Nonconforming mimetic methods for diffusion problems

Gianmarco Manzini

Joint collaborations with:

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Outline

1. The construction of an MFD method:
   - meshes;
   - degrees of freedom;
   - approximation of the bilinear form;
   - approximation of the loading term.

2. Consistency condition and degrees of freedom:
   - the conforming MFD formulation;
   - the non-conforming MFD formulation.

3. Building a bridge with VEM.

4. Convergence results and numerical experiments.
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The linear diffusion problem

- Differential formulation:

  \[-\text{div}(K \nabla u) = f \quad \text{in} \quad \Omega,\]
  \[u = g \quad \text{on} \quad \Gamma,\]

  (this talk: constant K)

- Variational formulation:

  \[\text{Find } u \in H^1_0(\Omega) \text{ such that:}\]
  \[\int_{\Omega} K \nabla u \cdot \nabla v \, dV = \int_{\Omega} f v \, dV \quad \forall v \in H^1_0(\Omega),\]
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  \]
Scheme construction in five steps
Steps 1 and 2

1. We decompose $\Omega$ into a mesh $\Omega_h$ of polygons (2-D) or polyhedrons (3-D);
   - admissible meshes may contain "crazy" cells (non-convex, "singular" as in AMR);
   - we need some regularity assumptions to avoid pathological cases and perform the convergence analysis;

2. degrees of freedom: $\mathcal{V}_h$

\[ u, v \in H^1_g(\Omega) \cap C^\alpha(\Omega) \quad \rightarrow \quad u_h, v_h \in \mathcal{V}_h, \quad \text{numbers!} \]

(with $\alpha \geq 0$).
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Scheme construction in five steps
Steps 3 and 4

3. **bilinear form**: $\mathcal{A}_h(\cdot, \cdot) : \mathcal{V}_h \times \mathcal{V}_h \to \mathbb{R}$

$$\mathcal{A}_h(u_h, v_h) \approx \int_{\Omega} K \nabla u \cdot \nabla v \, dV,$$

it is built by “mimicking” a fundamental relation of calculus (*integration by parts*);

4. **linear functional**: $(f, \cdot)_h : \mathcal{V}_h \to \mathbb{R}$

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becomes the “mimetic variational” formulation:

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\text{Find } u_h \in \mathcal{V}_{h,g} \text{ such that:}
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Meshes: why polygonal/polyhedral?

- The meshes should be easily adaptable to the geometric characteristics of the domain, but also to the solution:
  - non-conforming meshes (hanging nodes);
  - (local) adaptive refinements (AMR);
  - highly deformed cells;
  - non-convex cells;
  - curved faces;
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Meshes: academic examples

Examples: convex and non-convex polygonal cells
Meshes: academic examples
Examples: randomized quads and Adaptive Mesh Refinements (AMR)
Meshes: academic examples
Examples: locally refined, prismatic and random hexahedral meshes
Construction of $A_h(u_h, v_h)$

- $A_h(u_h, v_h)$ must be
  - symmetric, bounded and semi-positive;
  - locally defined through an assembly process (like FEM):
    $$A_h(u_h, v_h) = \sum_P A_{h,P}(u_{h,P}, v_{h,P})$$

where $u_{h,P} = u_h|_P$, $v_{h,P} = v_h|_P$;

- Any $A_{h,P}(u_{h,P}, v_{h,P})$ must be a local approximation:
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Construction of $A_h(u_h, v_h)$: consistency and stability

- **PROBLEM:** in MFD we do **not** have an approximation space (as in FEM, DG, VEM, etc) . . . only degrees of freedom!

- **Consistency:** exactness property on polynomials $\rightarrow$ accuracy
  
  Let $u, v \in P_k(P)$, $u_{h,P}, v_{h,P}$ their dofs:

  $$A_{h,P}(u_{h,P}, v_{h,P}) = \int_P K\nabla u \cdot \nabla v \, dV.$$ 

- **Stability:** well-posedness property $\rightarrow$ continuity and coercivity
  
  There exist two constants $\sigma_*, \sigma^*$ such that

  $$\sigma_* \| v_{h,P} \|_{1,h,P}^2 \leq A_{h,P}(v_{h,P}, v_{h,P}) \leq \sigma^* \| v_{h,P} \|_{1,h,P}^2$$

  (for some suitable norm $\| \cdot \|_{1,h,P}$ which mimics the energy norm on $P$)
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Construction of $A_h(u_h, v_h)$: consistency and stability

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  (for some suitable norm $\| \cdot \|_{1,h,P}$ which mimics the energy norm on P)
Let $K$ be constant on $P$. We integrate by parts on the polygonal cell $P$.

