(B)$ be a global function field and $X/F$ a smooth Fano variety. Let 

$$\pi : \mathcal{X} \rightarrow B$$

be a model. A point $x \in X(F)$ gives rise to a section $\tilde{x}$ of $\pi$. Let $\mathcal{L}$ be a very ample line bundle on $\mathcal{X}$. The height zeta function takes the form

$$Z(s) = \sum_{\tilde{x}} q^{-(\mathcal{L},\tilde{x})s}$$

$$= \sum_d \mathcal{M}_d(\mathbb{F}_q) q^{-ds},$$

where $d = (\mathcal{L}, \tilde{x})$ and $\mathcal{M}_d$ is the space of sections of degree $d$. 

Counting points
The dimension of $\mathcal{M}_d$ can be estimated, provided $\tilde{x}$ is unobstructed:

$$\dim \mathcal{M}_d \sim (-K_X, \tilde{x}), \quad \tilde{x} \in \mathcal{M}_d.$$
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Heuristic assumption:

$$\mathcal{M}_d(\mathbb{F}_q) = q^{\dim(\mathcal{M}_d)}$$

leads to a modified zeta function

$$Z_{\text{mod}}(s) = \sum q^{-(-\mathcal{L}, \tilde{x})s + (-K_X, \tilde{x})},$$

its analytic properties are governed by the ratio of the linear forms

$$(-K_X, \cdot) \quad \text{and} \quad (\mathcal{L}, \cdot)$$
The Batyrev–Manin conjecture

\[ N(X^\circ, \mathcal{L}, B) = c \cdot B^{a(\mathcal{L})} \cdot \log(B)^{b(\mathcal{L}) - 1}(1 + o(1)), \quad B \to \infty \]
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- \( a(L) = \inf \{ a \mid a[L] + [K_X] \in \Lambda_{\text{eff}}(X) \} \),
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- \(b(L) = \text{codimension of the face of } \Lambda_{\text{eff}}(X) \text{ containing } a(L)[\mathcal{L}] + [K_X]\),
- \(c(-K_X) = \alpha(X) \cdot \beta(X) \cdot \tau(K_X)\), the "volume" of the effective cone,
- \(c(L) = \sum y c(L|X^y)\), where \(X \to Y\) is a "Mori fiber space" – \(L\)-primitive fibrations of Batyrev–T.

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G. Segal 1979

\[ \text{Cont}_d(S^2, \mathbb{P}^n(\mathbb{C})) \sim \text{Hol}_d(S^2, \mathbb{P}^n(\mathbb{C})), \quad d \to \infty \]
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This was generalized to Grassmannians and toric varieties as target spaces by Kirwan 1986, Guest 1994, and others.

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Basic idea

\( \mathcal{M}_d(\mathbb{F}_q) \sim q^{\dim(M_d)}, \ for \ d \to \infty, \) provided the homology stabilizes.
Effective stabilization of homology of Hurwitz spaces

There exist $A, B, D$ such that

$$\dim H_d(Hur_G^C, n) = \dim H_d(Hur_G^C, n+D),$$

for $n \geq Ad + B$. 

This has applications to Cohen–Lenstra heuristics over function fields of curves.
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Applications in the context of height zeta functions?
Results over $\mathbb{Q}$

Extensive numerical computations confirming Manin’s conjecture, and its refinements, for Del Pezzo surfaces, hypersurfaces of small degree in dimension 3 and 4.
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Many recent theoretical results on asymptotics of points of bounded height on cubic surfaces and other Del Pezzo surfaces, via (uni)versal torsors (Browning, de la Breteche, Derenthal, Heath-Brown, Peyre, Salberger, Wooley, ...).
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Caution: counterexamples to Manin’s conjecture for cubic surface bundles over $\mathbb{P}^1$ (Batyrev-T. 1996). These are compactifications of affine spaces.

Counting points
Legendre: If \( ax^2 + by^2 = cz^2 \) is solvable mod \( p \), for all \( p \), then it is solvable in \( \mathbb{Z} \).
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Points of smallest height

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An *effective* bound on the error term in the circle method (or in the other asymptotic results) also gives an effective bound on \( H_{\min} \), the height of smallest solutions.
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An effective bound on the error term in the circle method (or in the other asymptotic results) also gives an effective bound on $H_{\text{min}}$, the height of smallest solutions.

