Incidentor coloring: methods and results

A.V. Pyatkin

"Graph Theory and Interactions"
Durham, 2013
4-color problem
Reduction to the vertex coloring
Vertex coloring problem
Edge coloring problem
Incidentor coloring

- *Incidentor* is a pair \((v, e)\) of a vertex \(v\) and an arc (edge) \(e\), incident with it.
- It is a half of an arc (edge) adjoining to a given vertex.

![Diagram of incidentor]

- initial
- final
- (mated incidentors)
Incidentor coloring problem

• Color all incidentors of a given multigraph by the minimum number of colors in such a way that the given restrictions on the colors of adjacent (having a joint vertex) and mated (having a joint arc) incidentors would be satisfied.
An example of incidentor coloring
Incidentor coloring generalizes both vertex and edge coloring
Incidentor coloring generalizes both vertex and edge coloring
Motivation

Central computer

All links capacities are equal to 1

1

... i j ...

i-th object must send to j-th one $d_{ij}$ units of information
There are two ways of information transmission:

• 1) Directly from $i$-th object to $j$-th one (during one time unit);

• 2) With memorizing in the central computer (receive the message from $i$-th object, memorize it, and transmit to $j$-th one later).
Reduction to the incidentor coloring

• Each object corresponds to a vertex of the multigraph \((n \text{ vertices})\).

• Each unit of information to transmit from \(i\)-th object to \(j\)-th one corresponds to the arc \(ij\) of the multigraph (there are \(d_{ij}\) arcs going from a vertex \(i\) to the a vertex \(j\)).

• The maximum degree \(\Delta\) equals the maximum load of the link.
Scheduling

- To each information unit two time moments should be assigned – when it goes via $i$-th and $j$-th links.

- These moments could be interpreted as the colors of the incidentors of the arc $ij$. 
Restrictions

- The colors of adjacent incidentors must be distinct.
- For every arc, the color of its initial incidentor is at most the color of the final incidentor, i.e. $a \leq b$. 
• It is required to color all incidentors by the minimum number of colors $\chi$ satisfying all the restrictions (the length of the schedule is $\chi$).

• For this problem $\chi = \Delta$. Such coloring can be found in $O(n^2\Delta^2)$ time.

• (P., 1995)
Sketch of the algorithm

• Consider an arc that is not colored yet
• Try to color its incidentors:
  • 1. In such a way that $a = b$
  • 2. In such a way that $a < b$
• Otherwise, modify the coloring (consider bicolored chains)
Construct a (1,3)-chain
Construct a (2,3)-chain
Construct a (1,2)-chain
Further investigations

• 1) Modifications of initial problem

• 2) Investigation of the incidentor coloring itself
Modifications of initial problem

1) Arbitrary capacities
2) Two sessions of message transmission
3) Memory restrictions
4) Problem of Melnikov & Vizing
5) Bilevel network
Memory restriction

- The memory of the central computer is at most $Q$
- If $Q=0$ then second way of transmission is impossible and we have the edge coloring problem
• If $Q \geq n$ then we can store each message in the central computer during 1 unit of time. Incidentor coloring problem with the following restriction on mated incidentors colors appears:

• $b - 1 \leq a \leq b$
• In this case $\chi = \Delta$
• (Melnikov, Vizing, P.; 2000).
\((k,l)\)-coloring of incidentors

- Let \(0 \leq k \leq l \leq \infty\). Restrictions:
  - 1) adjacent incidentors have distinct colors;
  - 2) mated incidentors colors satisfy \(k \leq b - a \leq l\).
- Denote the minimum number of colors by \(\chi_{k,l}(G)\).
• Case $k=0$ is solved:
  • $\chi_{0,0}(G)$ is an edge chromatic number
  • $\chi_{0,1}(G) = \chi_{0,\infty}(G) = \Delta$ (Melnikov, P., Vizing, 2000)

• Another solved case is $l=\infty$:
  • $\chi_{k,\infty}(G) = \max\{\Delta, k + \Delta^+, k + \Delta^-\}$ (P., 1999)
Vizing’s proof

• Let \( t = \max\{\Delta, k + \Delta^+, k + \Delta^-\} \)

• 1. Construct a bipartite interpretation \( H \) of the graph \( G \):
  
  • \( v \in V(G) \) corresponds to \( v^+, v^- \in V(H) \)
  • \( vu \in E(G) \) corresponds to \( v^+u^- \in E(H) \)
Vizing’s proof

• 2. Color the edges of $H$ by $\Delta(H)$ colors. Clearly, $\Delta(H) = \max\{\Delta^+(G), \Delta^-(G)\}$

• 3. If $v^+u^- \in E(H)$ is colored $a$, color $a$ the initial incidentor of the arc $vu \in E(G)$ and color $a+k$ its final incidentor
Vizing’s proof