- **IF** $u$ is a **linear polynomial** on $P$ $\implies K \nabla u$ is a **constant vector**;

**THEN**

$$
\int_P K \nabla u \cdot \nabla v \, dV = - \int_P \text{div}(K \nabla u) \, v \, dV + \sum_{e \in \partial P} K \nabla u \cdot n_{P,e} \int_e v \, dS
$$

**equal to zero!**

**THUS,**

$$
\int_P K \nabla u \cdot \nabla v \, dV = \sum_{e \in \partial P} K \nabla u \cdot n_{P,e} \int_e v \, dS.
$$
The local consistency condition: two options

The low-order setting, \( m = 1, \ d = 2 \)

1. we use a numerical integration rule on each edge \( e = (v', v'') \), we require the **exactness for linear polynomials**:

\[
\sum_{e \in \partial P} K \nabla u \cdot n_{P,e} \left( \int_e v \, dS \right) \approx \sum_{e \in \partial P} K \nabla u \cdot n_{P,e} |e| \frac{v(x_v') + v(x_v'')}{2}.
\]

**trapezoidal rule**

2. we introduce the 0-th order moment of \( v \) as a degree of freedom:

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\sum_{e \in \partial P} K \nabla u \cdot n_{P,e} \left( \int_e v \, dS \right) = \sum_{e \in \partial P} K \nabla u \cdot n_{P,e} |e| \mu_{e,0}(v)
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where:

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\mu_{e,0}(v) = \frac{1}{|e|} \int_e v \, dS.
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1. Conforming mimetic discretization

The low-order setting, \( m = 1, \ d = 2 \)

1. According to

\[
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\]

we require that

\[
A_{h,P}(u_{h,P}, v_{h,P}) = \sum_{e \in \partial P} K \nabla u \cdot n_{P,e} |e| \frac{v_{v'} + v_{v''}}{2}.
\]

when

- \( u_{h,P} \) is a discrete representation of the linear polynomial \( u \) on \( P \);
- \( v_{v'}, v_{v''} \) are the degrees of freedom of \( v_{h,P} \) at \( v', v'' \).

The dofs represent the vertex values of \( u_{h,P}, v_{h,P} \)
2. Non-conforming mimetic discretization

The low-order setting, $m = 1$, $d = 2$

2. As $|e| \mu_{e,0}(v) = \int_e v \, dS$, and according to:

$$\sum_{e \in \partial P} K \nabla u \cdot n_{P,e} \int_e v \, dS = \sum_{e \in \partial P} K \nabla u \cdot n_{P,e} |e| \mu_{e,0}(v)$$

we require that

$$\mathcal{A}_{h,P}(u_{h,P}, v_{h,P}) = \sum_{e \in \partial P} K \nabla u \cdot n_{P,e} |e| v_{e,0}$$

when

- $u_{h,P}$ is a discrete representation of the linear polynomial $u$ on $P$;
- $v_{e,0}$ is the degree of freedom of $v_{h,P}$ associated with edge $e$.

The dofs represent the zero-th order moments of $u_{h,P}$, $v_{h,P}$
Algebraic consistency: matrices $\mathbb{N}$ and $\mathbb{R}$

Low order setting, $m = 1$, $d = 2$

- **basis of** $\mathbb{P}_1(P) = \left\{ 1, (x - x_P), (y - y_P) \right\} = \{ u_1, u_2, u_3 \}$

  
  $( (x_P, y_P) \text{ is the barycenter of } P)$
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- **matrix $\mathbb{N}$:** degrees of freedom of the polynomial basis:
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- **matrix $\mathbb{N}$**: degrees of freedom of the polynomial basis:

  $$\mathbb{N} = \begin{pmatrix} 
  1 & (x_1 - x_P) & (y_1 - y_P) \\
  1 & (x_2 - x_P) & (y_2 - y_P) \\
  \vdots & \vdots & \vdots \\
  1 & (x_m - x_P) & (y_m - y_P) 
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- Diagram of $P$ with nodes labeled 1, 2, 3, m.
Algebraic consistency: matrices $\mathbb{N}$ and $\mathbb{R}$

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\vdots & \vdots & \vdots \\
1 & (x_m - x_P) & (y_m - y_P)
\end{pmatrix}
\]

- **matrix $\mathbb{R}$**: integration-by-parts for the polynomials $u_i$:

\[
\mathcal{A}_{h,P}(u_{ih,P}, v_{h,P}) = \sum_{f \in P} K \nabla u_i \cdot n_{P,e} \int_e v \, dS = v^T \mathbb{R}_i
\]
Algebraic consistency: $MN = R$

Low order setting, $m = 1$, $d = 2$

RECALL THAT

$$A_{h,P}(u_{ih,P}, v_{h,P}) = \sum_{f \in P} K \nabla u_i \cdot n_{P,e} \int_e v \, dS = v^T R_i$$