In particular, Manin’s and Peyre’s conjecture suggest that

$$H_{\text{min}} \leq \frac{1}{\tau}$$
Points of smallest height

There are extensive numerical data for smallest points on Del Pezzo surfaces, Fano threefolds. E.g.,

Elsenhans-Jahnel 2010
How are all of these related? (Hassett-T.)

Let $X$ be a Del Pezzo surface over $F = \mathbb{F}_q(t)$ and

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\pi : X \to \mathbb{P}^1.
$$

its integral model. Fix a height, and consider the spaces $\mathcal{M}_d$ of sections of $\pi$ of height $d$ (degree of the section).
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- there is a critical $d_0$, related to the height of $X$, such that $M_{d_0}$ is either birational to $IJ(X)$ or to a $\mathbb{P}^1$-bundle over $IJ(X)$
- for $d \geq d_0$, $M_d$ fibers over $IJ(X)$, with general fiber a rationally connected variety
We consider fibrations

\[ \pi : \mathcal{X} \to \mathbb{P}^1, \]

with general fiber a degree-four Del Pezzo surface and with square-free discriminant. In this situation, we have an embedding

\[ \mathcal{X} \subset \mathbb{P}(\pi_\ast \omega_{\pi}^{-1}). \]
Del Pezzo surfaces over $\mathbb{F}_q(t)$

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with general fiber a degree-four Del Pezzo surface and with square-free discriminant. In this situation, we have an embedding

$$\mathcal{X} \subset \mathbb{P}(\pi_*\omega_{\pi}^{-1}).$$

We have

$$\pi_*\omega_{\pi}^{-1} = \bigoplus_{i=1}^5 \mathcal{O}_{\mathbb{P}^1}(-a_i),$$

with

$$a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5,$$

occurring cases are discussed by Shramov (2006), in his investigations of rationality properties of such fibrations.
Del Pezzo surfaces over $\mathbb{F}_q(t)$

We assume that $\pi_*\omega^{-1}_\pi$ is generic, i.e., $a_5 - a_1 \leq 1$; we can realize

$$X \subset \mathbb{P}^1 \times \mathbb{P}^d, \quad d = 4, 5, \ldots, 8,$$

as a complete intersection.
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\textbf{Theorem (Hassett-T. 2011)}

Let $k$ be a finite field with at least $2^2 \cdot 17^4$ elements and $X$ a general Del Pezzo surface of degree 4 over $F = k(t)$ such that its integral model

$$\mathcal{X} \rightarrow \mathbb{P}^1$$

is a complete intersection in $\mathbb{P}^1 \times \mathbb{P}^4$ of two general forms of bi-degree $(1, 2)$. Then $X(F) \neq \emptyset$. 

\textit{Counting points}
Idea of proof

Write $\mathcal{X} \subset \mathbb{P}^1 \times \mathbb{P}^4$ as a complete intersection

$$P_1 s + Q_1 t = P_2 s + Q_2 t = 0,$$

where $P_i, Q_i$ are quadrics in $\mathbb{P}^4$. 
Idea of proof

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where $P_i, Q_i$ are quadrics in $\mathbb{P}^4$. The projection

$$\pi : \mathcal{X} \to \mathbb{P}^1$$

has 16 constant sections corresponding to solutions $y_1, \ldots, y_{16}$ of

$$P_1 = Q_1 = P_2 = Q_2 = 0.$$
Idea of proof

Projection onto the second factor gives a (nonrational) singular quartic threefold $\mathcal{Y}$:

$$P_1 Q_2 - Q_1 P_2 = 0,$$

with nodes at $y_1, \ldots, y_{16}$.
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Projection onto the second factor gives a (nonrational) singular quartic threefold $\mathcal{Y}$:

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Counting points
Idea of proof

Projection onto the *second factor* gives a (nonrational) *singular quartic threefold* $\mathcal{Y}$:

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with nodes at $y_1, \ldots, y_{16}$. The projection $\mathcal{X} \to \mathcal{Y}$ is a small resolution of the singularities of $\mathcal{Y}$. We analyse *lines* in the smooth locus of $\mathcal{Y}$.

**Main observation**

There exists an *irreducible* curve (of genus 289) of lines $l \subset \mathcal{Y}$, giving sections of $\pi$. 

Counting points