• 4. Shift colors at every vertex

• Initial: turn $a_1 < a_2 < \ldots < a_p$ into $1, 2, \ldots, p$

• Final: turn $b_1 > b_2 > \ldots > b_q$ into $t, t-1, \ldots, t-q+1$

• We get a required incidentor coloring of $G$ by $t$ colors
Example

\[ k = 1 \]
\[ \Delta = 3 \]
\[ \Delta^+ = \Delta^- = 2 \]
\[ t = 3 \]
Bipartite interpretation
Edge coloring
Shifting the colors
Equivalent problem in scheduling theory

- Job Shop with $n$ machines and $m$ jobs, each of which has two unit operations (at different machines), and there must be a delay at least $k$ and at most $l$ between the end of the first operation and the beginning of the second one.
• It is NP-complete to find out whether there is a \((1,1)\)-coloring of a multigraph by \(\Delta\) colors even for \(\Delta=7\) (Bansal, Mahdian, Sviridenko, 2006).

• Reduction from 3-edge-coloring of a 3-regular graph
Reduction from 3-edge-coloring

- Substitute each edge by the following gadget:
Reduction from 3-edge-coloring

- It can be verified that in any (1,1)-coloring by 7 colors the incidentors of the initial incidentors of the red edges must be colored by the same even color.
Results on \((k,l)\)-coloring

- \(\chi_{k,k}(G) = \chi_{k,\infty}(G)\) for \(k \geq \Delta(G) - 1\)
- \(\chi_{k,\Delta(G)-1}(G) = \chi_{k,\infty}(G)\) (Vizing, 2003)

- Let \(\chi_{k,l}(\Delta) = \max\{\chi_{k,l}(G) \mid \deg(G) = \Delta\}\)
- \(\chi_{k,\infty}(\Delta) = k + \Delta\)
- \(\chi_{k,k}(\Delta) \geq \chi_{k,l}(\Delta) \geq k + \Delta\)
- \(\chi_{0,1}(\Delta) = \Delta\)
Results on \((k,l)\)-coloring

- \(\chi_{k,l}(2) = k + 2\) except \(k = l = 0\)
- (Melnikov, P., Vizing, 2000)
- \(\chi_{k,l}(3) = k + 3\) except \(k = l = 0\) and \(k = l = 1\)
  (P., 2003)
- \(\chi_{k,l}(4) = k + 4\) except \(k = l = 0\)
- For \(l \geq \lceil \Delta/2 \rceil\), \(\chi_{k,l}(\Delta) = k + \Delta\)
- (P., 2004)
Results on (1,1)-coloring

- For odd $\Delta$, $\chi_{1,1}(\Delta) > \Delta + 1$ (P., 2004)
Results on (1,1)-coloring

- For odd $\Delta$, $\chi_{1,1}(\Delta) > \Delta + 1$ (P., 2004)
Results on (1,1)-coloring

- For odd $\Delta$, $\chi_{1,1}(\Delta) > \Delta + 1$ (P., 2004)
Results on \((1,1)\)-coloring

- For odd \(\Delta\), \(\chi_{1,1}(\Delta) > \Delta + 1\) (P., 2004)
Results on (1,1)-coloring

• For even $\Delta$, it is unknown whether there is a graph $G$ of degree $\Delta$ such that $\chi_{1,1}(G) > \Delta + 1$. If such $G$ exists, then it has degree at least 6.

• Theorem. $\chi_{1,1}(4) = 5$ (P., 2004)
Proof

• Consider an Eulerian route in a given 4-regular multigraph

• Say that an edge is red, if its orientation is the same as in the route and blue otherwise

• Construct a bipartite interpretation according to this route (it consists of the even cycles)
• Find an edge coloring $f: E \rightarrow \{1,2\}$ such that:

• 1) any two edges adjacent at the right side have distinct colors;

• 2) any two blue or red edges adjacent at the left side have distinct colors;

• 3) If a red edge $e$ meets a blue one $e'$ at the left side, then $f(e) \neq f(e') + 1$
• Construct an incidentor coloring $g$ in the following way:

1) For the right incidentor let $g = 2f$
2) For the left red incidentor let $g = 2f - 1$
3) For the left blue incidentor let $g = 2f + 1$

• We obtain an incidentor $(1,1)$-coloring of the initial multigraph by 5 colors
Example
Example
Example
Incidentor coloring of weighted multigraph

- Each arc $e$ has weight $w(e)$
- Coloring restrictions:
  1) adjacent incidentors have distinct colors;
  2) For every arc $e$, $w(e) \leq b - a$
Results on weighted coloring

• Problem is NP-hard in a strong sense for $\chi = \Delta$ (P., Vizing; 2005)