SINCE

$$A_{h,P}(u_{ih,P}, v_{h,P}) = v^T M N_i$$

THEN

$$MN_i = R_i \quad i = 1, 2, 3.$$ 

EQUIVALENTLY,

$$MN = R$$
Algebraic consistency: \( MN = R \)

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Algebraic consistency: $\mathbf{M} \mathbf{N} = \mathbf{R}$

Low order setting, $m = 1$, $d = 2$

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$$\mathcal{A}_{h,P}(\mathbf{u}_{ih,P}, \mathbf{v}_{h,P}) = \sum_{f \in P} K \nabla u_i \cdot \mathbf{n}_{P,e} \int_e \mathbf{v} \, dS = \mathbf{v}^T \mathbf{R}_i$$

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$$\mathbf{M} \mathbf{N}_i = \mathbf{R}_i \quad i = 1, 2, 3.$$ 

EQUIVALENTLY,

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Algebraic consistency: $MN = R$

Low order setting, $m = 1$

- The formula $MN = R$ is **ubiquitous** in the MFD method.

- Also,

  $$NT R_{ij} = \int_P K \nabla u_i \cdot \nabla u_j \, dV$$

  where $u_i, u_j \in \{1, x - x_p, y - y_p\}$

- The (one-parameter) formula for the stiffness matrix:

  $$M = R (N^T R)^\dagger R^T + \mu (I - N(N^T N)^{-1} N^T) M_0 + M_1$$

  The second term depends on the parameter $\mu$ and gives a (one-parameter) family of methods.
The stiffness matrix formula

The formula for the stiffness matrix:

\[ M = R(N^T R)^\dagger R^T + \mu (I - N(N^T N)^{-1} N^T) = M_0 + M_1 \]

### Remarks:

- The consistency term \( M_0 \) is responsible for the accuracy of the method.

- The stability term \( M_1 \) ensures the well-posedness of the method.

- The bilinear form \( A_{h,P} \) contains a **stabilization term** that depends on a set of parameters \( \Rightarrow \) **family of schemes**!

- Both terms can be given the same (algebraic) form of the corresponding terms in the VEM.
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- Both terms can be given the same (algebraic) form of the corresponding terms in the VEM.
Three-dimensional case: conforming MFD

The low-order setting, \( m = 1, d = 3 \)

- Recall that \( v_{h|v} := v_v \approx v(x_v') \) and
  \[
  \int_P K \nabla u \cdot \nabla v \, dV = \sum_{f \in \partial P} K \nabla u \cdot n_{P,f} \int_f v \, dS
  \]

- we assume that there exists a **quadrature rule** \( \{(x_{f,v}, \omega_{f,v})_{v \in \partial f}\} \) on each face \( f \in \partial P \) such that
  \[
  \int_f v \, dS \approx \sum_{v \in \partial f} \omega_{f,v} v(x_{f,v})
  \]
  is exact when \( v \) is a linear polynomial;

- we require that for every **linear polynomial** \( u \) and every discrete field \( v_h \) the bilinear form satisfies
  \[
  A_{h,P}(u_{h,P}, v_{h,P}) := \sum_{f \in \partial P} K \nabla u \cdot n_{P,f} \sum_{v \in \partial f} \omega_{f,v} v_v \quad [v_v \text{ represents } v(x_{f,v})].
  \]
Three-dimensional case: non-conforming MFD

The low-order setting, \( m = 1, d = 3 \)

Let \( K \) be constant on \( P \), \( u \) a linear polynomial, and integrate by parts.

- We use the **0-th order moment** of \( v \) as a degree of freedom:

\[
\int_P K \nabla u \cdot \nabla v \, dV = \sum_{f \in \partial P} K \nabla u \cdot n_{P,f} \int_f v \, dS = \sum_{f \in \partial P} K \nabla u \cdot n_{P,e}|e| \mu_{f,0}(v)
\]

where:

\[
\mu_{f,0}(v) = \frac{1}{|f|} \int_f v \, dS.
\]

- The local consistency condition is:

\[
A_{h,P}(u_{h,P}, v_{h,P}) = \sum_{f \in \partial P} K \nabla u \cdot n_{P,e}|f| v_{f,0} \quad [v_{f,0} \text{ represents } \mu_{f,0}(v)]
\]

For both formulations, we do the same as in 2D!
Three-dimensional case: non-conforming MFD

The low-order setting, $m = 1$, $d = 3$

Let $K$ be constant on $P$, $u$ a linear polynomial, and integrate by parts.

- We use the 0-th order moment of $v$ as a degree of freedom:

$$\int_P K \nabla u \cdot \nabla v \, dV = \sum_{f \in \partial P} K \nabla u \cdot n_{P,f} \int_f v \, dS = \sum_{f \in \partial P} K \nabla u \cdot n_{P,e} \left| e \right| \mu_{f,0}(v)$$

where:

$$\mu_{f,0}(v) = \frac{1}{\left| f \right|} \int_f v \, dS.$$

- The local consistency condition is:

$$A_{h,P} \left( u_{h,P}, v_{h,P} \right) = \sum_{f \in \partial P} K \nabla u \cdot n_{P,e} \left| f \right| v_{f,0} \quad \left[ v_{f,0} \text{ represents } \mu_{f,0}(v) \right]$$

For both formulations, we do the same as in 2D!
Summarizing the low-order formulation:

Low order setting, \( m = 1 \)

- Degrees of freedom:

  ![Conforming MFD](image1)
  ![Non-conforming MFD](image2)

- exactness for linear polynomials;

- both 2D and 3D formulations are available (same dofs);

- we only need to implement \( N \) and \( R \) and apply the stiffness matrix formula for \( M \).
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Conforming MFD

Non-conforming MFD

Summarizing the low-order formulation:
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Summarizing the low-order formulation:
Low order setting, \( m = 1 \)

- Degrees of freedom:

  ![Diagram of degrees of freedom](image)

  - exactness for linear polynomials;
  - both 2D and 3D formulations are available (same dofs);
  - we only need to implement \( N \) and \( R \) and apply the stiffness matrix formula for \( M \).
High order: towards a local consistency condition (2D)

The high-order setting, $m > 1$, $d = 2$

Let $K$ be constant and integrate by parts on the polygonal cell $P$:

$$
\int_P K \nabla u \cdot \nabla v \, dV = - \int_P \text{div}(K \nabla u) \, v \, dV + \sum_{e \in \partial P} \int_e K \nabla u \cdot n_{P,e} \, v \, dS.
$$

not zero!

not constant!

If $u$ is a polynomial of degree $m$ on $P$:

- $\text{div}(K \nabla u)$ is a polynomial of degree $m - 2$;
- $K \nabla u \cdot n_{P,e}$ is a polynomial of degree $m - 1$;
Divergence term
Internal degrees of freedom, $m > 1, d = 2$

- For the **conforming** and **non-conforming case**, we use the **moments** of $v$ to express the integral over $P$:

  if 
  $$\text{div}(K \nabla u) = a_0 1 + a_1 x + a_2 y + \ldots \in \mathbb{P}_{m-2}(P)$$

  then 
  $$\int_P \text{div}(K \nabla u) \, v \, dV = a_0 \int_P 1 \, v \, dV + a_1 \int_P x \, v \, dV + a_2 \int_P y \, v \, dV + \ldots$$ 

  $$= a_0 \hat{v}_{P,0} + a_1 \hat{v}_{P,1,x} + a_2 \hat{v}_{P,1,y} + \ldots$$

This choice suggests us to define

- $m(m - 1)/2$ **internal** degrees of freedom $\approx \hat{v}_{P,0}, \hat{v}_{P,1,x}, \hat{v}_{P,1,y}, \ldots$
Edge terms: conforming MFD
Nodal degrees of freedom, \( m > 1, d = 2 \)

- **We use a Gauss-Lobatto formula** with \( m + 1 \) nodes and weights \( \{(x_{e,q}, w_{e,q})\} \) on every (2D) edge \( e \in \partial P \) for:

\[
\int_e K \nabla u \cdot n_{P,e} \nu \, dS \approx \sum_{q=1}^{m+1} w_{e,q} K \nabla u(x_{e,q}) \cdot n_{P,e} \nu(x_{e,q}).
\]

This choice suggests us to define:

- **one degree of freedom per vertex**, \( \nu_{e,1} = \nu_{v'} \approx \nu(x_{v'}) \), \( \nu_{e,m+1} = \nu_{v''} \approx \nu(x_{v''}) \);

- **\((m-1)\) nodal degrees of freedom per edge of** \( P \), \( \nu_{e,q} \approx \nu(x_{e,q}) \) for \( q = 2, \ldots, m \).
High-order conforming MFD
The high-order setting, $m > 1$, $d = 2$

Local Consistency Condition:

Let $K$ be constant.