• For $\chi > \Delta$ the problem is NP-hard in a strong sense even for multigraphs on two vertices (Lenstra, Hoogevean,Yu; 2004)

• It can be solved approximately with a relative error $3/2$ (Vizing, 2006)
List incidentor coloring

• A weighted incidentor coloring where each arc $e$ has a list $L(e)$ of allowed colors for its incidentors

• **Conjecture.** If $|L(e)| \geq w(e) + \Delta$ for every arc $e$ then an incidentor coloring exists

• True for $|L(e)| \geq w(e) + \Delta + 1$. Proved for even $\Delta$ (Vizing, 2001) and for $\Delta = 3$ (P., 2007)
Total incidentor coloring

• Color incidentors and vertices in such a way that vertex coloring is correct and a color of each vertex is distinct from the color of all incidentors adjoining this vertex

• $\chi^T_{k,\infty}(G) \leq \chi_{k+1,\infty}(G)+1 \leq \chi_{k,\infty}(G)+2$

• $\chi^T_{0,\infty}(G) = \Delta+1$ (Vizing, 2000)

• Conjecture. $\chi^T_{k,\infty}(G) \leq \chi_{k,\infty}(G)+1$
Interval incidentor coloring

- The colors of adjacent incidentors must form an interval

- \( \chi^l_{0,\infty}(G) \leq \max\{\Delta, \Delta^+ + \Delta^- - 1\} \)

- \( \chi^l_{1,\infty}(G) \leq \Delta^+ + \Delta^- \)

- For \( k \geq 2 \) there could be no interval incidentor \((k,\infty)\)-coloring (e.g. directed cycle) (Vizing, 2001)
Undirected case

- Instead of $b-a$ use $|b-a|$ for colors of mated incidentors

- Undirected incidentor chromatic number is equal to the best directed ones taken among all orientations
Undirected case

- $\chi_{k,\infty}(G) = \max\{\Delta, \left\lceil\Delta/2\right\rceil + k\}$
- $\chi_{k,\infty}^T(G) \leq \chi_{k,\infty}(G) + 1$ (Vizing, Toft, 2001)

- If $k \geq \Delta/2$ then $\chi_{k,k}(G) = \left\lceil\Delta/2\right\rceil + k$
- If $\Delta = 2kr$ then $\chi_{k,k}(G) = \Delta$
- If $\Delta = 2kr + s$ then $\chi_{k,k}(G) \leq \Delta + k - \left\lfloor s/2\right\rfloor$
- (Vizing, 2005)
Undirected case

- For every regular multigraph $G$ with $\Delta \geq 2k$, $\chi_{k,l}(G) \in \{\Delta, \chi_{k,k}(G)\}$ depending only on $l$; in particular, $\chi_{k,l}(G) = \chi_{k,l}(H)$ for every two regular multigraphs $G$ and $H$ of degree $\Delta$ (Vizing, 2005)
Undirected case

- Interval incidentor coloring of undirected multigraphs always exists

\begin{align*}
\chi_{I,0,\infty}^I(G) = \chi_{I,1,\infty}^I(G) &= \Delta \\
\text{For } k \geq 2, \quad \chi_{I,k,\infty}^I(G) \geq \max\{\Delta, \min\{2k, \Delta + k\}\} \quad \text{and} \\
\chi_{I,k,\infty}^I(G) \leq 2\Delta + k(k - 1)/2 \\
\text{(Vizing, 2003)}
\end{align*}
Undirected case

- The incidentor coloring of weighted undirected multigraph is NP-hard in a strong sense even for $\chi = \Delta$
- It can be solved approximately with a relative error $5/4$
- (Vizing, P., 2008)
Open problems

• 1. Is it true that for every $k$ there is $l$ such that $\chi_{k,l}(\Delta) = \chi_{k,\infty}(\Delta) = k + \Delta$?

• Proved for $k=0$ ($l=1$). Incorrect for $\chi_{k,l}(G)$

• 2. What are the values of $\chi_{1,2}(5)$ and $\chi_{2,2}(5)$?
Open problems

3. Given a \( \Delta \)-regular bipartite graph with red and blue edges is there an edge coloring \( f: E \to \{1, 2, \ldots, \Delta\} \) such that:

1) any two edges adjacent at the right side have distinct colors;
2) any two blue or red edges adjacent at the left side have distinct colors;
3) If a red edge \( e \) meets a blue one \( e' \) at the left side, then \( f(e) \neq f(e') + 1 \)?
Open problems

• 4. Is it true that if $|L(e)| \geq w(e) + \Delta$ for every arc $e$ then a list incidentor coloring exists?

• 5. Is it true that $\chi^T_{k,\infty}(G) \leq \chi_{k,\infty}(G) + 1$?
Thanks for your attention!!!