- For every $u \in \mathbb{P}_m(P)$ ($m \geq 1$) and every discrete field $v_{h,P} \in \mathcal{V}_h$ we require that:

\[
A_{h,P}(u_{h,P}, v_{h,P}) := -\sum_{j=0}^{m(m-1)/2-1} a_j \hat{v}_{P,j} + \sum_{e \in \partial P} \sum_{q=1}^{m+1} w_{e,q} K \nabla u(x_{e,q}) \cdot n_{P,e} v_{e,q}.
\]

**(divergence)** \(a_j \hat{v}_{P,j}\) \(\sum_{j=0}^{m(m-1)/2-1}\)

**(boundary)** \(\sum_{e \in \partial P} \sum_{q=1}^{m+1} w_{e,q} K \nabla u(x_{e,q}) \cdot n_{P,e} v_{e,q}\).

(u_{h,P} are the dofs of $u$ for $P$; terms $a_j \hat{v}_{P,j}$ are conveniently renumbered).
Edge terms: non-conforming MFD

Edge degrees of freedom, $m > 1$, $d = 2$

We use the moments of $\mathbf{v}$ to express the integral over $e \in \partial P$:

If

$$(K \nabla u)_{|e} \cdot \mathbf{n}_{P,e} = b_0 1 + b_1 \xi + b_2 \xi^2 + \ldots \in P_{m-1}(e)$$

then

$$\int_e K \nabla u \cdot \mathbf{n}_{P,e} \, \mathbf{v} \, dS = b_0 \int_e 1 \, \mathbf{v} \, dS + b_1 \int_e \xi \, v \, dS + b_2 \int_e \xi^2 \, v \, dS + \ldots$$

$$= b_0 \hat{\mathbf{v}}_{e,0} + b_1 \hat{\mathbf{v}}_{e,1} + b_2 \hat{\mathbf{v}}_{e,2} + \ldots$$

This choice suggests us to define

- $m$ degrees of freedom per edge $\approx \hat{\mathbf{v}}_{e,0}, \hat{\mathbf{v}}_{e,1}, \hat{\mathbf{v}}_{e,2}, \ldots$
High-order non-conforming MFD
The high-order setting, \( m > 1, d = 2 \)

**Local Consistency Condition:**

Let \( K \) be constant.

- For every \( u \in \mathbb{P}_m(P) \) (\( m \geq 1 \)) and every discrete field \( v_{h,P} \in \mathcal{V}_h \) we require that:

\[
\mathcal{A}_{h,P}(u_{h,P}, v_{h,P}) := - \sum_{j=0}^{m(m-1)/2-1} a_j \hat{v}_{P,j} + \sum_{e \in \partial P} \sum_{j=0}^{m-1} b_j \hat{v}_{e,j}.
\]

\( (u_{h,P} \) are the dofs of \( u \) for \( P \); terms \( a_j \hat{v}_{P,j} \) are conveniently renumbered).
Degrees of freedom
Conforming/non-conforming case

Conforming

\( m=1 \)

\( m=2 \)

\( m=3 \)

Non-Conforming

\( m=1 \)

\( m=2 \)

\( m=3 \)
Algebraic consistency condition: $MN = R$

Let $M$ be a **symmetric** and **semi-positive definite** matrix such that

$$A_{h,P}(u_{h,P}, v_{h,P}) = v_{h,P}^T M u_{h,P}.$$ 

- For any $u \in \{1, x, y, x^2, xy, y^2, \ldots \}$ and any discrete field $v_{h,P}$
  - we write
  $$A_{h,P}(v_{h,P}, u_{h,P}) = v^T M N_u \quad \text{where} \quad N_u = [u_{h,P}] \quad \text{("dofs" of } u);$$
  - we impose the *local consistency condition*:
    $$A_{h,P}(u_{h,P}, v_{h,P}) = \ldots = v^T R_u$$
  - we obtain by comparison:
    $$M N_u = R_u$$
A family of schemes

- Using $\mathbf{N} = [N_1, N_2, \ldots], \mathbf{R} = [R_1, R_2, \ldots]$, we have:

\[
\mathbf{M N} = \mathbf{R} \quad \text{and} \quad (\mathbf{R}^T \mathbf{N})_{ij} = \int P K \nabla u_i \cdot \nabla u_j dV \quad \text{where} \quad u_i, u_j \in \{1, x, y, x^2, \ldots\}.
\]

- $\mathbf{M}$ (symmetric and semi-positive definite) is given by

\[
\mathbf{M} = \mathbf{R} (\mathbf{R}^T \mathbf{N})^{-1} \mathbf{R}^T + \hat{\mathbf{M}} \quad \text{with} \quad \mathbf{MN} = \mathbf{R} + \hat{\mathbf{M}},
\]

where $\hat{\mathbf{M}}$ is a symmetric matrix of parameters.

- A one-parameter ($\gamma$) choice for $\hat{\mathbf{M}}$ is given by:

\[
\hat{\mathbf{M}} = \gamma (\mathbb{I} - \mathbf{N} (\mathbf{N}^T \mathbf{N})^{-1} \mathbf{N}^T).
\]
The linear functional \((f, v_h)_h\)

The low-order case \(m = 1\)

Recall that \((f, v_h)_h \approx \int_{\Omega} fv \, dV\).

- We assemble \((f, v_h)_h\) from local contribution:

\[
(f, v_h)_h := \sum_P (f, v_h)_{h,P} \quad \text{where} \quad (f, v_h)_{h,P} \approx \int_P fv \, dV
\]

- We approximate the forcing term by its average on \(P\):

\[
f \approx \frac{1}{|P|} \int_P f \, dV =: \bar{f}_P;
\]

- We use a (first-order) quadrature based on vertex (conforming) or edge (non-conforming) values. Example: let \(\{(x_v, w_{P,v})\}\):

\[
\int_P fv \, dV \approx \bar{f}_P \int_P v \, dV \approx |P| \bar{f}_P \sum_{v \in \partial P} w_{P,v} v(x_v) \quad [\text{conforming}]
\]
The linear functional \((f, v_h)_h\)

The low-order case \(m = 1\)

- Recall that \((f, v_h)_h := \sum_P (f, v_h)_{h,P}\), where

\[(f, v_h)_{h,P} \approx \int_P fv \, dV, \quad \text{and} \quad \int_P fv \, dV \approx |P| \bar{f}_P \sum_{v \in \partial P} w_{P,v} v_v\]

- Thus, for every cell \(P\) we define

\[(f, v_h)_{h,P} := |P| \bar{f}_P \sum_{v \in \partial P} w_{P,v} v_v \quad \forall v_h \in V_h\]

\[|P| \bar{f}_P = \int_P f \, dV\]

\(w_{P,v}\) 1-st order integration weights.
The linear functional \((f, v_h)_h\)

High-order case \(m > 1\)

1. Again,
   \[
   (f, v_h)_h := \sum_P (f, v_h)_{h,P} \quad \text{where} \quad (f, v_h)_{h,P} \approx \int_P f v \, dV.
   \]

2. For \(m > 1\) we consider the **orthogonal projection** of \(f\) onto the polynomials of degree \(m - 2\):
   \[
   f \approx c_0 1 + c_1 x + c_2 y + \ldots \in \mathbb{P}_{m-2}(P)
   \]

3. and use the moments of \(v\) to express the r.h.s. integral:
   \[
   \int_P f v \, dV \approx c_0 \int_P 1 v \, dV + c_1 \int_P x v \, dV + c_2 \int_P y v \, dV + \ldots
   \]
   \[
   = c_0 \hat{v}_{P,0} + c_1 \hat{v}_{P,1,x} + c_2 \hat{v}_{P,1,y} + \ldots
   \]
The linear functional \((f, \mathbf{v}_h)_h\)

High-order case \(m > 1\)

- Recall that

\[
(f, \mathbf{v}_h)_h := \sum_{P} (f, \mathbf{v}_h)_{h,P} \quad \text{where} \quad (f, \mathbf{v}_h)_{h,P} \approx \int_P f \mathbf{v} \, dV.
\]

- thus, for every cell \(P\) we define

\[
(f, \mathbf{v}_h)_{h,P} := \sum_j c_j \hat{\mathbf{v}}_{P,j} \quad \forall \mathbf{v}_h \in \mathcal{V}_h
\]

\[
f \approx c_0 1 + c_1 x + c_2 y + \ldots \in \mathbb{P}_{m-2}(P)
\]

\[
(c_j) \quad \text{projection coefficients}
\]

\[
\hat{\mathbf{v}}_j \quad \text{moments, degrees of freedom of } \mathbf{v}_h
\]

(The terms \(c_j \hat{\mathbf{v}}_{P,j}\) are conveniently renumbered).
Extension to 3D and variable coefficients

3D formulation

- The 3D conforming formulation should have degrees of freedom associated to vertices, edges, faces and cells: too many!
- For the 3D non-conforming formulation: we use moments on the faces and on the cells as for the VEM method.

Variable coefficients (conforming/non-conforming)

- Modified consistency condition.
  If $u \in \mathbb{P}_m(P)$ and $K(X)$ is variable in $P$: 

$$\int_P K(x) \nabla u \cdot \nabla v \, dV \approx \int_P \Pi_{m-1}(K(x) \nabla u) \cdot \nabla v \, dV = \ldots$$

There exists a VEM counterpart using a modified projector $\tilde{\Pi} \nabla$. 
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  \]

  There exists a VEM counterpart using a **modified projector** \( \tilde{\Pi} \nabla \).
Building a bridge with the VEM

Conforming/non-conforming MFD, $m \leq 1$

- Let $\mathbf{N} = [1, \hat{N}], \mathbf{R} = [0, \hat{R}]$;

\[
\mathbf{N}^T \mathbf{R} = \begin{pmatrix}
0 & 0 \\
0 & \hat{N}^T \hat{R}
\end{pmatrix}
\quad \text{and} \quad
(\mathbf{N}^T \mathbf{R})^\dagger = \begin{pmatrix}
0 & 0 \\
0 & (\hat{N}^T \hat{R})^{-1}
\end{pmatrix}
\]

where $\hat{N}^T \hat{R}$ is symmetric and positive definite.

- Let $\mathbf{G} = -[|\mathbf{P}| \hat{N}^T \hat{R}]^{-\frac{1}{2}} \mathbf{R}^T$. Then,

\[
\mathbf{M}_0 = \mathbf{R}(\mathbf{N}^T \mathbf{R})^\dagger \mathbf{R}^T = \hat{R}(\hat{N}^T \hat{R})^{-1} \hat{R}^T = \mathbf{G}^T \mathbf{G} |\mathbf{P}|.
\]

- $\mathbf{G} u_{h,P} \approx -K \nabla u$ is the flux operator such that

\[
\mathbf{u}^T \mathbf{M}_0 \mathbf{v} = (\mathbf{G} u_{h,P})^T \mathbf{G} v_{h,P} |\mathbf{P}| \approx \int_{\mathbf{P}} K \nabla \nabla (u) \cdot \nabla \nabla (v) \, dV
\]
Building a bridge with the VEM

Conforming/non-conforming MFD, \( m \leq 1 \)

- Let \( N = [1, \hat{N}], \ R = [0, \hat{R}] \);

\[
N^T R = \begin{pmatrix} 0 & 0 \\ 0 & \hat{N}^T \hat{R} \end{pmatrix}
\quad \text{and} \quad
(N^T R)^\dagger = \begin{pmatrix} 0 & 0 \\ 0 & (\hat{N}^T \hat{R})^{-1} \end{pmatrix}
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where \( \hat{N}^T \hat{R} \) is symmetric and positive definite.

- Let \( G = -[|P| \hat{N}^T \hat{R}]^{-\frac{1}{2}} R^T \). Then,

\[
M_0 = R (N^T R)^\dagger R^T = \hat{R} (\hat{N}^T \hat{R})^{-1} \hat{R}^T = G^T G |P|.
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- \( G u_{h,P} \approx -K \nabla u \) is the flux operator such that

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u^T M_0 v = (G u_{h,P})^T G v_{h,P} |P| \approx \int_P K \nabla \nabla (u) \cdot \nabla \nabla (v) \, dV
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Building a bridge with the VEM
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where $\hat{N}^T \hat{R}$ is symmetric and positive definite.

- Let $G = -[|P| \hat{N}^T \hat{R}]^{-\frac{1}{2}} R^T$. Then,

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$$u^T M_0 v = (G u_{h,P})^T G v_{h,P} |P| \approx \int_P K \nabla \nabla (u) \cdot \nabla \nabla (v) \, dV$$
Building a bridge with the VEM

Similarities and differences:

For both the conforming and the non-conforming MFD and VEM formulations we can prove that:

- the **degrees of freedom** are the same;

- the **consistency** term is the same:
  - in the MFD setting it relates to an exactness property;
  - in the VEM setting it is the projection of the bilinear form on polynomials;

- the **stabilization** term of VEM forms a **subset** of those of MFD:
  - in the MFD setting it gives the proper rank of the stiffness matrix;
  - in the VEM setting it relates to the non-computable part of the bilinear form;

- the formulation is different: VEM has the advantage of being a FEM!
Building a bridge with the VEM

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  - in the VEM setting it is the projection of the bilinear form on polynomials;

- the **stabilization** term of VEM forms a **subset** of those of MFD:
  - in the MFD setting it gives the proper rank of the stiffness matrix;
  - in the VEM setting it relates to the non-computable part of the bilinear form;

- the formulation is different: VEM has the advantage of being a FEM!
Building a bridge with the VEM

Similarities and differences:

For both the conforming and the non-conforming MFD and VEM formulations we can prove that:

- the **degrees of freedom** are the same;

- the **consistency** term is the same:
  - in the MFD setting it relates to an exactness property;
  - in the VEM setting it is the projection of the bilinear form on polynomials;

- the **stabilization** term of VEM forms a **subset** of those of MFD:
  - in the MFD setting it gives the proper rank of the stiffness matrix;
  - in the VEM setting it relates to the non-computable part of the bilinear form;

- the formulation is different: VEM has the advantage of being a FEM!
MFD and VEM: much more than a bridge!

For the Poisson equation (in primal form) we have:

- **Conforming MFD**
  - 2009 *low-order, 2D-3D*: Brezzi, Buffa, Lipnikov (M2AN);
  - 2011 *high-order, 2D*: Beirao da Veiga, Lipnikov, M. (SINUM);

- **Conforming VEM**
  - 2013 *any order, 2D*: ”volley” team (M3AS);

- **Non-conforming MFD**
  - 2014 *any order, 2D-3D*: Lipnikov, M., (JCP);

- **Non-conforming VEM**
A mesh-dependent norm

Conforming case

We consider the mesh-dependent norm

$$\| v_h \|_{1,h}^2 = \sum_{P \in \Omega_h} \| v_h \|_{1,h,P}^2$$

that mimics the \( | \cdot |_{1,\Omega} \) semi-norm;

- for the low-order method \((m = 1, d = 2, 3), e = (v', v'')\) being an edge,

$$\| v_h \|_{1,h,P}^2 = \| \text{GRAD}_h(v_h) \|_{h,P}^2 = h_P \sum_{e \in \partial P} | v_{v''} - v_v |^2 ;$$

- for the high-order method \((m > 1, d = 2), e = (v', v'')\) being an edge,

$$\| v_h \|_{1,h,P}^2 = h_P \sum_{e \in \partial P} \left\| \frac{\partial v_h}{\partial S} \right\|_{L^2(e)}^2 + \left[ \text{"moments"} \right]$$
Convergence results

Conforming case

The **consistency** and the **stability** conditions allow us to determine a **family of mimetic schemes**:

- for the **low-order** method $m = 1$:
  \[
  \| u^I - u_h \|_{1,h} < Ch( |f|_{0,\Omega} + |u|_{1,\Omega} + |u|_{2,\Omega} );
  \]
  \((\text{Brezzi, Buffa, Lipnikov, M2AN (2009)})\),

- for the **high-order** method $m > 1$:
  \[
  \| u^I - u_h \|_{1,h} < Ch^m \| u \|_{m+1,\Omega};
  \]
  \((\text{Beirao da Veiga, Lipnikov, M., SINUM (2011); VEM, Brezzi et. al. M3AS ("volley" paper)})\)

(For the non-conforming case refer to the talk of Blanca A.).
Conforming MFD method

Meshes with randomized quadrilaterals

- **Meshes:**

- **Exact solution:** \( u(x, y) = (x - e^{2(x-1)})(y^2 - e^{3(y-1)}) \)

- **Diffusion tensor**

\[
K = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]
Conforming MFD method

Randomized quadrilaterals, $||\cdot||_{1,h}$ errors, constant $K$

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Conforming MFD method

Meshes with non-convex polygons

- Meshes:

  \[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

- Exact solution: \( u(x, y) = e^{-2\pi y} \sin(2\pi x) \)

- Diffusion tensor

  \[ K = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad K(x, y) = \begin{pmatrix} (x + 1)^2 + y^2 & -xy \\ -xy & (x + 1)^2 \end{pmatrix} \]
Conforming MFD method
Non-convex polygons, $\| \cdot \|_1, h$ errors, constant $K$

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G. Manzini
Conforming MFD method
Non-convex polygons, $\| \cdot \|_{1,h}$ errors, non-constant $K$

### $m = 2$

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### $m = 5$

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</table>
Non-conforming MFD method

Meshes with random hexahedra

- Meshes:

- Exact solution: \( u(x, y, z) = x^3 y^2 z + x \sin(2\pi xy) \sin(2\pi yz) \sin(2\pi z) \)

- Diffusion tensor

\[
K = \begin{pmatrix}
1 + y^2 + z^2 & -xy & -xz \\
-xz & 1 + x^2 + z^2 & -yz \\
-xy & -yz & 1 + x^2 + y^2
\end{pmatrix}
\]
Non-conforming MFD method

Meshes with random hexahedra

\[ L^2(\Omega) \text{ relative error} \]

\[ H^1(\Omega) \text{ relative error} \]

The error is given by \( u - \Pi_m^\nabla(u_h) \).
Conclusions

The conforming and non-conforming MFD methods are such that:

(i) the low-order formulation uses either vertex or edge values to represent linear polynomials; it works in 2-D and 3-D;

(ii) the high-order formulation uses edge nodal values and moments to represent $m$-degree polynomials; it works in 2-D and 3-D (only non-conforming).

(iii) a reformulation as finite element exists in the virtual element framework.

Possible future developments:

(i) more complex operators (+convection, +reaction);
(ii) exploit the strong connection with the VEM;
(iii) curved faces;
(iv) …
Conclusions

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  (ii) exploit the strong connection with the VEM;
  
  (iii) curved faces;
  
  (iv) …
A few references...


